

# Credit Cycles with Renegotiation\*

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PRELIMINARY AND INCOMPLETE

## Abstract

The significant amplification effect of collateral constraints in Kiyotaki and Moore (1997) for adverse shocks depends on the assumption that there is no default ex post, even when the outstanding debt is larger than the value of collateral assets. We show that allowing for default eliminates the dynamic amplification effects in their setup. However, when there is specificity in the use of productive assets, borrowers will prefer to keep them. This creates a wedge between the inside and outside value of assets setting the stage for bargaining over debt repayments. In this case we do observe amplification effects for moderate shocks, while for larger shocks debt renegotiation dampens amplification. Furthermore, shocks are more persistent the more specific the productive assets. The association found between the degree of asset specificity and the form and intensity of the financial accelerator of economic downturns is consistent with features of observed behavior in some macroeconomic crises. An empirical test of the model's transmission mechanism is presented.

KEYWORDS: Credit cycles; balance sheet recessions; default; renegotiation.

JEL CLASSIFICATION CODE: E32, E44, G01.

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# 1 Introduction

In times of crisis, financial intermediation may have major effects in the propagation and amplification of large macroeconomic fluctuations.<sup>1</sup> Among the mechanisms through which these effects take place, changes in the supply of credit for productive uses resulting from shifts in collateral constraints seem particularly salient. A negative shock that affects the net worth of levered borrowers has persistent effects as it can take time for productive agents to return to their original production plan with resources generated through retained earnings. If productive agents are forced to sell capital to less efficient users this depresses asset prices. Current prices are further depressed by the negative impact that the persistence of temporary shocks has on future prices, leading potentially to significant amplification of the initial shock. Such a “dynamic multiplier” effect was formalized in the celebrated model of Kiyotaki and Moore (1997) (hereafter KM). A question arises, both for analytical and practical concerns, about the conditions that generate that remarkable result and, more generally, about those that may determine the intensity of financial multipliers.

In KM, productive firms face a credit limit given by the expected value of their capital stock. They cannot commit their effort to production, so that lenders do not consider output prospects as adequate collateral. At the same time, impatient business owners, acting on the belief that their expectations are error free, have incentives to borrow up to the collateral constraint.<sup>2</sup> If now, after work has been engaged, a wholly unanticipated aggregate “rare event” disturbs for one period the production of levered firms, the promised debt repayments exceed the post shock value of their capital. By construction, the debt contract is assumed to be specified unconditionally in terms of quantities of output. Then, in order to fulfill its promises, the firm not only needs to deliver back its available capital, but must also use some of its current output for debt servicing. Interestingly, in this setup, the potential of default constrains *ex ante* the supply of credit below the capital stock, but once the shock is realized, debtors honor their commitments in full even when *ex post* the value of capital turns out to be insufficient.

This asymmetry is crucial for the large multiplier effect. The strong financial amplification is caused by the combination of fears of default that limit the supply of credit with the absence of actual default when the economy is hit by the unforeseen shock. In contrast we show that, if debt repayment is bounded by capital (e.g., because the legal environment is such that creditors can appropriate fixed assets, but are unable to get hold of produced goods unless willingly delivered), so that the whole volume of current output remains available to borrowers, a transitory fall in productivity has second order effects on asset prices, which moderates the multiplier linking the negative shock to the fall in output.<sup>3</sup> Thus, when the collateral constraint applies both *ex ante* and *ex post*,

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<sup>1</sup>See for example the empirical findings of Aikman et al. (2010), and Jordà et al. (2011).

<sup>2</sup>This is a result of the simplifying assumptions in KM. In a more general setting producers would hold liquidity buffers to partially insure against the risk of hitting the borrowing constraint. See e.g. Brunnermeier et al. (2011).

<sup>3</sup>However, the strong amplification would still hold for positive displacements, as long as productive agents do not hold a liquidity buffer, i.e. as long as shocks are zero probability events.

the financial accelerator is much weaker. Default on the seemingly unconditional contract smoothes the adjustment to the negative shock. However, this result depends on the costlessness for debtors of limiting repayments below the promised amounts.

Even without outside sanctions, and without reputation concerns, borrowers may have incentives to avoid situations where the liquidation of their firm brings about their separation from the productive assets that they have been holding. It seems reasonable to assume that entrepreneurs are more productive running a firm they are “familiar” with, than they would be working with other assets having the same market value. This *specificity* increases the bargaining power of creditors in potential repayment negotiations, by strengthening the threat of liquidation.<sup>4</sup> Thus, a shock would create a tension between the non-appropriability of output by creditors and their ability to extract resources from debtors who try to maintain access to the assets that they have learned to operate effectively. This added feature implies that the financial multiplier is larger than in the costless default case, but there is a bound such that further amplification does not occur. For small shocks,  $0 < \Delta < \bar{\Delta}$  (where  $-\Delta$  is the percentage change in productivity), the desire to keep their assets induces productive entrepreneurs to repay in full their debts (which tends to depress asset prices), while for larger shocks,  $\bar{\Delta} < \Delta$ , the threat of default is credible and induces lenders to renegotiate debt repayments. The threshold  $\bar{\Delta}$  is higher, so that more deleveraging takes place and asset prices fall more intensely, the stronger the specificity of assets.

We obtain the main results for the case in which asset specificity affects the productivity of non-tradable output. We then show in two extensions the effects of asymmetric information about default costs, and the effects of reduced productivity of tradable output. In the former we find that the more dispersed the probable costs for a borrower, the less willing is a lender to renegotiate the debt contract as this will result in capital losses with little impact on the margin of borrower that would accept the new terms. Thus more asymmetric information results in *less* amplification of the initial shock. In the latter we show that more asset specificity leads to more persistence since in this case the borrower has less output to pledge on the market, and thus needs more time to rebuild her initial working capital.

Economies that have reached states where the business sector has accumulated excessive debts can adapt in several ways. Leaving aside government interventions (like bailouts or legislated write offs), the process of debt reduction may take two polar forms: a gradual deleveraging and a sudden burst of defaults and debt restructurings. In the first instance, firms go through a prolonged period where they use current income for the purpose of servicing debts; consequently, their access to productive resources is diminished. In the other extreme, after bankruptcy procedures have been completed, the restoration of solvency would bring about a relief of the financial constraints on production and investment. Our model can depict in broad terms these two scenarios, and specifies conditions (linked to asset specificity) that may make an economy go one way or another. However, the

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<sup>4</sup>Note that this concept is broader than the usual one for “asset specificity” (related to their market liquidity) since it involves the specificity of the relation between a given entrepreneur and the assets she uses for production. Alternatively one can interpret this friction as a “learning by doing” effect in the production process.

simplifying assumption of immediate and costless debt renegotiations results in a overly benign picture of the case of default, which leaves aside the drama and the macroeconomic disruptions associated with large-scale redrawings of assets and liabilities.<sup>5</sup>

The association found between the degree of asset specificity and the form and intensity of the financial propagators of economic downturns allows parallels with features of observed behaviors in macroeconomic crises. A long-drawn process of corporate debt reductions in a stagnant economy with, at the same time, few breakdowns or restructurings of large business firms or groups, like that experienced in the 1990's in Japan (e.g. Koo, 2003; Chang, 2006a; Ahmadjian, 2006) would seem to correspond to a case with highly specific productive assets, which motivates strong efforts to repay debts on the parts of owner/managers of assets and, as a counterpart, accentuates the effect of financial multipliers. By contrast, large-scale transfers of property and control took place in other cases, such as that of Indonesia in the late 1990's, with a macroeconomic crisis characterized by a very agitated period with numerous renegotiations of private debts. This pattern would roughly correspond to a low specificity scenario. It has been suggested (e.g. Chang, 2006b; Hanani, 2006) that the shifts of assets that happened in the Indonesian case were often linked to political influences: political contacts are more "portable" across business activities than technological or managerial expertise.

Beyond this qualitative analysis of balance sheet recessions, our model provides testable implications of its transmission mechanism, i.e., of the relation between the degree of ex post capital losses incurred by creditors (affected by the level of protection of creditor rights, and the degree of asset specificity in a given industry) and the behavior of credit constrained firms over the business cycle. We intend to test these relations using U.S. firm level data.<sup>6</sup> For this we will exploit heterogeneity in the levels of debtor protection in personal bankruptcy procedures across U.S. states, and in real asset redeployability across industries.<sup>7</sup> It is well known that personal bankruptcy law affects small firms' access to credit (see e.g. Berkowitz and White (2004) or Berger et al. (2011)). Combining this result with the model's predictions of more debt renegotiation in industries with low asset specificity, and in environments with less protection of creditor rights, we expect to find differential effects in small firms' response to their business cycle (as measured by the fluctuations in output of large firms) across industries and states.

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The large multiplier effects found in KM have been discussed in the literature. Kocherlakota (2000) builds an economy with a representative agent who has an incentive to

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<sup>5</sup>A more realistic treatment of the complications of default and renegotiation may extend the model to encompass relevant effects of disorganization (to recall, in another context, the notions presented in Blanchard and Kremer (1997)), and permit to incorporate in the analysis transitional output costs of redefining contracts.

<sup>6</sup>For example, the Longitudinal Business Database described by Jarmin and Miranda (2002) that has a long series of annual observations of employment and payroll for private firms, or the National Establishment Time Series, described in Neumark et al. (2007).

<sup>7</sup>We use real asset redeployability as a proxy of asset specificity. We take this measure from Beutler and Grob ty (2011).

smooth consumption, making it optimal to hold liquid assets to self insure against negative shocks. In that setting, the quantitative significance of the amplification effects generated by collateral constraints hinges on the parameters of the economy, in particular on factor shares. In the same spirit, Cordoba and Ripoll (2004) show in a calibrated closed economy with heterogeneous agents that the amplification and propagation generated by credit constraints need not be quantitatively large if preferences allow for risk aversion and the capital share in production has values similar to those in actual economies. Moreover they find that there exists a trade-off between amplification and persistence. In contrast, Liu et al. (2009) show that when economic shocks impact directly upon the price of collateral assets, quantitatively significant amplification is restored.

More recently, Brunnermeier and Sannikov (2010) build a continuous time two sector model which is solved without using a log-linear approximation. The resulting dynamics feature highly non-linear amplification effects and, unlike KM, multipliers are asymmetric, stronger for negative shocks. Entrepreneurs are aware that they might experience adverse shocks, and choose to hold liquidity buffers. This mitigates moderate shocks dampening their amplification. However, in response to more severe shocks, entrepreneurs reduce their asset demand, affecting their price and triggering amplification loops, thus resembling a balance sheet crisis. The stationary distribution has a double-humped shape suggesting that crises are highly persistent.

Our model is related to the literature on the macroeconomic implications of financial frictions (See e.g. Bernanke and Gertler, 1989; Bernanke et al., 1999; Carlstrom and Fuerst, 1997), and of limited enforceability (see Cooley et al. (2004)). For a survey of these literatures see Brunnermeier et al. (2011), Freixas and Rochet (1997), or Bhattacharya et al. (2004).

The remainder of the paper is organized as follows. Section 2 shows that when there are no costs to default, productive agents prefer to renege on their debts after negative shocks, thus eliminating KM dynamic amplification effects. In section 3 we introduce endogenous default costs by making new assets less productive. This gives an incentive to entrepreneurs to repay their debts in the face of moderate adverse shocks, and relates the degree of asset specificity to the form and intensity of the financial multiplier. Section 4 develops an extension with asymmetric information about default costs. In section 5 we show that asset specificity also leads to an increase in shock persistence. Section 6 presents an empirical strategy to test the model's transmission channel using U.S. firm level data. Section 7 concludes.

## 2 Credit Cycles with Default

A salient feature of KM is the existence of a dynamic multiplier that amplifies productivity shocks through collateral prices. In this section, we show that this result depends on the distribution of the debt burden. In KM, farmers have to fully repay their debt after the shock. We change their model in just this crucial but essential aspect: we allow farmers to default. This modeling assumption implies an asymmetric effect of shocks, as it shuts down the amplification mechanism after a negative shock.

We briefly outline here the setup of KM. There are two types of producers, farmers and gatherers, of measure one and  $m$  respectively. Both are risk neutral and maximize their expected utility given by

$$E_t \sum_{s=0}^{\infty} \beta^s x_{t+s} \quad \text{and} \quad E_t \sum_{s=0}^{\infty} \beta'^s x'_{t+s},$$

where  $0 < \beta < 1$ ,  $0 < \beta' < 1$ ,  $x_{t+s}$  and  $x'_{t+s}$  are their respective discount factors and consumption of a perishable good, fruit, in period  $t + s$ .

Farmers have access to a linear production technology that yields  $a$  units of tradable fruit and  $c$  units of non-tradable fruit per unit of land. Gatherers' production technology exhibit decreasing returns to scale in the use of this asset:

$$y'_{t+1} = G(k'_t), \text{ where } G' > 0, G'' < 0, G'(0) > aR > G' \left( \frac{\bar{K}}{m} \right),$$

where  $R$  is the equilibrium interest rate, and  $\bar{K}$  is the total amount of land in the economy, assumed to be in fixed supply. There is a financial friction in the model, farmers cannot precommit to work, and as a result find themselves credit-constrained by the value of their collateral. Specifically, if at date  $t$  a farmer has land  $k_t$ , then she can borrow  $b_t$  in total, as long as the repayment does not exceed the market value of her land at date  $t + 1$ :

$$Rb_t \leq q_{t+1}k_t,$$

where  $q$  is the asset's market price. KM explicitly make two assumptions that deliver their results: farmers are relatively more impatient than gatherers, and non-tradable output is sufficiently large to ensure farmers always want to reinvest all their tradable output. Mathematically,

$$\begin{aligned} \beta &< \beta', \\ c &> \left( \frac{1}{\beta} - 1 \right) a. \end{aligned}$$

The intuition for making these assumptions is the following. Since gatherers have linear utility functions and are not credit-constrained, the equilibrium interest rate reflects their rate of impatience ( $R = \frac{1}{\beta'}$ ). Even though farmers are relatively impatient and want to borrow all they can at this interest rate, they would rather invest the funds collected than consume because the return of investing in their linear technology is even higher than their own rate of impatience.<sup>8</sup>

There is, however, an implicit assumption: an enforcement technology exists that precludes default *after* production has been realized. Although KM allow farmers to repudiate their debt contracts at any time, they assume that the unexpected shock happens after farmers' labor supply decision was made (see footnote 13 in KM). Thus implicitly

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<sup>8</sup>This is true around the steady state. If land prices were very high and expected to decline, the capital losses of holding land could outweigh the "technological" benefits of investing.

they are assuming that farmers' debt repayments can be collected against both the assets and tradable output. We lift this assumption and allow farmers to default while keeping the whole of their current production. This implies the farmers' and the gatherers' flow-of-funds are respectively given by

$$\begin{aligned} q_t k_t + I_{ND} (Rb_{t-1} - q_t k_{t-1}) + x_t &= (a + c)k_{t-1} + b_t, \\ q_t k'_t + I_{ND} (Rb'_{t-1} - q_t k'_{t-1}) + x'_t &= G(k'_{t-1}) + b'_t, \end{aligned} \tag{1}$$

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where  $I_{ND}$  is an indicator variable that takes the value one when the farmer does not default.<sup>9</sup> As argued, under these assumptions farmers want to invest all their income in the neighborhood of the steady state. With perfect foresight this implies

$$Rb_{t-1} = q_t k_{t-1}$$

Therefore, having the option to default in the steady state equilibrium is irrelevant: the value of debt always matches the value of the collateral. However, our modification is essential to understand the dynamics around the steady state. Clearly, after a positive (negative) shock, the value of the collateral will be above (below) the value of debt. Since in this simple model there is no cost of default, this implies that, given the opportunity, farmers will always choose to default after a negative shock. This has important consequences for the dynamic behavior of asset prices and farmers' net worth.

Figure 1: Default dampens the amplification mechanism.

To illustrate this point, consider an unexpected negative shock of percentage size  $\Delta$  on farmers' tradable output productivity,  $a$ . In this case, farmers always default. As in KM, farmers consume only their non-tradable fruit around the steady state ( $x_t = ck_t$ )

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<sup>9</sup>Note that this is a control variable for the farmer but not for the gatherer.

and max out their credit constraint ( $b_t = \frac{q_{t+1}k_t}{R}$ ). In appendix 8.1 we show that, in this case, the deviations of total farmers' asset holdings ( $\widehat{K}_t$ ) and asset price ( $\widehat{q}_t$ ) with respect to their steady state values are respectively given by

$$\begin{aligned}\widehat{K}_t &= -\frac{\eta}{\eta+1}\Delta, \\ \widehat{q}_t &= -\frac{R-1}{\frac{R-1}{\eta} + R}\Delta,\end{aligned}\tag{2}$$

where  $\eta \equiv \frac{d \log u(K)}{d \log K_t} \Big|_{K=K^*}$  is the elasticity of the residual supply of the asset with respect to the user cost,  $u$ , evaluated at the steady state (the user cost of land depends, through the gatherers' technology, on the distribution of the asset among farmers and gatherers:  $u(K_t) = q_t - \frac{q_{t+1}}{R} = \frac{1}{R}G' \left( \frac{\bar{K}-K_t}{m} \right)$ , thus is represented as a function of farmers' total land holdings). As (2) shows, prices fall less than  $\frac{R-1}{R}\Delta$ . In contrast, in KM farmers, having to fully repay their debt, depress prices further ( $\widehat{q}_t = -\frac{1}{\eta}\Delta$  in KM). Since farmers are more productive than gatherers, this generates an amplification mechanism by significantly reducing the amount of land in hands of farmers ( $\widehat{K}_t = -\frac{\eta}{\eta+1}(1 + \frac{1}{\eta}\frac{R}{R-1})\Delta$  in KM). It is clear that with default the “dynamic” amplification effect through collateral prices is not operative. Hence, the amplification mechanism depends crucially on the fact that debtors bear the full burden of the capital losses. Figure 1 compares the evolution of output when default is possible and when borrowers pay the full value of their debts.

### 3 Amplification and land specificity

In the model of the previous section farmers face no costs of default and, thus, repudiate their debt after any unforeseen negative shock. In this section we introduce a cost of default related to asset specificity. We assume that new assets yield less non-tradable output the first period of usage (thus one can think of this friction as a “learning by doing” effect in farmers' technology).<sup>10</sup> Hence, if a farmer defaults, she must buy new land, which is less valuable to her. This implies that, on the steady state, the farmer is strictly better off by avoiding default. Formally, land yields  $\alpha c$  non-tradable output the first period of usage

$$y_{t+1} = F(k_t^{new}, k_t^{old}) = (a + \alpha c)k_t^{new} + (a + c)k_t^{old},$$

where  $0 < \alpha < 1$  can be interpreted as a measure of “asset specificity”.<sup>11</sup> In industries in which changing assets is more (less) costly to the farmer/entrepreneur, more (less) non-tradable output is lost the first period if she decides to default and leave her land/capital to buy a new one. This translates into a lower (higher)  $\alpha$ .

<sup>10</sup>Assuming that productivity is reduced for only one period results in small quantitative effects. We will later explore the case that it takes more time for productivity of new assets to converge to its long run value.

<sup>11</sup>To avoid spurious differences with KM the fruit  $(1 - \alpha)c$  is not lost, but redistributed uniformly among the rest of the farmers. This will lead to the same total output while affecting farmers' incentive to default.



As in KM, this model presents a steady state where farmers reinvest all their proceeds, while keeping their asset holdings constant. However, unlike the case of the previous section, now farmers *strictly* prefer not to default. If they did so, their assets would be less productive the following period while having no capital gains (since  $Rb^* = q^*k^*$ ). Therefore there is a region of productivity shocks ( $0 < \Delta < \bar{\Delta}$ ) where nobody wants to default and, hence, the KM amplification mechanism operates.

We enrich the model further to include the possibility of renegotiation. To prevent farmers from declaring default, gatherers may make an offer to pardon a fraction  $\varphi$  of the debt. Then, farmers decide whether to take the offer or default. Thus, a farmer's flow-of-funds-constraint is given by:

$$q_t k_t + I_{ND} ((1 - \varphi) R b_{t-1} - q_t k_{t-1}) + x_t - c(\alpha k_{t-1}^{new} + k_{t-1}^{old}) = a k_{t-1} + b_t, \quad (3)$$

On the other hand, the gatherer's flow-of-funds-constraint is given by:

$$q_t k'_t + I_{ND} ((1 - \varphi) R b'_{t-1} - q_t k'_{t-1}) + x'_t = G(k'_{t-1}) + b'_t.$$

When farmers and gatherers renegotiate, they share the burden of the shock.<sup>12</sup> Clearly, the larger the shock, the more attractive the outside option of the farmer. Intuitively, as shocks become larger, the burden of the shock is transferred from the debtor to the creditor. This, in turn, will affect the size of the amplification effect and the response of the economy to these shocks.

### 3.1 Response to an unexpected negative shock

Suppose now that the economy was on the steady state and is hit by an unforeseen negative productivity shock ( $\Delta$  in percentage terms) that reduces the quantity of tradable output produced by farmers in period  $t$ . This affects the farmers' flow-of-funds (3), which becomes:

$$q_t k_t + I_{ND} ((1 - \varphi) R b^* - q_t k^*) + x_t - c k^* = (a - \Delta a) k^* + b_t. \quad (4)$$

To ensure farmers do not want to consume more than their bruised fruit, even after defaulting, we make the following assumption which echoes KM's Assumption 2<sup>13</sup>

$$(\alpha + \beta(1 - \alpha)) c > \left( \frac{1}{\beta} - 1 \right) a.$$

In appendix 8.2 we show that this assumption guarantees farmers choose to invest all available funds below the steady state ( $x_t = c k^*$ ), which, in turn, implies farmers' credit constraint is binding every period, in particular  $R b^* = q^* k^*$ . Replacing in (4),

$$q_t k_t + I_{ND} ((1 - \varphi) q^* k^* - q_t k^*) = (a - \Delta a) k^* + \frac{q_{t+1} k_t}{R}.$$

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<sup>12</sup>As in KM we give the lenders full bargaining power ex post. This ensures that the debt renegotiation offer leaves farmers indifferent between accepting it or defaulting, thus simplifying the analysis.

<sup>13</sup>Since the output of bruised fruit is lower for new land, our framework requires a more stringent assumption on the returns of investment. Actually, this assumption ensures that investing as much as possible is the optimal strategy around the steady state, even after defaulting. Note, additionally, that negative shocks make investing even more attractive.

Manipulating terms, and using the user cost,  $u(K_t) = q_t - \frac{q_{t+1}}{R}$ ,

$$u(K_t)k_t = (a - \Delta a)k^* + I_{ND}(q_t - (1 - \varphi)q^*)k^*.$$

Note that prices depend on aggregate capital, which is affected by the share of farmers that default,  $\mu$ . Thus, in making a decision the farmer will have to form expectations of this share  $\mu$ .

If the farmer does not default,

$$u(K_t)k_t^{ND} = (a - \Delta a + q_t - q^* + \varphi q^*)k^*.$$

Log-linearizing this equation around the steady state,

$$\frac{1}{\eta}\widehat{K}_t + \widehat{k}_t^{ND} = \frac{R}{R-1}\widehat{q}_t - \Delta + \frac{R}{R-1}\varphi. \quad (5)$$

If the farmer defaults,

$$u(K_t)k_t^D = (a - \Delta a)k^*.$$

Log-linearizing this equation around the steady state,

$$\frac{1}{\eta}\widehat{K}_t + \widehat{k}_t^D = -\Delta. \quad (6)$$

As in KM, we assume there are no more shocks to the economy. Since agents have perfect foresight  $Rb_t = q_{t+1}k_t^i$  (for  $i = D, ND$ ). Thus, in the following periods the flow of fund becomes

$$u(K_{t+s})k_{t+s}^i = ak_{t+s-1}^i.$$

Log-linearizing this equation around the steady state,

$$\frac{1}{\eta}\widehat{K}_{t+s} + \widehat{k}_{t+s}^i = \widehat{k}_{t+s-1}^i. \quad (7)$$

Note from the previous equation that the rate of convergence to the steady state after a shock is independent of the degree of asset specificity.<sup>14</sup>

### 3.1.1 Dynamics when a proportion $\mu$ of the farmers defaults

A fraction  $1 - \mu$  of the farmers follows (5) in  $t$  and a fraction  $\mu$  follows (6). Summing both laws of motion weighted by their proportions yields

$$\left(\frac{1}{\eta} + 1\right)\widehat{K}_t = (1 - \mu)\frac{R}{R-1}(\widehat{q}_t + \varphi) - \Delta,$$

in  $t$  and, summing the corresponding equations (7)

$$\left(\frac{1}{\eta} + 1\right)\widehat{K}_{t+s} = \widehat{K}_{t+s-1}. \quad (8)$$

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<sup>14</sup>A different assumption on asset specificity would change this result, to have new assets produce less tradable output. We explore this later.

Using the transversality condition,  $u(K_t) = q_t - \frac{q_{t+1}}{R}$ , and linearizing around the steady state yields

$$\hat{q}_t = \frac{1}{\eta} \frac{R-1}{R} \frac{1}{1 - \frac{1}{R} \frac{\eta}{1+\eta}} \hat{K}_t.$$

Solving for  $\hat{K}_t$  and  $\hat{q}_t$ ,

$$\hat{K}_t = - \left( 1 + \frac{(1-\mu) \frac{R}{\eta}}{R-1 + \frac{\mu}{\eta} R} \right) \frac{\eta}{\eta+1} \left( \Delta - (1-\mu) \frac{R}{R-1} \varphi \right), \quad (9)$$

$$\hat{q}_t = -\frac{1}{\eta} \left( \frac{R-1}{R-1 + \frac{\mu}{\eta} R} \right) \left( \Delta - (1-\mu) \frac{R}{R-1} \varphi \right). \quad (10)$$

Using the aggregate law of motion of capital and prices and equations (5) and (6) we can obtain the law of motion of individual capital in each case,

$$\hat{k}_t^{ND} = -\frac{\eta}{\eta+1} \left( 1 + \frac{1}{\eta^2} \frac{R(\eta+\mu)}{R-1 + \frac{\mu}{\eta} R} \right) \left( \Delta - \frac{(1-\mu)R}{R-1} \varphi \right) + \frac{\mu R}{R-1} \varphi, \quad (11)$$

$$\hat{k}_t^D = -\frac{\eta}{\eta+1} \left( \Delta - \frac{(1-\mu)R}{R-1} \varphi \right) \left( 1 - \left( \frac{\frac{R}{\eta^2} (1-\mu)}{R-1 + \frac{\mu}{\eta} R} \right) \right) - \frac{(1-\mu)R}{R-1} \varphi \quad (12)$$

### 3.1.2 Implied Utilities

Now we are ready to compute the implied utilities of default and renegotiation, given the level of the shock,  $\Delta$ , expectations about the general level of default,  $\mu$ , and the proposed debt-reduction,  $\varphi$ . If the farmer decides to accept the proposed debt write-off, then in the future she only pays a penalty for the new assets that she buys. In this case her expected utility is given by

$$\begin{aligned} U^{ND}(\mu; \Delta, \varphi) &= cK^* + \beta c k_t^{ND} + \beta^2 (c\alpha(k_{t+1}^{ND} - k_t^{ND}) + c k_t^{ND}) \\ &\quad + \dots + \lim_{s \rightarrow \infty} \beta^s (c\alpha(k_{t+s-1}^{ND} - k_{t+s-2}^{ND}) + c k_{t+s-2}^{ND}). \end{aligned}$$

Let  $\tilde{U}^{ND}(\cdot) = \frac{U^{ND}(\cdot)}{\beta c K^*} - \frac{1}{\beta}$ . In appendix 8.3 we show  $\tilde{U}^{ND}(\cdot)$  can be expressed as a function of individual and aggregate capital:

$$\tilde{U}^{ND}(\mu; \Delta, \varphi) = \frac{1}{1-\beta} \left( 1 + \hat{k}_t^{ND} - \beta \hat{K}_t \frac{(\alpha + \beta(1-\alpha))}{(1 + \eta(1-\beta))} \right).$$

On the other hand, if the farmer defaults, all her land in  $t+1$  is new. Hence, she faces a tougher productivity penalty. In this case her expected utility is given by

$$\begin{aligned} U^D(\mu; \Delta, \varphi) &= cK^* + \beta c \alpha k_t^D + \beta^2 (c\alpha(k_{t+1}^D - k_t^D) + c k_t^D) \\ &\quad + \dots + \lim_{s \rightarrow \infty} \beta^s (c\alpha(k_{t+s-1}^D - k_{t+s-2}^D) + c k_{t+s-2}^D). \end{aligned}$$

Let  $\tilde{U}^D(\cdot) = \frac{U^D(\cdot)}{\beta c K^*} - \frac{1}{\beta}$ . In appendix 8.3 we show  $\tilde{U}^D(\cdot)$  can be expressed as a function of individual and aggregate capital:

$$\tilde{U}^D(\mu; \Delta, \varphi) = \frac{\alpha + \beta(1 - \alpha)}{1 - \beta} \left( 1 + \hat{k}_t^D - \beta \hat{K}_t \frac{(\alpha + \beta(1 - \alpha))}{(1 + \eta(1 - \beta))} \right).$$

Since lenders moves first, and have all the bargaining power ex post, each one will offer the lowest debt reduction such that the farmer is indifferent between defaulting and renegotiating. Note that

$$U^D = U^{ND} \Leftrightarrow (\alpha + \beta(1 - \alpha)) \left( 1 + \hat{k}_t^D \right) = 1 + \hat{k}_t^{ND}. \quad (13)$$

As we have already shown,  $\hat{k}_t^D$  and  $\hat{k}_t^{ND}$  depend on  $\Delta$ ,  $\mu$  and  $\varphi$ . Although at first sight it may seem quite complex to derive optimal combinations of  $\varphi$  and  $\mu$  for each  $\Delta$ , the following proposition simplifies the problem.

**Proposition 1.** There is no equilibrium with  $\mu > 0$ .

*Proof.* First we will show that there cannot be any equilibrium where  $0 < \mu < 1$ . Suppose  $0 < \mu < 1$  for some  $\tilde{\varphi}$ . If this is an equilibrium, then farmers must be indifferent between renegotiating and defaulting. This implies that, if a farmer were offered  $\varphi > \tilde{\varphi}$ , he would not default (*ceteris paribus* aggregate  $\tilde{\varphi}$ ). Hence, an individual gatherer knows that if he offers  $\tilde{\varphi} + \epsilon$  with a tiny  $\epsilon > 0$  he can avoid the capital losses he expects to face with probability  $\mu$  when offering  $\tilde{\varphi}$ . Therefore, any individual gatherer would have an incentive to deviate from such an equilibrium.

Now we will show that  $\mu = 1$  is not possible either. Using (11) and (12) evaluated at  $\mu = 1$  and replacing in (13) we can derive the  $\tilde{\varphi}$  the gatherer would need to offer the farmer in order to make her accept his renegotiation offer:

$$\tilde{\varphi} = \max \left[ 0, \frac{1}{\eta} \left( \frac{R - 1}{R - 1 + \frac{R}{\eta}} \right) \Delta - \frac{(1 - \alpha)(1 - \beta)(R - 1)}{R} \left( 1 - \Delta \frac{\eta}{\eta + 1} \right) \right].$$

Note that  $\Delta \leq 1$  (so that  $a(1 - \Delta) \geq 0$ ), which implies  $\tilde{\varphi} < 1$ . If the gatherer lets the farmer default, he gets the collateral  $q_t k^*$  while, if he offers a debt pardon of  $\tilde{\varphi}$ , he gets  $(1 - \tilde{\varphi})q^* k^*$ . Then, using the fact that  $q_t = (1 + \hat{q}_t)q^*$ , the gatherer will offer  $\varphi < \tilde{\varphi}$  (prompting default) if and only  $\tilde{\varphi} > -\hat{q}_t$ . Evaluating (10) at  $\mu = 1$  yields

$$\hat{q}_t = -\frac{1}{\eta} \left( \frac{R - 1}{R - 1 + \frac{1}{\eta} R} \right) \Delta.$$

Since  $\Delta \leq 1$ ,  $\tilde{\varphi} < -\hat{q}_t$ . Thus, whenever the farmer prefers to default, the gatherer has an incentive to improve his debt-relief offer.  $\square$

Proposition 1 states that there are no equilibria where a share of farmers wants to default. Thus, if an equilibrium exists no farmer would find it optimal to default. Using

(11) and (12) evaluated at  $\mu = 0$  and replacing in (13) we can derive the lowest  $\tilde{\varphi}$  the gatherer must offer the farmer to make her accept his renegotiation offer:<sup>15</sup>

$$\begin{aligned} \varphi &= \max \left[ 0, \frac{R-1}{\left(\alpha + \beta(1-\alpha) + A \frac{\eta}{\eta+1}\right) R} \left( \frac{\eta}{\eta+1} A \Delta - B \right) \right], \\ A &\equiv 1 - (\alpha + \beta(1-\alpha)) + \frac{1}{\eta^2} \frac{R}{R-1} ((\alpha + \beta(1-\alpha)) + \eta) > 0, \\ B &\equiv 1 - (\alpha + \beta(1-\alpha)) > 0. \end{aligned} \quad (14)$$

Clearly, the larger the shock, the larger the debt-reduction required to make farmers indifferent between defaulting and renegotiating. Moreover, as we had anticipated, it follows directly from (14) that there is a lower bound for productivity shocks  $\bar{\Delta}$  such that, when  $\Delta < \bar{\Delta}$ ,  $\varphi$  hits the zero lower bound. Hence, there is a region of productivity shocks ( $0 < \Delta < \bar{\Delta}$ ) where gatherers do not need to reduce farmers' debt to ensure repayment. The threshold  $\bar{\Delta}$  is given by

$$\bar{\Delta} = \frac{B \eta + 1}{A \eta}.$$

From the expressions for  $A$  and  $B$  above it is clear that  $\partial \bar{\Delta} / \partial \alpha < 0$ , i.e. industries characterized by higher asset specificity will present significant amplification for a larger range of initial negative shocks. This is illustrated in figure 2, which shows the level of debt reduction as a function of the size of the productivity shock for an economy with low asset specificity and for another with high asset specificity. Figure 3 displays the initial change in GDP for these two economies.

The stylized analysis of the previous sections suggests that the nature and the extent of the financial accelerator depend not only on the size of the shock, but also on the level of protection of creditor rights, and the specificity of productive assets. From the point of view of the consequences of the disturbance, the argument points to a mechanism that may limit the financial multiplier, by placing bounds on the fall of asset prices. The moderating effects relying on costless debt renegotiations.

Although the model is highly stylized, it is a matter of observation that in some debt crises, either through government intervention or bankruptcy procedures that preserve financially troubled firms, constraints on the access to resources for production get somehow relieved (even if this happens after a period of turmoil). At the same time, other "balance sheet crises" show long periods of deleveraging where firms hold to their capital but investment and production are financially restricted and asset prices stay depressed. This corresponds qualitatively with the KM argument, and in the context of our model it would be the case of strong asset specificity, where debtors have incentives to use current proceeds to diminish their obligations while maintaining business arrangements.

The long recession of Japan in the 1990's fits that description. As indicated by Koo (2003), many firms made long and sustained efforts to reduce their debts. Furthermore,

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<sup>15</sup>Note that, since  $\varphi$  is increasing in  $\Delta$  and  $\varphi(1) < 1$ , we do not need to worry about the natural upper bound of  $\varphi$ .

Figure 2: Debt-reduction and asset-specificity.

Ahmadjian (2006) has found, through an analysis of ownership ties within business groups, that although some peripheral relationships were disrupted, the links between core firms remained robust. The image is that of a collection of physical and relational capital that incumbents find costly to take apart, and prefer to go through the painful process of debt reduction to preserve them.

The evidence of other crises points in the other direction: that of disaggregations of assets and business relationships. Thus, for example, Gomez (2006) finds that in the Malaysian crisis some major capitalists lost control of their corporate assets since they were burdened with enormous debts and depended too much on a state leader, while business groups with better political connections thrived. Hanani (2006) paints a similar picture for Indonesia. In those instances, the degree of asset-entrepreneur specificity appears to have been affected by political influences.

## 4 Asymmetric Information about Default Costs

In this section we extend the model to allow for heterogeneity among farmers' default costs, which are now private information. The population distribution of  $\alpha$  is known and given by a cumulative distribution function  $F(x) \in C^2$  with support  $[0, 1]$ . Since gatherers don't know the type of farmers they have lent to, they face a trade-off in the event of an unforeseen negative shock: a higher level of debt-pardon makes more borrowers willing to accept, but the rent extracted from each farmer is lower.

We keep the timing of the previous section: first gatherers make an offer and then farmers decide whether to accept it or not.<sup>16</sup> Hence, we will solve it by backward induction.

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<sup>16</sup>Given gatherers' risk neutrality, we proceed as if each one of them faces a continuum of farmers. This makes the amount of borrowers that default for a given debt level deterministic from the point of view of the gatherer, not only at an aggregate level.

Figure 3: Initial output drop and asset specificity.

First, a farmer must decide whether to accept or decline her debt-pardon offer  $\varphi$ , taking as given the dynamics of aggregate capital and prices (which, from (9) and (10), depend on aggregate default share,  $\tilde{\mu}$ , and aggregate debt-offer,  $\tilde{\varphi}$ ).

Equation (13) shows that a farmer is indifferent between defaulting and renegotiating when

$$(\alpha + \beta(1 - \alpha)) (1 + \hat{k}_t^D(\varphi; \tilde{\varphi}, \tilde{\mu})) = (1 + \hat{k}_t^{ND}(\varphi; \tilde{\varphi}, \tilde{\mu}))$$

Manipulating terms we can derive a productivity threshold  $\hat{\alpha}$  for a given  $\varphi$  such that if a farmer has  $\alpha > \hat{\alpha}(\varphi)$ , given  $\tilde{\mu}$  and  $\tilde{\varphi}$ , then she will prefer to default on its debt:

$$\hat{\alpha}(\varphi) = \frac{1}{1 - \beta} \left( \frac{1 + \hat{k}_t^{ND}(\varphi; \tilde{\varphi}, \tilde{\mu})}{1 + \hat{k}_t^D(\varphi; \tilde{\varphi}, \tilde{\mu})} - \beta \right) \quad (15)$$

This implies that the gatherer making debt pardon offer  $\varphi$  (given  $\tilde{\mu}$  and  $\tilde{\varphi}$ ) faces a default probability  $\mu(\varphi) = 1 - F(\hat{\alpha}(\varphi))$ . Next, note that from equations (5) and (6):

$$\begin{aligned} \hat{k}_t^{ND}(\varphi; \tilde{\varphi}, \tilde{\mu}) &= \frac{R}{R-1} \hat{q}_t(\tilde{\mu}, \tilde{\varphi}) - \Delta + \frac{R}{R-1} \varphi - \frac{1}{\eta} \hat{K}_t(\tilde{\mu}, \tilde{\varphi}), \\ \hat{k}_t^D(\varphi; \tilde{\varphi}, \tilde{\mu}) &= -\Delta - \frac{1}{\eta} \hat{K}_t(\tilde{\mu}, \tilde{\varphi}). \end{aligned}$$

Replacing in (15) and rearranging,

$$\hat{\alpha}(\varphi) = \frac{\frac{R}{R-1} (\hat{q}_t(\tilde{\mu}, \tilde{\varphi}) + \varphi)}{(1 - \beta)(1 - \Delta - \frac{1}{\eta} \hat{K}_t(\tilde{\mu}, \tilde{\varphi}))} + 1, \quad (16)$$

$$\frac{d\hat{\alpha}(\varphi)}{d\varphi} = \frac{\frac{R}{R-1}}{(1 - \beta)(1 - \Delta - \frac{1}{\eta} \hat{K}_t(\tilde{\mu}, \tilde{\varphi}))} \quad (17)$$

Moreover, we can invert (16)

$$\varphi(\hat{\alpha}) = -\hat{q}_t(\tilde{\mu}, \tilde{\varphi}) - \frac{(1 - \hat{\alpha})(1 - \beta)}{\frac{R}{R-1}} \left(1 - \Delta - \frac{1}{\eta} \hat{K}_t(\tilde{\mu}, \tilde{\varphi})\right) \quad (18)$$

Thus,  $\varphi(\hat{\alpha})$  is the required debt-pardon to generate a threshold of  $\hat{\alpha}$ .

Finally, taking this into account the gatherer wishes to reduce his losses to a minimum. For a given debt pardon offer  $\varphi$ , he faces capital losses (in percentage terms) given by  $-\hat{q}_t(\tilde{\mu}, \tilde{\varphi})$  on a fraction  $1 - F(\hat{\alpha}(\varphi))$  of his borrowers, whereas if renegotiation takes place he loses  $\varphi$  (in percentage terms) on the complementary fraction  $F(\hat{\alpha}(\varphi))$ . Since  $\hat{q}_t$  is a constant for the gatherer - there is a continuum of them - we can write the gatherers' problem as

$$\min_{\varphi} (\varphi + \hat{q}_t)F(\hat{\alpha}(\varphi))$$

In an interior optimum the first-order condition is

$$F(\hat{\alpha}(\varphi)) + f(\hat{\alpha}(\varphi)) \frac{d\hat{\alpha}(\varphi)}{d\varphi} (\varphi + \hat{q}_t) = 0.$$

From (17), noting that this implies  $\frac{d\hat{\alpha}(\varphi)}{d\varphi} (\varphi + \hat{q}_t) = -(1 - \hat{\alpha}(\varphi))$ , replacing in the first order condition, and dividing the whole expression by  $f(\hat{\alpha}(\varphi))$  yields

$$\frac{F(\hat{\alpha}(\varphi))}{f(\hat{\alpha}(\varphi))} - (1 - \hat{\alpha}(\varphi)) = 0 \quad (19)$$

We now assume  $F(x)$  and its corresponding density,  $f(x)$ , satisfy the following properties:

A1.  $F(x)$  has a continuous increasing Mills' ratio and satisfies

$$\frac{F(x)}{f(x)} \rightarrow 0 \ (\infty) \text{ when } x \rightarrow 0 \ (1).$$

A2.  $f(x) \rightarrow 0$  when  $x \rightarrow 0 \ (1)$ .

A symmetric Beta distribution with parameter  $\gamma > 1$  satisfies these two conditions, as we show in Appendix 8.4. We will derive several results for this particular distribution, whose density is given by

$$f(x; \gamma) = \frac{x^{\gamma-1}(1-x)^{\gamma-1}}{B(\gamma, \gamma)},$$

where  $B(\gamma, \gamma)$  is the beta function evaluated at  $(\gamma, \gamma)$ .

Note that under assumptions A1 and A2, as  $\alpha \rightarrow 0$ , the LHS goes to  $-1$  while, as  $\alpha \rightarrow 1$ , the LHS goes to  $+\infty$ . Since the Mills' ratio is continuous and increasing, the LHS is strictly increasing with  $\alpha$ . By continuity there is a unique  $0 < \alpha^* < 1$  such that the first-order-condition is satisfied with equality. Under assumptions A1 and A2,  $\alpha^*$  is in fact a global minimum (see Appendix 8.5).

Unfortunately, it is not possible to characterize any further the solution without resorting to numerical methods, unless we consider a uniform distribution. Equation (19)



allows us to find the optimal  $\alpha^*$ , which only depends on characteristics of the distribution,  $F(\cdot)$ . This in turn implies that the share of defaulting farmers in equilibrium is also independent of aggregate capital and prices:  $\mu = 1 - F(\alpha^*)$ . Using (18) we can recover the equilibrium level of debt pardon. In a symmetric equilibrium  $\tilde{\varphi} = \varphi(q(\tilde{\varphi}, \mu^*), K(\tilde{\varphi}, \mu^*))$ . Hence, using equilibrium expressions for aggregate capital and prices, (9) and (10),

$$\tilde{\varphi} = \frac{1}{\eta} \left( \frac{R-1}{R-1+\frac{\mu^*}{\eta}R} \right) \left( \Delta - (1-\mu^*)\frac{R}{R-1}\tilde{\varphi} \right) - \frac{(1-\alpha^*)(1-\beta)}{\frac{R}{R-1}} \left( 1 - \Delta + \frac{1}{\eta} \left( 1 + \frac{(1-\mu^*)\frac{R}{\eta}}{R-1+\frac{\mu^*}{\eta}R} \right) \frac{\eta}{\eta+1} \left( \Delta - (1-\mu^*)\frac{R}{R-1}\tilde{\varphi} \right) \right)$$

Let  $A \equiv \frac{1}{\eta} \left( \frac{R-1}{R-1+\frac{\mu^*}{\eta}R} \right) > 0$ ,  $B \equiv \frac{(1-\alpha^*)(1-\beta)}{\frac{R}{R-1}} > 0$  and  $C \equiv \frac{1}{\eta} \left( 1 + \frac{(1-\mu^*)\frac{R}{\eta}}{R-1+\frac{\mu^*}{\eta}R} \right) \frac{\eta}{\eta+1}$ . Replacing,

$$\tilde{\varphi} = (A - BC)\Delta - B(1 - \Delta) + (BC - A)(1 - \mu^*)\frac{R}{R-1}\tilde{\varphi}$$

Solving for  $\tilde{\varphi}$

$$\tilde{\varphi} = \frac{(A - B(C - 1))\Delta - B}{1 + (A - BC)(1 - \mu^*)\frac{R}{R-1}}.$$

It is straightforward to verify that a more severe negative shock leads to a higher level of debt forgiveness,  $\frac{d\tilde{\varphi}}{d\Delta} > 0$ , and that there is a threshold level,  $\bar{\Delta}$ , such that shocks below this level meet no renegotiation offer from lenders. The more severe is the problem of asymmetric information the more default is observed in equilibrium and thus there is less amplification of the initial shock. Finally, in the limit of perfect information,  $\gamma \rightarrow \infty$ , the level of debt pardon is the same as the one obtained in the previous section when  $\alpha = \frac{1}{2}$ .

## 5 Renegotiation and persistence

In this section we assume that the productivity penalty of buying new land reduces the amount of tradable output available in the first period by an amount  $\epsilon$ , increasing other farmers' nontradable output by the same amount.<sup>17</sup> Hence, if farmers are expanding production, they must do so at a lower rate than before, since new land now provides less tradable output that may be used in market transactions.<sup>18</sup> In this context, using the fact that the farmer consumes her bruised fruit, a farmer's flow-of-funds-constraint is given by:

$$q_t k_t + I_{ND}((1 - \varphi)Rb_{t-1} - q_t k_{t-1}) = a k_{t-1} - \epsilon k_{t-1}^{new} + b_t, \quad (20)$$

Gatherer's flow-of-funds-constraint is unaffected.

<sup>17</sup>As before, this assumption is made to have no effects on aggregate output directly due to asset specificity.

<sup>18</sup>This is consistent with the data, as crisis that are resolved without debt renegotiations on average take longer to recover.

## 5.1 Response to an unexpected negative shock

As usual, we assume that the economy is at the steady state when hit by an unforeseen negative productivity shock ( $\Delta$  in percentage terms) that reduces the quantity of tradable output produced by farmers in period  $t$ . This affects the farmers' flow-of-funds (20), which becomes:

$$q_t k_t + I_{ND}((1 - \varphi)Rb^* - q_t k^*) + x_t - ck^* = (a - \Delta a)k^* + b_t. \quad (21)$$

As before, we make the assumption that preferences guarantee that farmers choose to invest all available funds below the steady state ( $x_t = ck^*$ ), which, in turn, implies farmers' credit constraint is binding every period, in particular  $Rb^* = q^*k^*$ . Replacing in (21),

$$q_t k_t + I_{ND}((1 - \varphi)q^*k^* - q_t k^*) = (a - \Delta a)k^* + \frac{q_{t+1}k_t}{R}.$$

As in KM, we assume there are no more shocks to the economy. Since agents have perfect foresight  $Rb_t = q_{t+1}k_t^i$  (for  $i = D, ND$ ). In the periods following the shock, when farmers seek to increase their landholdings, they incur a cost as new land has lower productivity of tradable output. If the farmer does not default, in  $t + 1$  all his land is old

$$u(K_{t+1})k_{t+1}^{ND} = ak_t^{ND}$$

while, if he does default, all his land is new:

$$u(K_{t+1})k_{t+1}^D = (a - \epsilon)k_t^D.$$

Log-linearizing these equations:

$$\frac{1}{\eta}\widehat{K}_{t+1} + \widehat{k}_{t+1}^{ND} = a\widehat{k}_t^{ND}$$

$$\frac{1}{\eta}\widehat{K}_{t+1} + \widehat{k}_{t+1}^D = \left(\frac{a - \epsilon}{a}\right)\widehat{k}_t^D$$

In  $t + 2$  and beyond, some land is new and some old. Given that convergence is monotonic and log-linearizing:

$$\frac{1}{\eta}\widehat{K}_{t+s} + \widehat{k}_{t+s}^i = \frac{(a - \epsilon)}{a}\widehat{k}_{t+s-1}^i + \frac{\epsilon}{a}\widehat{k}_{t+s-2}^i \quad (22)$$

Intuitively we can see that convergence will be slower the larger  $\epsilon$ , since this gives a higher weight on land holdings in  $t + s - 2$ , which are smaller.

### 5.1.1 Dynamics when nobody defaults

Now we assume  $\mu = 0$  (ie. nobody defaults). In this case, aggregate capital follows

$$\left(\frac{1}{\eta} + 1\right)\widehat{K}_{t+1} = \left(\frac{a - \epsilon}{a}\right)\widehat{K}_t \quad (23)$$

$$\left(\frac{1}{\eta} + 1\right) \widehat{K}_{t+s} = \left(\frac{a-\epsilon}{a}\right) \widehat{K}_{t+s-1} + \frac{\epsilon}{a} \widehat{K}_{t+s-2} \quad \text{for } s > 1 \quad (24)$$

Note that (24) is an homogeneous difference equation of second order. Hence, it has a solution of the form

$$\widehat{K}_s = A_1 b_1^s + A_2 b_2^s \quad (25)$$

The roots are given by

$$b_{1,2} = \frac{\frac{a-\epsilon}{a} \pm \sqrt{\left(\frac{a-\epsilon}{a}\right)^2 + 4\frac{\eta+1}{\eta} \frac{\epsilon}{a}}}{2\frac{\eta+1}{\eta}}$$

Using the fact that  $b_1 b_2 < 0$  and that  $b_1 + b_2 > 0$ , we know there is a positive root,  $b_1$ , which is the largest in absolute value and a negative one. In appendix 8.6 we show that  $b_1 < 1$ , which implies convergent dynamics, that  $b_1 \leq \frac{\eta}{1+\eta}$ , which implies that the convergence rate is lower than the case with default costs in non-tradable output, and we solve for  $A_1(\widehat{K}_t) = \xi_1 \widehat{K}_t$  and  $A_2(\widehat{K}_t) = \xi_2 \widehat{K}_t$ .

Next we need to solve for the dynamics of individual capital if a farmer decides to default. After  $t+2$ , this is given by (22), which we repeat here for the sake of exposition,

$$\frac{1}{\eta} \widehat{K}_{t+s} + \widehat{k}_{t+s}^i = \frac{(a-\epsilon)}{a} \widehat{k}_{t+s-1}^i + \frac{\epsilon}{a} \widehat{k}_{t+s-2}^i$$

Note that now there is a non-homogeneous term given by  $\widehat{K}_{t+s}$ , which we can substitute using (25):

$$\widehat{k}_{t+s}^i = \frac{(a-\epsilon)}{a} \widehat{k}_{t+s-1}^i + \frac{\epsilon}{a} \widehat{k}_{t+s-2}^i - \frac{1}{\eta} \left( \xi_1 \widehat{K}_t b_1^s + \xi_2 \widehat{K}_t b_2^s \right) \quad (26)$$

In appendix 8.7 we solve for this dynamics.

## 5.2 Implied Utilities

Now we are ready to compute the implied utilities of default and renegotiation, given the level of the shock,  $\Delta$ , expectations about the general level of default,  $\mu$ , and the proposed debt-reduction,  $\varphi$ . The farmer's expected utility is given by

$$U^i(\mu; \Delta, \varphi) = cK^* + \beta c k_t^i + \beta^2 c k_{t+1}^i + \dots + \lim_{s \rightarrow \infty} \beta^s c k_{t+s-1}^i.$$

$$U^i(\mu; \Delta) = cK^* + c\beta \sum_{s=0}^{\infty} \beta^s K^* (1 + \widehat{k}_{t+s}^i)$$

Since lenders moves first, and have all the bargaining power ex post, each one will offer the lowest debt reduction such that the farmer is indifferent between defaulting and renegotiating. Note that

$$U^D = U^{ND} \Leftrightarrow \sum_{s=0}^{\infty} \beta^s (1 + \widehat{k}_{t+s}^D) = \sum_{s=0}^{\infty} \beta^s (1 + \widehat{k}_{t+s}^{ND}). \quad (27)$$

Since nobody defaults  $\widehat{k}_{t+s}^{ND} = \widehat{K}_{t+s}^{ND}$ . Using the solution to the differential equations, we can write

$$\begin{aligned}
& 1 + \left(-\Delta + \frac{1}{\eta}\widehat{K}_t\right) + \beta \left(1 + \left(\frac{a-\epsilon}{a} - \frac{\eta}{\eta+1}\right)\frac{1}{\eta}\widehat{K}_t - \left(\frac{a-\epsilon}{a}\right)\Delta\right) + \\
& \beta^2 \sum_{s=0}^{\infty} \beta^s (\xi_3 \widehat{K}_t b_1^s + \xi_4 \widehat{K}_t b_2^s + \xi_5 \widehat{K}_t + \xi_6 \widehat{K}_t \left(-\frac{\epsilon}{a}\right)^s) = \\
& 1 + \widehat{K}_t + \beta \left(1 + \frac{\eta}{\eta+1}\right) \widehat{K}_t + \beta^2 \sum_{s=0}^{\infty} \beta^s (\xi_1 \widehat{K}_t b_1^s + \xi_2 \widehat{K}_t b_2^s)
\end{aligned}$$

## 6 Empirical evidence

### 6.1 Testable implications of the transmission mechanism

According to our model, the strength of the amplification mechanism depends on the distribution of the ex post capital losses between debtors and creditors, which is affected by the level of protection of creditor rights, and the degree of asset specificity. In this section, we posit an empirical strategy that would allow us to test the implications of this transmission mechanism using U.S. data.

We explore the first aspect, related to the protection of creditor rights, using state variability in the homestead exemption in the personal bankruptcy procedures under chapter 7 in the U.S. Hynes et al. (2011) show that the only relevant determinant of current exemption levels is inertia. Thus, in the context of our model, heterogeneity in exemption levels can be seen as a source of exogenous variation in the distribution of the debt burden, at least for small firms for which chapter 7 procedures are relevant. Those states with lower exemption levels put the burden on the debtor, thus amplifying their losses. Conversely, states with unlimited exemption levels put the burden on creditors thus shutting down the dynamic amplification mechanism.

Berger et al. (2011) show that the variation in exemption levels affects the access to credit for proprietorships, while this is not an issue for big firms. In this sense, proprietorship are affected by the “collateral” constraint, given by exemption levels. Hence, we expect proprietorships to behave like farmers in our model. Since big firms are not affected by this legislation, we will use them to control for the corresponding (exogenous) productivity trend. Using U.S. firm level data, such as the Longitudinal Business Database described by Jarmin and Miranda (2002) or the NETS database described by Neumark et al. (2007), we would like to test whether the response of small firms over the business cycle (controlling by the cyclical fluctuations of big firms) is more volatile in low-exemption states.<sup>19</sup> Additionally, we would follow Berger et al. (2011) and control for economic and institutional differences across states using the state median income and a

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<sup>19</sup>Arguably, this result could also follow if the pool of small firms is less risky in high-exemption states. However, previous works suggest the opposite is true (Berger et al., 2005). Moreover, the NETS database also provides credit ratings for each establishment, which would allow us to control for firm riskiness.

dummy equal to one for those states where lenders must go through courts to foreclose on a property.

Furthermore, our model provides a testable implication between the degree of asset specificity in a given industry and the behavior of credit constrained firms over the business cycle. We will take Beutler and Grobéty (2011)’s measure of real asset redeployability as a proxy for asset specificity. Beutler and Grobéty (2011) use cross country data to estimate the impact of the degree of debt contract enforcement on economic activity through the collateral channel. For this they exploit the heterogeneity of expenditures in new and used capital across U.S. industries, constructing “technological” indicators of the ease with which real assets used in one industry can be transferred to alternative uses. We expect to see more amplification over the business cycle for firms whose assets are less redeployable. Moreover, our model also implies that the rate of births and deaths should be smaller in high specificity sectors, an implication we could also take to the data.<sup>20</sup>

## 7 Conclusions

We showed that Kiyotaki and Moore (1997)’s “dynamic amplification” result depends on the assumption that ex post debtors must pay their debts in full, even when the value of assets posted as collateral is below the value of debt. In contrast, we lift this assumption, which seems reasonable in the presence of bankruptcy procedures that limit the ability of creditors to collect debts, or the fact that debt renegotiations are common after aggregate shocks. Doing this highlights the relation between the amplification of shocks and the degree of asset specificity, understood as the differential in productivity of an ongoing firm and a new one that an entrepreneur must set up after defaulting and delivering her previous assets to her creditors.

To study the implications of implicit default costs on the business cycle we considered three simple model variations from the basic Kiyotaki and Moore (1997)’s model. First, we had defaulting firms incurring reduced productivity on non-tradable output. We find that amplification is larger for firms with higher costs, and that persistence is unaffected. Second, we considered the case when these default costs are private information and found that amplification is larger when there are less problems of asymmetric information, because amplification is inversely related to the amount of equilibrium default and this is higher when there is more uncertainty about default costs. Finally we considered the case when defaulting firms face a reduction in productivity of tradable output. In this case the more severe is this problem the more persistence after a negative shock.

Our model provides testable implications of its transmission mechanism. We described an empirical strategy to test these relations using U.S. firm level data exploiting heterogeneity in the levels of debtor protection in personal bankruptcy procedures across U.S. states, and in real asset redeployability across industries. We expect to find differential effects in small firms’ response to their business cycle (as measured by the fluctuations in

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<sup>20</sup>As a robustness check we could build a variable that summarizes the difficulty of establishing a new business (measured in time, capital, and technological requirements). This proxy measure for the model’s asset specificity would serve as a robustness check, since it relates to the cost of starting a new business.

output of large firms) across industries and states.

The basic model that we consider here is apt to be extended in several ways. The treatment of the shock as a zero probability bolt from the blue is a useful analytical device in first approximation, but it certainly leaves aside behaviors that can modify the results of the model, especially by moderating the impacts of small disturbances if, for instance, debtors find it convenient to leave a slack in the collateral constraint (as in Brunnermeier and Sannikov (2010)). Then, the macroeconomic responses may show two types of non-linearities: one, an increase in the financial multipliers with the size of the shock (evocative of a corridor effect) as the blow gets harder and, in the other extreme, a moderation as debts are renegotiated in the event of a very strong shock.

A representation of deep macroeconomic crises would require a closer attention to the ways in which parties in financial contracts, and the legal system itself, process situations of widespread broken promises. In particular we would like to introduce a cost to renegotiation, e.g. modeled as a war of attrition between lenders and borrowers, that might produce stronger immediate output drops followed by steeper recoveries.

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## 8 Appendix

### 8.1 Computation of $\hat{q}_t$ and $\hat{K}_t$ in KM with default.

If the economy is in a steady state and is hit by an unforeseen negative shock, the farmer defaults and loses the collateral  $q_t k^*$  but does not have to pay her debt  $Rb^*$ . Thus, her flow-of-funds, given by (1), becomes:

$$q_t k_t = (a - \Delta)k^* + \frac{q_{t+1}k_t}{R}.$$

Defining, as KM, the user cost of land  $u_t \equiv q_t - \frac{q_{t+1}}{R}$ ,

$$u_t k_t = (a - \Delta)k^*.$$

In equilibrium, this user cost of land depends, through the gatherer's production function, on the amount of land in the hands of gatherers,

$$u(K_t) = \frac{1}{R} G' \left( \frac{1}{m} (\bar{K} - K_t) \right),$$

where  $K_t$  is the aggregate amount of land owned by farmers. Since all farmers behave in the same way in equilibrium:

$$u_t(K_t)K_t = (a - \Delta)K^*.$$

Log-linearizing,

$$\left(1 + \frac{1}{\eta}\right) \hat{K}_t = -\Delta, \quad (28)$$

where  $\eta$  is, as in KM,  $\frac{d \log u(K)}{d \log K_t} |_{K=K^*}$ . Land prices are equal to the discounted stream of the user cost of land  $u_t$ . Using the transversality condition, the relation  $u(K_t) = q_t - \frac{q_{t+1}}{R}$ , and linearizing around the steady state yields

$$\hat{q}_t = \frac{1}{\eta} \frac{R-1}{R} \frac{1}{1 - \frac{1}{R} \frac{\eta}{1+\eta}} \hat{K}_t.$$

From the expression for  $\hat{K}_t$ , given by (28), this gives

$$\hat{q}_t = -\frac{R-1}{\frac{R-1}{\eta} + R} \Delta.$$

### 8.2 Investing is optimal for the farmer close to the steady state

When the farmer decides to default, she has to pay a productivity penalty in  $t+1$ . Nevertheless, under our assumption 2 the farmer still decides to consume only his bruised fruit. To see this, consider a marginal unit of fruit at date  $t$ . If she invests it to buy  $\frac{1}{u_t}$  units of land, tomorrow she will have  $\alpha c \frac{1}{u_t}$  units of nontradable fruit and  $a \frac{1}{u_t}$  units

of tradable fruit. This fruit allows the farmer to increase her land-holdings in  $t + 1$  to  $a \frac{1}{u_t} \frac{1}{u_{t+1}}$ . Some of this land is old ( $\frac{1}{u_t}$ ) and some ( $a \frac{1}{u_t} \frac{1}{u_{t+1}} - \frac{1}{u_t}$ ) is new. In  $t + 2$ , this land yields  $a^2 \frac{1}{u_t} \frac{1}{u_{t+1}}$  tradable fruit and  $(1 - \alpha)c \frac{1}{u_t} + \alpha c a \frac{1}{u_t} \frac{1}{u_{t+1}}$  bruised fruit. Reinvesting the tradable fruit enables the farmer to increase her land holdings to  $a^2 \frac{1}{u_t} \frac{1}{u_{t+1}} \frac{1}{u_{t+2}}$ , and so on. Evaluating this utility in steady state,

$$\begin{aligned} u(\text{invest}) &= \beta \sum c \beta^s \frac{\alpha}{a} + \beta^2 \sum \frac{c}{a} \beta^s (1 - \alpha), \\ u(\text{invest}) &= \frac{\beta}{(1 - \beta)} \frac{c \alpha}{a} + \frac{\beta^2}{(1 - \beta)} \frac{c}{a} (1 - \alpha), \\ u(\text{invest}) &= \frac{c}{a} \frac{\beta}{(1 - \beta)} (\alpha + \beta(1 - \alpha)). \end{aligned}$$

Saving yields

$$u(\text{saving}) = R \beta u(\text{invest}),$$

and consumption

$$u(\text{consumption}) = 1.$$

In equilibrium  $R = \frac{1}{\beta}$ . Thus, investment is preferred to saving. Moreover, by assumption 2, investment is better than consumption. This means that, in the neighborhood of the steady state the farmer will decide to consume only her bruised fruit.

### 8.3 Computation of $\tilde{U}^D(\mu; \Delta, \varphi)$ and $\tilde{U}^{ND}(\mu; \Delta, \varphi)$

#### 8.3.1 Case of no default

$$\begin{aligned} U^{ND}(\mu; \Delta) &= cK^* + \beta c k_t + \beta^2 (c \alpha (k_{t+1} - k_t) + c k_t) \\ &\quad + \dots + \lim_{s \rightarrow \infty} \beta^s (c \alpha (k_{t+s-1} - k_{t+s-2}) + c k_{t+s-2}), \end{aligned}$$

$$\begin{aligned} U^{ND}(\mu; \Delta) &= cK^* + \beta c K^* (1 + \hat{k}_t) (1 - \alpha) \\ &\quad + \beta c (\alpha + \beta(1 - \alpha)) \sum_{s=0}^{\infty} \beta^s K^* (1 + \hat{k}_{t+s}), \end{aligned}$$

$$\begin{aligned} U^{ND}(\mu; \Delta) &= cK^* + \beta c K^* (1 + \hat{k}_t) (1 - \alpha) \\ &\quad + \beta c (\alpha + \beta(1 - \alpha)) \sum_{s=0}^{\infty} \beta^s K^* (1 + \hat{k}_{t+s}). \end{aligned}$$

$$\text{Let } \tilde{U}^{ND}(\Delta) = \frac{U^{ND}(\Delta)}{\beta c K^*} - \frac{1}{\beta}$$

$$\tilde{U}^{ND}(\mu; \Delta) = (1 + \hat{k}_t) (1 - \alpha) + (\alpha + \beta(1 - \alpha)) \sum_{s=0}^{\infty} \beta^s (1 + \hat{k}_{t+s}),$$

$$\begin{aligned}\tilde{U}^{ND}(\mu; \Delta) &= (1 + \hat{k}_t)(1 - \alpha) \\ &\quad + (\alpha + \beta(1 - \alpha)) \sum_{s=0}^{\infty} \beta^s (1 + \hat{k}_t - \sum_{j=1}^s \frac{1}{\eta} \hat{K}_{t+j}),\end{aligned}$$

$$\begin{aligned}\tilde{U}^{ND}(\mu; \Delta) &= (1 + \hat{k}_t)(1 - \alpha) + \frac{(\alpha + \beta(1 - \alpha))}{1 - \beta} (1 + \hat{k}_t) \\ &\quad - \frac{(\alpha + \beta(1 - \alpha))}{\eta} \sum_{s=0}^{\infty} \beta^s \sum_{j=1}^s \hat{K}_{t+j},\end{aligned}$$

Using equation (8),

$$\begin{aligned}\tilde{U}^{ND}(\mu; \Delta) &= (1 + \hat{k}_t) \left( 1 - \alpha + \frac{(\alpha + \beta(1 - \alpha))}{1 - \beta} \right) \\ &\quad - \frac{(\alpha + \beta(1 - \alpha)) \hat{K}_t}{\eta} \sum_{s=0}^{\infty} \beta^s \sum_{j=1}^s \left( \frac{\eta}{\eta + 1} \right)^j,\end{aligned}$$

$$\tilde{U}^{ND}(\mu; \Delta) = \frac{1 + \hat{k}_t}{1 - \beta} + \frac{(\alpha + \beta(1 - \alpha)) \hat{K}_t}{\eta + 1} \sum_{s=0}^{\infty} \beta^s \sum_{j=0}^{s-1} \left( \frac{\eta}{\eta + 1} \right)^j,$$

$$\tilde{U}^{ND}(\mu; \Delta) = \frac{1 + \hat{k}_t}{1 - \beta} + \frac{(\alpha + \beta(1 - \alpha)) \hat{K}_t}{\eta + 1} \sum_{s=0}^{\infty} \beta^s \frac{\left( \frac{\eta}{\eta + 1} \right)^s - 1}{\frac{\eta}{\eta + 1} - 1},$$

$$\tilde{U}^{ND}(\mu; \Delta) = \frac{1 + \hat{k}_t}{1 - \beta} + (\alpha + \beta(1 - \alpha)) \hat{K}_t \sum_{s=0}^{\infty} \beta^s \left( 1 - \left( \frac{\eta}{\eta + 1} \right)^s \right),$$

$$\tilde{U}^{ND}(\mu; \Delta) = \frac{1 + \hat{k}_t}{1 - \beta} + (\alpha + \beta(1 - \alpha)) \hat{K}_t \left( \frac{1}{1 - \beta} - \frac{1}{1 - \frac{\beta\eta}{\eta + 1}} \right),$$

$$\tilde{U}^{ND}(\mu; \Delta) = \frac{1 + \hat{k}_t}{1 - \beta} + \hat{K}_t \left( \frac{1}{1 - \beta} - \frac{\eta + 1}{1 + \eta(1 - \beta)} \right),$$

$$\tilde{U}^{ND}(\mu; \Delta) = \frac{1 + \hat{k}_t}{1 - \beta} + \hat{K}_t \left( \frac{1 + \eta(1 - \beta) - (1 - \beta)(\eta + 1)}{(1 - \beta)(1 + \eta(1 - \beta))} \right),$$

$$\tilde{U}^{ND}(\mu; \Delta) = \frac{1 + \hat{k}_t}{1 - \beta} + \hat{K}_t \left( \frac{(\alpha + \beta(1 - \alpha))\beta}{(1 - \beta)(1 + \eta(1 - \beta))} \right),$$

### 8.3.2 Case of default

$$U^D(\mu; \Delta, \varphi) = cK^* + \beta c\alpha k_t^D + \beta^2(c\alpha(k_{t+1}^D - k_t^D) + ck_t^D) \\ + \dots + \lim_{s \rightarrow \infty} \beta^s(c\alpha(k_{t+s-1}^D - k_{t+s-2}^D) + ck_{t+s-2}^D),$$

$$U^D(\mu; \Delta, \varphi) = cK^* + c\beta k_t^D(\alpha + \beta(1 - \alpha)) \\ + \dots + c \lim_{s \rightarrow \infty} k_{t+s-1}^D \beta^s(\alpha + \beta(1 - \alpha)),$$

$$U^D(\mu; \Delta, \varphi) = cK^* + \beta c(\alpha + \beta(1 - \alpha)) \sum_{s=0}^{\infty} \beta^s k_{t+s-1}^D,$$

$$U^D(\mu; \Delta, \varphi) = cK^* + \beta c(\alpha + \beta(1 - \alpha)) \sum_{s=0}^{\infty} \beta^s (K^*(1 + \widehat{k}_{t+s}^D)),$$

$$U^D(\mu; \Delta, \varphi) = cK^* + \beta cK^*(\alpha + \beta(1 - \alpha)) \sum_{s=0}^{\infty} \beta^s (1 + \widehat{k}_t^D - \sum_{j=1}^s \frac{1}{\eta} \widehat{K}_{t+j}).$$

Let  $\widetilde{U}^D(\cdot) = \frac{U^D(\cdot)}{\beta c K^*} - \frac{1}{\beta}$ ,

$$\widetilde{U}^D(\mu; \Delta, \varphi) = \frac{(\alpha + \beta(1 - \alpha))(1 + \widehat{k}_t^{ND})}{1 - \beta} - \frac{(\alpha + \beta(1 - \alpha))}{\eta} \sum_{s=0}^{\infty} \beta^s \sum_{j=1}^s \widehat{K}_{t+j}.$$

Using equation (8),

$$\widetilde{U}^D(\mu; \Delta, \varphi) = \frac{(\alpha + \beta(1 - \alpha))(1 + \widehat{k}_t^D)}{1 - \beta} - \frac{(\alpha + \beta(1 - \alpha))\widehat{K}_t}{\eta} \sum_{s=0}^{\infty} \beta^s \sum_{j=1}^s \left(\frac{\eta}{\eta + 1}\right)^j,$$

$$\widetilde{U}^D(\mu; \Delta, \varphi) = \frac{(\alpha + \beta(1 - \alpha))(1 + \widehat{k}_t^D)}{1 - \beta} - \frac{(\alpha + \beta(1 - \alpha))\widehat{K}_t}{\eta} \sum_{s=0}^{\infty} \beta^s \sum_{j=1}^s \left(\frac{\eta}{\eta + 1}\right)^j,$$

$$\widetilde{U}^D(\mu; \Delta, \varphi) = \frac{(\alpha + \beta(1 - \alpha))(1 + \widehat{k}_t^D)}{1 - \beta} - \frac{1}{1 - \beta} \left( \beta \widehat{K}_t \frac{(\alpha + \beta(1 - \alpha))}{(1 + \eta(1 - \beta))} \right),$$

$$\widetilde{U}^D(\mu; \Delta, \varphi) = \frac{\alpha + \beta(1 - \alpha)}{1 - \beta} \left( 1 + \widehat{k}_t^D - \beta \widehat{K}_t \frac{(\alpha + \beta(1 - \alpha))}{(1 + \eta(1 - \beta))} \right). \quad (29)$$

## 8.4 Monotonicity of Mills ratio for Symmetric Beta Function

Suppose  $\alpha \sim \text{Beta}(\gamma, \gamma)$  with  $\gamma > 1$ . Since both parameters coincide, the distribution is symmetric and, since  $\gamma > 1$ , the distribution is unimodal. The density is given by

$$f(\alpha; \gamma) = \frac{\alpha^{\gamma-1}(1 - \alpha)^{\gamma-1}}{B(\gamma, \gamma)},$$

where  $B(\gamma, \gamma)$  is the beta function evaluated at  $(\gamma, \gamma)$ , i.e.  $\int_0^1 t^{\gamma-1}(1-t)^{\gamma-1} dt$ .

Let  $\psi(\alpha; \gamma) \equiv \frac{F(\alpha; \gamma)}{f(\alpha; \gamma)}$ . Then,

$$\Psi(\alpha, \gamma) = \int_0^\alpha \left( \frac{t(1-t)}{\alpha(1-\alpha)} \right)^{\gamma-1} dt.$$

$$\begin{aligned} \frac{\partial \Psi(\alpha, \gamma)}{\partial \alpha} &= 1 - \frac{(\gamma-1)(1-2\alpha)}{\alpha(1-\alpha)} \int_0^\alpha \left( \frac{t(1-t)}{\alpha(1-\alpha)} \right)^{\gamma-1} dt \\ \frac{\partial \Psi(\alpha, \gamma)}{\partial \alpha} &= 1 - \frac{(\gamma-1)(1-2\alpha)}{(\alpha(1-\alpha))^\gamma} B_\alpha(\gamma, \gamma), \end{aligned}$$

where  $B_\alpha(\gamma, \gamma)$  is the incomplete beta function evaluated at  $\alpha$ . For  $\alpha \geq \frac{1}{2}$ , the derivative is clearly positive. To show the result for  $\alpha < \frac{1}{2}$ , we resort to a result that links the beta incomplete function with the hypergeometric function (from <http://dlmf.nist.gov/8.17>):

$$\frac{\partial \Psi(\alpha, \gamma)}{\partial \alpha} = 1 - \frac{(\gamma-1)(1-2\alpha)}{\gamma} F(2\gamma, 1; \gamma+1; \alpha). \quad (30)$$

Now we will show  $F(2\gamma, 1; \gamma+1; \alpha) < \frac{1}{1-2\alpha}$ . This ensures  $\frac{\partial \Psi(\alpha, \gamma)}{\partial \alpha} > 0$ . By definition,

$$F(2\gamma, 1; \gamma; \alpha) = 1 + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \prod_{i=0}^{n-1} \frac{(2\gamma+i)(1+i)}{(\gamma+i)}$$

Noting that  $\prod_{i=0}^{n-1} (1+i) = n!$ ,

$$F(2\gamma, 1; \gamma; \alpha) = 1 + \sum_{n=1}^{\infty} \alpha^n \prod_{i=0}^{n-1} \left( \frac{2\gamma+i}{\gamma+i} \right).$$

Rearranging,

$$F(2\gamma, 1; \gamma; \alpha) = 1 + \sum_{n=1}^{\infty} (2\alpha)^n \prod_{i=0}^{n-1} \left( \frac{\gamma + \frac{i}{2}}{\gamma+i} \right).$$

Hence,

$$F(2\gamma, 1; \gamma; \alpha) < \sum_{n=0}^{\infty} (2\alpha)^n.$$

Since  $\alpha < \frac{1}{2}$ ,

$$F(2\gamma, 1; \gamma+1; \alpha) < \frac{1}{1-2\alpha}.$$

Thus,  $\frac{\partial \Psi(\alpha, \gamma)}{\partial \alpha} > 0 \forall \alpha \in [0, 1]$ .

## 8.5 Uniqueness of Cutoff with Beta Distribution

Let  $L(\varphi) = (\varphi + \hat{q}_t)F(\hat{\alpha}(\varphi))$ . The first derivative yields

$$\frac{dL(\varphi)}{d\varphi} = F(\hat{\alpha}(\varphi)) + f(\hat{\alpha}(\varphi))\frac{d\hat{\alpha}(\varphi)}{d\varphi}(\varphi + \hat{q}_t)$$

From (16),  $\frac{d\hat{\alpha}(\varphi)}{d\varphi} = \frac{\frac{R}{R-1}}{(1-\beta)(1-\Delta-\frac{1}{\eta}\hat{K}_t(\bar{\mu}, \bar{\varphi}))}$ . Note that this implies  $\frac{d\hat{\alpha}(\varphi)}{d\varphi}(\varphi + \hat{q}_t) = -(1 - \hat{\alpha}(\varphi))$ .

Replacing,

$$\frac{dL(\varphi)}{d\varphi} = F(\hat{\alpha}(\varphi)) - f(\hat{\alpha}(\varphi))(1 - \hat{\alpha}(\varphi)).$$

Rewrite it as

$$\frac{dL(\varphi)}{d\varphi} = \left( \frac{F(\hat{\alpha}(\varphi))}{f(\hat{\alpha}(\varphi))} - (1 - \hat{\alpha}(\varphi)) \right) f(\hat{\alpha}(\varphi)).$$

The second order condition yields

$$\frac{d^2L(\varphi)}{d\varphi^2} = \frac{d\alpha}{d\varphi} \left( \left( \frac{d\frac{F(\hat{\alpha}(\varphi))}{f(\hat{\alpha}(\varphi))}}{d\alpha} + 1 \right) f(\hat{\alpha}(\varphi)) + \left( \frac{F(\hat{\alpha}(\varphi))}{f(\hat{\alpha}(\varphi))} - (1 - \hat{\alpha}(\varphi)) \right) \frac{df(\hat{\alpha}(\varphi))}{d\alpha} \right) \quad (31)$$

In an interior optimum the second term vanishes,

$$\frac{d^2L(\varphi)}{d\varphi^2} = \frac{d\alpha}{d\varphi} f(\hat{\alpha}(\varphi)) \left( \frac{d\frac{F(\hat{\alpha}(\varphi))}{f(\hat{\alpha}(\varphi))}}{d\alpha} + 1 \right)$$

By assumption A1, the Mills' ratio is (weakly) increasing in  $\alpha$ , which in turn implies that any  $\alpha$  in the interior of the support that solves the first-order-condition describes a local minimum. By assumption A2,  $f(0) = 0$ ,  $\frac{F(\hat{\alpha}(\varphi))}{f(\hat{\alpha}(\varphi))} \rightarrow 0$  when  $\hat{\alpha}(\varphi) \rightarrow 0$  and  $\frac{df(\hat{\alpha}(\varphi))}{d\alpha} > 0$ . This implies  $\alpha = 0$  is a local maximum. Since  $f(1) = 0$ , the first term vanishes at  $\alpha = 1$ ,  $\frac{F(\hat{\alpha}(\varphi))}{f(\hat{\alpha}(\varphi))} \rightarrow \infty$  when  $\alpha \rightarrow 1$  and  $\frac{df(1)}{d\alpha} < 0$ . This implies  $\alpha = 1$  is also a local maximum. Hence,  $\alpha^*$  is the global minimum.

## 8.6 Solution of difference equation

To prove that  $b_1 < 1$ , first note that  $b_1 = \frac{\eta}{\eta+1} < 1$  if  $\epsilon = 0$ , while  $b_1 = \sqrt{\frac{\eta}{\eta+1}} < 1$  if  $\epsilon = a$ . Second, we compute the derivative of  $b_1$  with respect to  $\epsilon$  :

$$\frac{\partial b}{\partial \epsilon} = -\frac{1}{2a\frac{\eta+1}{\eta}} + \frac{2\left(\frac{a-\epsilon}{a}\right)\left(-\frac{1}{a}\right) + 4\frac{\eta+1}{\eta}\frac{1}{a}}{4\frac{\eta+1}{\eta}\sqrt{\left(\frac{a-\epsilon}{a}\right)^2 + 4\frac{\eta+1}{\eta}\frac{\epsilon}{a}}}$$

$$\frac{\partial b}{\partial \epsilon} \frac{2a\eta}{\eta+1} = -1 + \frac{\left(\frac{a-\epsilon}{a}\right) + 2\frac{\eta+1}{\eta}}{\sqrt{\left(\frac{a-\epsilon}{a}\right)^2 + 4\frac{\eta+1}{\eta}\frac{\epsilon}{a}}}$$

$$\begin{aligned}
\sqrt{\left(\frac{a-\epsilon}{a}\right)^2 + 4\frac{\eta+1}{\eta}\frac{\epsilon}{a}} &< \sqrt{\left(\frac{a-\epsilon}{a}\right)^2 + \left(2\sqrt{\frac{\eta+1}{\eta}}\sqrt{\frac{\epsilon}{a}}\right)^2 + 4\frac{a-\epsilon}{a}\sqrt{\frac{\eta+1}{\eta}}\sqrt{\frac{\epsilon}{a}}} \\
\sqrt{\left(\frac{a-\epsilon}{a}\right)^2 + 4\frac{\eta+1}{\eta}\frac{\epsilon}{a}} &< \frac{a-\epsilon}{a} + 2\sqrt{\frac{\eta+1}{\eta}}\sqrt{\frac{\epsilon}{a}} \\
\frac{\left(\frac{a-\epsilon}{a}\right) + 2\frac{\eta+1}{\eta}}{\sqrt{\left(\frac{a-\epsilon}{a}\right)^2 + 4\frac{\eta+1}{\eta}\frac{\epsilon}{a}}} &> \frac{\left(\frac{a-\epsilon}{a}\right) + 2\frac{\eta+1}{\eta}}{\frac{a-\epsilon}{a} + 2\sqrt{\frac{\eta+1}{\eta}}\sqrt{\frac{\epsilon}{a}}} > 1
\end{aligned}$$

This implies  $\frac{\partial b}{\partial \epsilon} > 0 \forall \epsilon \in [0, a]$ . Thus,  $b_1 < 1 \forall \epsilon \in [0, a]$ .

Next we have to determine the constants  $A_1$  and  $A_2$ . Note that (25) is valid from  $t+2$  onwards. To determine  $A_1$  and  $A_2$  we need two initial conditions:  $\widehat{K}_t$  and  $\widehat{K}_{t+1}$ . From (23) we can write  $\widehat{K}_{t+1} = \frac{\eta}{\eta+1}\widehat{K}_t$ . Assume wlog that  $s = 0$  in (25). Hence, in  $t+2$  and  $t+3$ :

$$\begin{aligned}
\widehat{K}_{t+2} &= A_1 + A_2 = \frac{\eta}{\eta+1}\frac{\epsilon}{a}\widehat{K}_t + \left(\frac{\eta}{\eta+1}\right)^2\left(\frac{a-\epsilon}{a}\right)\widehat{K}_t, \\
\widehat{K}_{t+3} = A_1b_1 + A_2b_2 &= \left(\frac{\eta}{\eta+1}\right)^2\frac{\epsilon}{a}\widehat{K}_t + \left(\frac{\eta}{\eta+1}\right)^2\frac{\epsilon}{a}\left(\frac{a-\epsilon}{a}\right)\widehat{K}_t + \left(\frac{\eta}{\eta+1}\right)^3\left(\frac{a-\epsilon}{a}\right)^2\widehat{K}_t.
\end{aligned}$$

We have a system of two equations that allows us to solve for  $A_1(\widehat{K}_t)$  and  $A_2(\widehat{K}_t)$ .

$$\begin{aligned}
A_1(\widehat{K}_t) &= \xi_1\widehat{K}_t, \\
A_2(\widehat{K}_t) &= \xi_2\widehat{K}_t, \\
\xi_1 &\equiv \frac{1}{b_1 - b_2} \left( \frac{\eta}{\eta+1} \left( \frac{\eta}{\eta+1} \left( \frac{\epsilon}{a} + \left( \frac{a-\epsilon}{a} \right) \left( \frac{\epsilon}{a} - b_2 + \frac{\eta(a-\epsilon)}{(\eta+1)a} \right) \right) - \frac{\epsilon}{a}b_2 \right) \right), \\
\xi_2 &\equiv \frac{1}{b_2 - b_1} \left( \frac{\eta}{\eta+1} \left( \frac{\eta}{\eta+1} \left( \frac{\epsilon}{a} + \left( \frac{a-\epsilon}{a} \right) \left( \frac{\epsilon}{a} - b_1 + \frac{\eta(a-\epsilon)}{(\eta+1)a} \right) \right) - \frac{\epsilon}{a}b_1 \right) \right).
\end{aligned}$$

## 8.7 Dynamics of land holdings for a farmer that defaults

To solve for the dynamics of individual land holdings if a farmer decides to default, the roots of the homogeneous equation in 26 are given by

$$b_{1,2}^i = \frac{\frac{(a-\epsilon)}{a} \pm \sqrt{\left(\frac{a-\epsilon}{a}\right)^2 + 4\frac{\epsilon}{a}}}{2}$$

Note that  $b_1^i = 1$  and  $b_2^i = -\frac{\epsilon}{a}$ .

Now we try the following particular solution:  $B_1b_1^s + B_2b_2^s$ . Replacing in (26):

$$\frac{1}{\eta} \left( \xi_1\widehat{K}_tb_1^s + \xi_2\widehat{K}_tb_2^s \right) + B_1b_1^s + B_2b_2^s = \frac{(a-\epsilon)}{a} (B_1b_1^{s-1} + B_2b_2^{s-1}) + \frac{\epsilon}{a} (B_1b_1^{s-2} + B_2b_2^{s-2})$$

$$\left(\frac{1}{\eta}\xi_1\widehat{K}_t + B_1\left(1 - \frac{1}{b_1}\frac{(a-\epsilon)}{a} - \frac{1}{b_1^2}\frac{\epsilon}{a}\right)\right)b_1^s + \left(\frac{1}{\eta}\xi_2\widehat{K}_t + B_2\left(1 - \frac{1}{b_2}\frac{(a-\epsilon)}{a} - \frac{1}{b_2^2}\frac{\epsilon}{a}\right)\right)b_2^s = 0$$

Let  $\xi_3 \equiv -\frac{\xi_1}{\eta\left(1 - \frac{1}{b_1}\frac{(a-\epsilon)}{a} - \frac{1}{b_1^2}\frac{\epsilon}{a}\right)}$  and  $\xi_4 \equiv -\frac{\xi_2}{\eta\left(1 - \frac{1}{b_2}\frac{(a-\epsilon)}{a} - \frac{1}{b_2^2}\frac{\epsilon}{a}\right)}$ . Then,

$$B_1 = \xi_3\widehat{K}_t$$

$$B_2 = \xi_4\widehat{K}_t$$

Hence, the general solution starting from  $t + 2$  is

$$\widehat{k}_{t+2+s}^i = B_1b_1^s + B_2b_2^s + A_1^i + A_2^i\left(-\frac{\epsilon}{a}\right)^s$$

where the constants  $A_1^i$  and  $A_2^i$  still need to be determined. To compute these constants we use the "initial conditions"  $k_t^i$  and  $k_{t+1}^i$ . We will write everything as a function of  $\widehat{K}_t$ , which will prove useful later, using equations XX:

$$\widehat{k}_t^D = -\Delta + \frac{1}{\eta}\widehat{K}_t$$

$$\widehat{k}_{t+1}^D = \left(\frac{a-\epsilon}{a} - \frac{\eta}{\eta+1}\right)\frac{1}{\eta}\widehat{K}_t - \left(\frac{a-\epsilon}{a}\right)\Delta$$

Hence, we have the following system of equations

$$\widehat{k}_{t+2}^i = \frac{(a-\epsilon)}{a}\left(\left(\frac{a-\epsilon}{a} - \frac{\eta}{\eta+1}\right)\frac{1}{\eta}\widehat{K}_t - \left(\frac{a-\epsilon}{a}\right)\Delta\right) + \frac{\epsilon}{a}\left(-\Delta + \frac{1}{\eta}\widehat{K}_t\right) - \frac{1}{\eta}(A_1 + A_2) = \xi_3\widehat{K}_t + \xi_4\widehat{K}_t + A_1^i + A_2^i\left(-\frac{\epsilon}{a}\right)^s$$

$$\widehat{k}_{t+3}^i = \frac{(a-\epsilon)}{a}\left(\frac{(a-\epsilon)}{a}\left(\left(\frac{a-\epsilon}{a} - \frac{\eta}{\eta+1}\right)\frac{1}{\eta}\widehat{K}_t - \left(\frac{a-\epsilon}{a}\right)\Delta\right) + \frac{\epsilon}{a}\left(-\Delta + \frac{1}{\eta}\widehat{K}_t\right)\right) + \frac{\epsilon}{a}\left(\left(\frac{a-\epsilon}{a} - \frac{\eta}{\eta+1}\right)\frac{1}{\eta}\widehat{K}_t - \left(\frac{a-\epsilon}{a}\right)\Delta\right) + \frac{\epsilon}{a}\left(-\Delta + \frac{1}{\eta}\widehat{K}_t\right) - \frac{1}{\eta}(A_1 + A_2) = \xi_5\widehat{K}_t + \xi_6\widehat{K}_t + A_1^i + A_2^i\left(-\frac{\epsilon}{a}\right)^s$$

Let  $\xi_5 \equiv$  and  $\xi_6 \equiv$ . Then, solving this system yields

$$A_1^d = \xi_5\widehat{K}_t$$

$$A_2^d = \xi_6\widehat{K}_t$$

## 8.8 Implied utilities

Now we try the solution  $\widehat{k}_{t+s}^{ND} = A_{ND}\lambda^s$  and  $\widehat{k}_{t+s}^D = A_D\lambda^s$ . Replacing

$$\left(\frac{1-\mu}{\eta} + 1\right)A_{ND}\lambda^{s+2} + \left(\frac{\mu}{\eta}\right)A_D\lambda^{s+2} - \frac{(a-\epsilon)}{a}A_{ND}\lambda^{s+1} - \frac{\epsilon}{a}A_{ND}\lambda^s = 0$$

$$\left(\frac{1-\mu}{\eta}\right)A_{ND}\lambda^{s+2} + \left(\frac{\mu}{\eta} + 1\right)A_D\lambda^{s+2} - \frac{(a-\epsilon)}{a}A_D\lambda^{s+1} - \frac{\epsilon}{a}A_D\lambda^s = 0$$

System (XX) will yield a non-trivial solution for  $A_{ND}$  and  $A_D$  if and only if its determinant is zero, namely



$$\begin{vmatrix} \left(\frac{1-\mu}{\eta} + 1\right) \lambda^2 - \frac{(a-\epsilon)}{a} \lambda - \frac{\epsilon}{a} & \frac{\mu}{\eta} \lambda^2 \\ \left(\frac{\mu}{\eta} + 1\right) \lambda^2 - \frac{(a-\epsilon)}{a} \lambda - \frac{\epsilon}{a} & \frac{1-\mu}{\eta} \lambda^2 \end{vmatrix} = 0$$

The polynomial form of the determinantal equation XX is

$$\begin{aligned} & \left( \left( \frac{1-\mu}{\eta} + 1 \right) \lambda^2 - \frac{(a-\epsilon)}{a} \lambda - \frac{\epsilon}{a} \right) \left( \frac{1-\mu}{\eta} \lambda^2 \right) - \left( \left( \frac{\mu}{\eta} + 1 \right) \lambda^2 - \frac{(a-\epsilon)}{a} \lambda - \frac{\epsilon}{a} \right) \left( \frac{\mu}{\eta} \lambda^2 \right) = 0 \\ & \left( \frac{1-\mu}{\eta} + 1 \right) \left( \frac{1-\mu}{\eta} \right) \lambda^4 - \left( \frac{a-\epsilon}{a} \right) \left( \frac{1-\mu}{\eta} \right) \lambda^3 - \frac{\epsilon}{a} \left( \frac{1-\mu}{\eta} \right) \lambda^2 - \left( \frac{\mu}{\eta} + 1 \right) \frac{\mu}{\eta} \lambda^4 + \frac{(a-\epsilon)}{a} \frac{\mu}{\eta} \lambda^3 + \frac{\epsilon}{a} \frac{\mu}{\eta} \lambda^2 = 0 \\ & \left( \left( \frac{1-\mu}{\eta} + 1 \right) \left( \frac{1-\mu}{\eta} \right) - \left( \frac{\mu}{\eta} + 1 \right) \frac{\mu}{\eta} \right) \lambda^2 + \left( \frac{\mu}{\eta} - \left( \frac{1-\mu}{\eta} \right) \right) \frac{(a-\epsilon)}{a} \lambda + \left( \frac{\mu}{\eta} - \left( \frac{1-\mu}{\eta} \right) \right) \frac{\epsilon}{a} = 0 \end{aligned}$$

Hence, the roots are

$$\lambda_{1,2} = \frac{-(\mu - (1 - \mu)) \frac{(a-\epsilon)}{\eta a} \pm \sqrt{\left( \left( \frac{\mu}{\eta} - \left( \frac{1-\mu}{\eta} \right) \right) \frac{(a-\epsilon)}{a} \right)^2 - 4 \left( \left( \frac{1-\mu}{\eta} + 1 \right) \left( \frac{1-\mu}{\eta} \right) - \left( \frac{\mu}{\eta} + 1 \right) \frac{\mu}{\eta} \right) \left( \frac{\mu}{\eta} - \left( \frac{1-\mu}{\eta} \right) \right) \frac{\epsilon}{a}}{2 \left( \left( \frac{1-\mu}{\eta} + 1 \right) \left( \frac{1-\mu}{\eta} \right) - \left( \frac{\mu}{\eta} + 1 \right) \frac{\mu}{\eta} \right)}$$

Note that, when  $\mu = 0$ , this simplifies to

$$\begin{aligned} \lambda_{1,2} &= \frac{\frac{1}{\eta} \frac{(a-\epsilon)}{a} \pm \sqrt{\left( -\frac{1}{\eta} \frac{a-\epsilon}{a} \right)^2 - 4 \left( \frac{1}{\eta} + 1 \right) \left( \frac{1}{\eta} \right) \left( -\frac{1}{\eta} \right) \frac{\epsilon}{a}}{2 \left( \frac{1}{\eta} + 1 \right) \left( \frac{1}{\eta} \right)} \\ \lambda_{1,2} &= \frac{\frac{a-\epsilon}{a} \pm \sqrt{\left( \frac{a-\epsilon}{a} \right)^2 + 4 \frac{\eta+1}{\eta} \frac{\epsilon}{a}}{2 \frac{\eta+1}{\eta}} \end{aligned}$$

This coincides with our previous answer for the aggregate-no default case. The roots of the dynamic in the default case are the same, because they are a system. The solution has the form

$$\begin{aligned} \widehat{k}_{t+s+2}^D &= A_1^D \lambda_1^s + A_2^D \lambda_2^s \\ \widehat{k}_{t+s+2}^{ND} &= A_1^{ND} \lambda_1^s + A_2^{ND} \lambda_2^s \end{aligned}$$

Both differ only in the constants. To find the constants we need two initial conditions, given by  $\widehat{k}_t^i$  and  $\widehat{k}_{t+1}^i$  in equations (XX). We

### 8.8.1 Equations

- (1)  $\frac{1}{\eta} (\mu \widehat{k}_t^D + (1 - \mu) \widehat{k}_t^{ND}) + \widehat{k}_t^{ND} = \frac{R}{R-1} \widehat{q}_t - \Delta + \frac{R}{R-1} \varphi$
- (2)  $\frac{1}{\eta} (\mu \widehat{k}_t^D + (1 - \mu) \widehat{k}_t^{ND}) + \widehat{k}_t^D = -\Delta.$
- (3)  $\frac{1}{\eta+1} (\mu \widehat{k}_t^D + (1 - \mu) \widehat{k}_t^{ND}) + \widehat{k}_{t+1}^{ND} = a \widehat{k}_t^{ND}$

$$\begin{aligned}
(4) \quad & \frac{1}{\eta+1}(\mu\widehat{k}_t^D + (1-\mu)\widehat{k}_t^{ND}) + \widehat{k}_{t+1}^D = \left(\frac{a-\epsilon}{a}\right)\widehat{k}_t^D \\
(5) \quad & \widehat{k}_{t+2}^D = A_1^D + A_2^D \\
(6) \quad & \widehat{k}_{t+3}^D = A_1^D\lambda_1 + A_2^D\lambda_2 \\
(7) \quad & \widehat{k}_{t+2}^{ND} = A_1^{ND} + A_2^{ND} \\
(8) \quad & \widehat{k}_{t+3}^{ND} = A_1^{ND}\lambda_1 + A_2^{ND}\lambda_2 \\
(9) \quad & \widehat{q}_t = \frac{R-1}{R}\frac{1}{\eta}\sum_{s=0}^{\infty}\frac{1}{R^s}(\mu\widehat{k}_{t+s}^D + (1-\mu)\widehat{k}_{t+s}^{ND})
\end{aligned}$$

We need to solve for  $\widehat{q}_t$ ,  $A_1^D$ ,  $A_2^D$ ,  $A_1^{ND}$ ,  $A_2^{ND}$ ,  $\widehat{k}_t^D$  and  $\widehat{k}_t^{ND}$ .