Regularity of competitive equilibria in a production economy with externalities

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Object of our analysis: externalities and regular economies.

The model: a production economy with externalities.

Basic assumptions.

An example with a continuum of equilibria.

Perturbation of the production sets.

Two additional assumptions.

Our main result: Almost all perturbed economies are regular in the space of endowments and perturbations.
We consider a production economy with consumption and production externalities.

Our model of externalities is based on the seminal works by


where individual preferences and production technologies depend on the choices of all individuals and firms.
Some examples of externalities à la Laffont.

• Building a mall in a residential area is an example of positive (or negative) externalities created by a firm on the preferences of people living in that area.

• Externalities may be created by consumers on firms. For instance, an over consumption of air-conditioning and consequently of electricity might produce an electrical breakdown, decreasing all the production activities.

• Finally, externalities may be created by firms on firms. Indeed, a firm that extracts oil from a land can affect another firm that extracts oil from a nearby land whenever the oil comes from an joint underground reservoir.
Our goal is to give sufficient conditions for the regularity of a production economy with externalities à la Laffont.

**Why do we care about regular economies?**

Comparative static analysis. In a *regular economy*,

- the number of equilibria is finite and
- equilibria locally depend on the parameters describing the economy in a continuous or differentiable manner.

So, one can perform comparative static analysis.
Regular economies: Two key aspects

1. Pareto improving policies

In the presence of externalities, competitive equilibria are not necessarily Pareto optimal. It is an open and important issue to study Pareto improving policies.

The “generic set” of **regular economies is the starting point to implement Pareto improving analysis.**

2. Testable restrictions

Before implementing any public policy, one should verify whether the observed data are consistent with the economic model.

One of the methodologies to check whether or not a model is testable is based on **regular economies.**

So, our work is a first step to study testable restrictions and Pareto improving policies in production economies with externalities.
The approach: Our basic references

Following recent contributions by


We prove our regularity results using *Smale’s extended approach*, i.e., equilibria are described in terms of first order conditions and market clearing conditions.
1. Villanaci and Zenginobuz (2005) focus on a specific kind of externalities, namely public goods.

2. In Bonnisseau and del Mercato (2010), only consumption externalities are considered.

3. The model of Mandel (JME, 2008) allows for non-convexity on the production side.

   • The author uses an approach based on the aggregate excess demand function. Consequently, the author enlarges the commodity space treating externalities as additional variables.

   • In order to get a regularity result for almost all initial endowments, the author makes an assumption which implicitly involves the Lagrange multipliers, that is the equilibrium prices.
The model

A finite number $C$ of commodities, $\mathbb{R}^C$ is the commodity space, $p \in \mathbb{R}_+^C$ is a price system.

A finite set $\mathcal{H}$ of households, $h \in \mathcal{H}$, a finite set $\mathcal{J}$ of firms, $j \in \mathcal{J}$.

- $x_h$ is the consumption of household $h$ and $x = (x_h)_{h \in \mathcal{H}}$.
- $e_h$ is the endowment of household $h$, $s_{hj}$ is its share on the profit of firm $j$.
- $y_j$ is the production plan of firm $j$ and $y = (y_j)_{j \in \mathcal{J}}$.
- $x_{-h} = (x_k)_{k \neq h}$ is the consumption of all households other than $h$, $(x_{-h}, y)$ represents the \textbf{externality} created by the others on $h$.
- $y_{-j} = (y_z)_{z \neq j}$ is the production plan of all firms other than $j$, $(y_{-j}, x)$ represents the \textbf{externality} created by the others on $j$. 
As usual, in a production economy with externalities,

1. Individual preferences are affected by the choices of all the other agents. Individual preferences are represented by a utility function

   \[ u_h(x_h, x_{-h}, y) \]

2. The production technology is affected by the choices of all the other agents. The production set of firm \( j \) is given by

   \[ Y_j(y_{-j}, x) \subseteq \mathbb{R}^C \]

In order to model smooth production sets, we describe each production set by an inequality on a differentiable function \( t_j \) called the transformation function. That is,

\[ Y_j(y_{-j}, x) = \left\{ y_j \in \mathbb{R}^C : t_j(y_j, y_{-j}, x) \geq 0 \right\} \]
Competitive equilibrium à la Nash

**Definition** \((x^*, y^*, p^*)\) is a *competitive equilibrium* if

1. for all firm \(j\), \(y^*_j\) solves

\[
\max_{y_j \in \mathbb{R}^C} p^* \cdot y_j \\
\text{subject to } t_j(y_j, y^*_j, x^*) \geq 0
\]

2. for all household \(h\), \(x^*_h\) solves

\[
\max_{x_h \in \mathbb{R}_+^C} u_h(x_h, x^*_h, y^*) \\
\text{subject to } p^* \cdot x_h \leq p^* \cdot e_h + \sum_{j \in J} s_{hj} p^* \cdot y^*_j
\]

3. \((x^*, y^*)\) satisfies Market Clearing Conditions, i.e.

\[
\sum_{h \in \mathcal{H}} x_h^* = \sum_{h \in \mathcal{H}} e_h + \sum_{j \in J} y_j^*
\]
Basic assumptions

Assumptions on production sets
• Production sets are closed and smooth.
• Fix the externalities, standard assumptions on the production set $Y_j(y_{-j}, x)$, i.e. possibility of inaction, free-disposal, and convexity.
• At the aggregate level, asymptotic irreversibility and no free lunch (i.e. the set of feasible allocations is compact).

Assumptions on utility functions
• $u_h$ is continuous and $C^2$ in the interior of its domain.
• Fix the externalities, standard assumptions on $u_h$, i.e. strictly increasing, strictly quasi-concave, and the upper contour sets do not intersect the boundary.
We provide an example of a production economy with externalities where \textbf{for all endowments} one gets a \textit{continuum of equilibria}.

Two commodities, one household, two firms. Each firm uses commodity 2 to produce commodity 1.

- $y_1 = (y_1^1, y_1^2)$ is a production plan of firm 1 and $y_2 = (y_2^1, y_2^2)$ is a production plan of firm 2.

- The production set of firm 1 is given by

$$Y_1(y_2^1) = \{y_1 \in \mathbb{R}^2 : y_1^2 \leq 0, t_1(y_1, y_2^1) = 2\sqrt{-y_1^2 - y_1^1 - y_2^1} \geq 0\}$$

- The production set of firm 2 is given by

$$Y_2(y_1^1) = \{y_2 \in \mathbb{R}^2 : y_2^2 \leq 0, t_2(y_2, y_1^1) = 2\sqrt{-y_2^2 - y_2^1 - y_1^1} \geq 0\}$$
• The utility function of the household is \( u(x^1, x^2) = \frac{1}{2} \ln x^1 + \frac{1}{2} \ln x^2 \) and \( e = (e^1, e^2) \gg 0 \) is his initial endowment.

At equilibrium, one gets

• (Firm 1’s supply) \( y_1^* = 2p^* - y_2^* \) and \( y_1^* = -(p^*)^2 \)

• (Firm 2’s supply) \( y_2^* = 2p^* - y_1^* \) and \( y_2^* = -(p^*)^2 \)

• \( p^* = \frac{1}{8} \left( \sqrt{(e^1)^2 + 16e^2} - e^1 \right), x^1 = \frac{1}{2p^*} (p^* e^1 + e^2), x^2 = \frac{1}{2} (p^* e^1 + e^2) \).

So, for all initial endowment \( e = (e^1, e^2) \gg 0 \) any bundle

\[ ((p^*, 1), x^*, y_1^*, y_2^*) \] with \( y_2^* \in [0, 2p^*] \)

is a competitive equilibrium for the economy \( e \).
Why does regularity fail in our example?

First, note that without externalities at all, the transformation function of the firm $j$ is

$$t_j(y_j^1, y_j^2) := 2\sqrt{-y_j^2 - y_j^1} \geq 0$$

Consequently, the output supply of firm $j$ is given by

$$y_{j*1}^1 = 2p^*$$

Thus, if the output price increases, then the output supply of both firms increases too.

Since the equilibrium price is determined, equilibria are completely determined.
In our example,

for given $y_{2}^{*1}$, if $p^{*}$ increases by $k$ units then the output supply of firm 1 given by

$$y_{1}^{*1} = 2p^{*} - y_{2}^{*1}$$

increases by $2k$ units.

Consequently, the output supply $y_{2}^{*1}$ of firm 2 does not change since the price increase is compensated by firm 1’s output increase, that is

$$y_{2}^{*1} = 2(p^{*} + k) - (y_{1}^{*1} + 2k) = 2p^{*} - y_{1}^{*1}$$

So, the output supply of firm 2 is indeterminate, since the two effects resulting from changes in prices and externalities offset each others.
In order to overcome the previous effect, we consider a *displacement* of the boundary of the production sets, i.e. simple perturbations of the transformation functions. So, for all firms $j$ we consider the transformation function

$$t_j + b_j \text{ with } b_j \in \mathbb{R}_{++}$$

We remark that in our example,

1. the perturbations of the production sets are sufficient to control the *first-order external effects*,

2. there are no *second-order external effects* since the derivatives of the marginal productions with respect to the choices of the others are equal to zero.
As shown by Bonnisseau and del Mercato (2010), regularity may fail **whenever the second-order external effects are too strong.**

So, in order to control the second-order effects, we assume that

**Assumption 1.** The external effect of firms other than $j$ on the marginal production $D_{y_j} t_j(y_j, y_{-j}, x_h)$ is “dominated” by the direct effect of firm $j$.

**Assumption 2.** The external effect of household other than $h$ on the marginal utility $D_{x_h} u_h(u_h, x_{-h}, y_j)$ is “dominated” by the direct effect of household $h$. 
Regularity with externalities: Our main results

Fix utility and transformation functions \( t = (t_j)_{j \in J} \). A perturbed economy \((t + b, e)\) is parametrized by transformation levels \( b = (b_j)_{j \in J} \in \mathbb{R}_+^J \) and initial endowments \( e = (e_h)_{h \in H} \gg 0 \).

Using standard techniques, i.e. *Regular Value Theorem* and *Transversality Theorem*, we prove that

**Theorem.** The set \( \mathcal{R} \) of perturbed regular economies is an open and full measure subset of the space of endowments and transformation levels.

**Corollary.** Using the \( C^2 \) Whitney topology, the set of regular economies is an open subset of the space of endowments and transformation functions and it contains the set

\[
\bigcup_{t \in \mathcal{T}} \{(t + b, e) : (b, e) \in \mathcal{R}\}
\]
Thanks!
Two additional Assumptions

Assumption 1. Let \((x, y, z)\) such that \(z_j \cdot D_{y_j} t_j(y_j, y_{-j}, x) = 0\) for every \(j\) and \(\sum_{j \in J} z_j = 0\), then \(z_j \sum_{f \in J} D^2_{y_f y_j} t_j(y_j, y_{-j}, x)(z_f) < 0\) if \(z_j \neq 0\).

Indeed, the absolute value of \(z_j D^2_{y_j} t_j(y_j, y_{-j}, x)(z_j)\) is larger than the remaining term \(z_j \sum_{f \neq j} D^2_{y_f y_j} t_j(y_j, y_{-j}, x)(z_f)\).

Assumption 2

Let \((x, v, y, z)\) such that \(v_h \cdot D_{x_h} u_h(x_h, x_{-h}, y) = 0\) for every \(h\), \(z_j \cdot D_{y_j} t_j(y_j, y_{-j}, x) = 0\) for every \(j\) and \(\sum_{h \in H} v_h = \sum_{j \in J} z_j\), then

1. \(v_h \sum_{k \in H} D^2_{x_k x_h} u_h(x_h, x_{-h}, y)(v_k) < 0\) whenever \(v_h \neq 0\),

2. \(z_j \sum_{k \in H} D^2_{x_k y_j} t_j(y_j, y_{-j}, x)(v_k) \leq 0\) whenever \(v_k \cdot D_{y_j} t_j(y_j, y_{-j}, x) = 0\) for every \(j\) and for every \(k \in H\).
Theorem (Regular Value Theorem) Let $M, N$ be $C^r$ manifolds of dimensions $m$ and $n$, respectively. Let $f : M \to N$ be a $C^r$ function. Assume $r > \max\{m - n, 0\}$. If $y \in N$ is a regular value for $f$, then

1. if $m < n$, $f^{-1}(y) = \emptyset$,

2. if $m \geq n$, either $f^{-1}(y) = \emptyset$, or $f^{-1}(y)$ is an $(m - n)$-dimensional submanifold of $M$.

Corollary Let $M, N$ be $C^r$ manifolds of the same dimension. Let $f : M \to N$ be a $C^r$ function. Assume $r \geq 1$. Let $y \in N$ a regular value for $f$ such that $f^{-1}(y)$ is non-empty and compact. Then, $f^{-1}(y)$ is a finite subset of $M$. 
**Theorem** (Transversality Theorem) Let $M$, $\Omega$ and $N$ be $C^r$ manifolds of dimensions $m$, $p$ and $n$, respectively. Let $f : M \times \Omega \to N$ be a $C^r$ function. Assume $r > \max\{m - n, 0\}$. If $y \in N$ is a regular value for $f$, then there exists a full measure subset $\Omega^*$ of $\Omega$ such that for any $\omega \in \Omega^*$, $y \in N$ is a regular value for $f_\omega$, where

$$f_\omega : \xi \in M \to f_\omega(\xi) := f(\xi, \omega) \in N$$

**Theorem** (Implicit Function Theorem) Let $M$, $N$ be $C^r$ manifolds of the same dimension. Assume $r \geq 1$. Let $(X, \tau)$ be a topological space, and $f : M \times X \to N$ be a continuous function such that $D_\xi f(\xi, x)$ exists and it is continuous on $M \times X$. If $f(\xi, x) = 0$ and $D_\xi f(\xi, x)$ is onto, then there exist an open neighborhood $I$ of $x$ in $X$, an open neighborhood $U$ of $\xi$ in $M$ and a continuous function $g : I \to U$ such that $g(x) = \xi$, $f(\xi', x') = 0$ holds for $(\xi', x') \in U \times I$ if and only if $\xi' = g(x')$, and $D_\xi f(\xi', x')$ is onto for every $(\xi', x') \in U \times I$ such that $f(\xi', x') = 0$. 

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Externalities in Production Economies
Let $B := \mathbb{R}^{CJ} \times \mathbb{R}^{CH}_{++}$. We are interested on $C^2$ functions defined on $B$. $C^2(B, \mathbb{R})$ denotes the set of $C^2$ functions from $B$ to $\mathbb{R}$.

Since $C^2(B, \mathbb{R})$ is a linear space, in order to define the $C^2$ Whitney topology on $C^2(B, \mathbb{R})$, it is enough to define neighborhood basis of the function zero.

Let $\delta : B \rightarrow \mathbb{R}$ be a continuous and strictly positive function. The open neighborhood $N(0, \delta)$ of the function $0 \in C^2(B, \mathbb{R})$ is defined as

$$N(0, \delta) := \{ g \in C^2(B, \mathbb{R}) : |g(z)| < \delta(z) \ \forall \ z \in B \ \text{and} \ \\ \|D^k g(z)\| < \delta(z) \ \forall \ z \in B \ \text{and} \ \forall \ k = 1, 2 \}$$
Neighborhoods of $f \in C^2(B, \mathbb{R})$, for $f \neq 0$ can be constructed for translation, that is the neighborhood of $f$ determined by $\delta$ is given by $N(f, \delta) = f + N(0, \delta)$.

For every $f \in C^2(B, \mathbb{R})$, the collection of $\{N(f, \delta)\}_{\delta \in C^0(B, \mathbb{R}^{++})}$ forms a neighborhood basis of the function $f$ in the $C^2$ Whitney topology, where $\delta$ varies in the space of all continuous and strictly positive functions.

The $C^2$ Whitney topology on $C^2(B, \mathbb{R})$ is not necessarily metrizable. However, if the set $B$ is compact, the $C^2$ Whitney topology coincides with the topology of the $C^2$ uniform convergence on compacta which is metrizable.