An Economic Model of Friendship: Homophily, Minorities and Segregation[*]

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Abstract

We develop a model of friendship formation that sheds light on segregation patterns observed in social and economic networks. Individuals have types and see type-dependent benefits from friendships. We examine the properties of a steady-state equilibrium of a matching process of friendship formation. We use the model to understand three empirical patterns of friendship formation: (i) larger groups tend to form more same-type ties and fewer other-type ties than small groups, (ii) larger groups form more ties per capita, and (iii) all groups are biased towards same-type relative to demographics, with the most extreme bias coming from middle-sized groups. We show how these empirical observations can be generated by biases in preferences and biases in meetings. We also illustrate some welfare implications of the model.

Keywords: Networks, Homophily, Segregation, Friendships, Social Networks, Integration, Diversity, Minorities

JEL Classification Numbers: A14, D85, I21, J15

1   Introduction

The network structure of social interactions influences a variety of behaviors and economic outcomes, including the formation of opinions, decisions of which products to buy, investment in education, access to jobs, and social mobility, just to name a few. The extent to which a society is segregated across different groups can be critical in determining things like how quickly information diffuses, and the extent to which there is under-investment in human capital, among other things. In this paper we examine a fundamental and pervasive phenomenon of social networks which is known as "homophily".[1] This refers to a tendency of various types of individuals to associate with others who are similar to themselves. Homophily has been documented quite broadly, across characteristics such as age, race, gender, religion, profession and is generally a quite strong and robust observation (see McPherson, Smith-Lovin and Cook (2001) for an overview of research on homophily). Given the importance of social networks, developing models that help us to understand homophily is essential.

We begin our analysis by providing some detailed observations about patterns of homophily. Specifically, we identify three patterns of homophily in the 'Adolescent Health' data set (described in detail below), which examines friendship patterns in a representative sample of U.S. high schools.[2] We then build a model of friendship formation and show that the model can generate the observed patterns of homophily, and we trace the different patterns of observed homophily to different aspects of the model.

The three main observations that we point out from the data are summarized as follows. Consider a high school and the patterns of friendship within it. In particular, let the type of an individual be his/her race. The first empirical observation is a basic one, which we refer to as relative homophily: larger groups (measured as a fraction of the population of their respective schools) form a greater fraction of their friendships with people of their same type. This observation echoes previous studies. The second main empirical observation that we document is that larger groups form significantly more friendships per capita. More specifically, members of a group that comprise a small minority in a school form roughly

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[1]The etymology of the term is simple: homo = self and philìa = love. Homophily is a term coined by Lazarsfeld and Merton (1954).

[2]Add Health is a program project designed by J. Richard Udry, Peter S. Bearman, and Kathleen Mullan Harris, and funded by a grant P01-HD31921 from the National Institute of Child Health and Human Development, with cooperative funding from 17 other agencies. Persons interested in obtaining data files from Add Health should contact Add Health, Carolina Population Center, 123 W. Franklin Street, Chapel Hill, NC 27516-2524 (addhealth@unc.edu).

6 friendships per capita, while members of groups that comprise large majorities (close to one hundred percent of a school) form on average more than 8 friendships. The third main empirical observation that we document concerns a tendency for groups to form same-type friendships at rates that exceed the relative fractions in the population. That is, groups tend to "inbreed" (a term from the literature on homophily). More specifically, the new observation is that this inbreeding is essentially absent for groups that comprise very small or very large fractions of their school, while the inbreeding is very significant for groups that comprise a *middle-ranged* fraction of their school.

With these three observations in mind, we develop a simple model of friendship formation. In the model, individuals come into a society and form friendships when they enter. Friends are met through a process of random matching, and each matching entails a fixed cost. There are diminishing returns to forming friendships, and so eventually an individual exits the process. The critical determinants of an individual's strategy are his or her preferences over the mix of types of friends that he or she forms as well as the mix that he or she faces in the matching process.

The implications of the model are summarized as follows. We begin by noting that if agents' preferences over friendships are insensitive to type, so that agents only care about total number of friends and not on the composition of types, then all agents form the same number of friendships under any matching process such that all agents meet the same expected number of friends per unit of search. Next, we note that any matching process such that types are matched in frequencies in proportion to their relative stocks in the matching process cannot generate the observed evidence of generalized inbreeding, regardless of type-sensitivity of preferences. Given these two observations, we explore the implications of type-sensitivity of preferences and bias in matching. We first examine type-sensitivity of preferences and show that if agents see higher marginal returns when forming a mix of friendships that is biased towards same-type, then larger groups form more friendships per capita in equilibrium. Second, we show that a bias in the matching process, so that all groups can simultaneously be meeting their own type at rates faster than their fraction in the matching pool, generates inbreeding homophily patterns that match the observed patterns (where middle-sized groups are the most inbred). Such a bias in matching could be attributed to a variety of factors such as tracking and membership in various clubs and activities, as well as meeting friends through friends.[3] We then fit a parametric version of the model to the data, and show that a model with both a type-sensitive preferences and a matching bias fits all of the observations, while just type-sensitive preferences generates the observation of larger

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[3]See Feld (1981) for some discussion of such factors.

groups forming more per capita friendships but not the inbreeding homophily, while just a matching bias does the reverse.

We conclude the analysis with a brief consideration of welfare issues. While the model is too stark to take seriously for policy implications, the welfare analysis points out that average welfare depends in sensitive ways to the structure of preferences and matching. This suggests that it is vital to build richer models of friendship formation in order to better understand homophily and self-segregation within schools (and other applications) and to develop well-founded policies.

Regarding contributions to the literature, homophily has been noted throughout history, and is seen in such adages as "birds of a feather, flock together." which dates to at least the seventeenth century.[4] And as mentioned above, there is a large literature examining homophily along a wide variety of dimensions including ethnicity (e.g., see Fong and Isajiw (2000) and Baerveldt et al. (2004) for examples of studies focusing on ethnicity, and McPherson, Smith-Lovin and Cook (2001) for a wider survey). Researchers are well aware that homophily patterns could be influenced by both the meeting process and preferences. For example, Moody (2001, page 680) notes that: "friendship segregation results from the multilevel influences of mixing opportunity and individual preference". However, our model seems to be the first systematic investigation of the roles of preferences and matchings in determining the emerging patterns of homophily in social ties.[5] In terms of the specific observations that we examine, the contributions are as follows. The observation regarding relative homophily, and pointing out that this essentially follows from having friendships balance across types, is something that has roots in what is known as "contact theory." For example, Blau (1977) (see also Allport (1954)) points out that since each cross-group friendship must involve a member from each group, then smaller groups must have more cross-group friendships on a per capita basis. In our model, however, overall relative homophily is not implied by this alone, and also requires that preferences have diminishing returns. The second observation that larger groups form more relationships is an empirical finding that has been noted before, for instance by Marsden (1987) in a study of advice networks. Showing how this, at least in the context of our model, is generated by type-sensitive preferences, is new. As we show, absent sensitivity to types in preferences, to explain the

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[4]Lazarsfeld and Merton (1954) attribute this quote to Robert Burton in 1621, but a version of it appears in Philemon Holland's (1601) translation of "Livy's Roman history," and seems to come from folklore before that.

[5]In fact, even more generally, most models of network formation fall either into a class of random graph and/or statistical models governed by random attachment, or game theoretic models governed by preferences. See Jackson (2006) for some discussion and background.

larger number of friendships formed by larger groups, the matching process would need to be one of increasing returns so that types that make up a larger fraction of a school or society systematically meet more people per unit of time. We discuss such an alternative mechanism in Section 7.3. The third observation that inbreeding has a specific pattern, and is maximal for middle-sized groups also appears to be new, as does the explanation that this is tied to biases in matching process.[6] Finally, the welfare analysis that accompanies the model is new as well.

Finally, this paper is related to the search literature (e.g., see Diamond (1982), Mortensen and Pissarides (1994), Acemoglu and Shimer (1999), among others). Our analysis differs from most of that literature in allowing for each agent to form many relationships, and so results in a many-to-many matching. We also examine biased meeting processes, so that different types meet at different rates. Although this leads to some increased generality, it also pushes us to simplify on other fronts. In particular, we focus on settings where agents accept all matches so that their decision variable are stopping times.[7] We also provide (in Appendix A) some foundations for our continuum model in terms of viewing it as the limit of a large finite matching model.

We begin the paper by defining homophily indices, then present observations on the Add Health data, the model, and the analysis.

2 Measuring Homophily

Before taking a closer look at homophily patterns, we define some indexes of homophily.

Let N_i denote the number of type i individuals in the population, and let $w_i = \frac{N_i}{N}$ be the relative fraction of type i in the population, where $N = \sum_k N_k$.

Let s_i denote the average number of friendships that agents of type i have with agents who are of the same type, and let d_i be the average number of friendships that type i agents

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[6]There is some prior evidence that the percentage of cross-group links can vary nonlinearly and non monotonically with measures of population heterogeneity (e.g., see Blalock (1967) and Moody (2001)), although not in the explicit form found here. There have been explanations proposed for why inbreeding might vary with population mixes, for instance, based on power differences and competition (e.g., Giles and Evans (1986)). Moody (2001) suggests that when two groups are close to a majority their tendency to inbreed increases, but because of competing groups concerned about sociological factors, which are absent from our preference-based model.

[7]In supplementary material we present some results on an extension of the model where agents can experience satiation with respect to some types and not others, which leads to substantial complications in the model.

form with agents of types different than i.

The obvious basic homophily index, H_i, measures the fraction of the ties of individuals of type i that are with that same type.

DEFINITION 1 *The homophily index H_i is defined by*

$$H_i = \frac{s_i}{s_i + d_i}.$$

When the index H_i is increasing with relative group size, then we say that the pattern of friendships satisfies relative homophily.

DEFINITION 2 *A profile $(s, d) = (s_1, d_1, s_2, d_2, ..., s_K, d_K)$ satisfies* **relative homophily** *if $w_i > w_j$ implies $H_i > H_j$.*

As mentioned above, the index H_i is partly determined by relative populations, and so a benchmark is the relative size of a group, which corresponds to the expectation of the fraction of same-type friendships in a world where friendships are distributed uniformly at random among pairs of agents in the population.

DEFINITION 3 *The profile $(s, d) = (s_1, d_1, s_2, d_2, ..., s_K, d_K)$ satisfies* **baseline homophily** *if for all i*

$$H_i = w_i.$$

Note that baseline homophily immediately implies relative homophily, while the reverse is not true.

The observed tendency of friendships to be biased towards own types beyond the effect of relative population sizes has been referred to in the sociological literature as "inbreeding homophily" (see, e.g., Coleman (1958), Marsden (1987), and McPherson Smith-Lovin and Cook (2001)).

DEFINITION 4 *The profile (s, d) satisfies* **inbreeding homophily** *for type i if*

$$H_i > w_i.$$

One can also have the reverse condition where friendships are biased towards having different-type friendships.

DEFINITION 5 *The profile (s, d) satisfies* **heterophily** *for type i if*

$$H_i < w_i.$$

Generally, there is a difficulty in simply measuring homophily according to H_i. For example, consider a group that comprises 95 percent of a population. Suppose that its same-type friendships are 96 percent of its friendships. Compare this to a group that comprises 5 percent of a population and has 96 percent of its friendships being same-type. Although both have the same homophily index, they are very different in terms of how homophilous they are relative to how homophilous they could be. Comparing the homophily index, H_i, to the baseline, w_i, provides some information, but even that does not fully capture the idea of how biased a group is compared to how biased it could potentially be. In order to take care of this we use the measure developed by Coleman (1958) that normalizes the homophily index by the potential extent to which a group could be biased.[8]

DEFINITION 6 *The **inbreeding homophily** of type i is*

$$IH_i = \frac{H_i - w_i}{1 - w_i}.$$

This index measure the amount of bias with respect to baseline homophily as it relates to the maximum possible bias (the term $1 - w_i$). It can be easily checked that we have inbreeding homophily for type i if and only if $IH_i > 0$, and inbreeding heterophily for type i if and only if $IH_i < 0$. The index of inbreeding homophily is 0 if there is pure baseline homophily, and 1 if a group completely inbreeds.[9,10]

3   Patterns of Friendships and Homophily

Table 1 presents friendships in an American high school, from the Add Health data set.[11]

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[8]As will be clear from the data, this also corrects an issue that the raw homophily index is heteroskedastic when viewed relative to group fraction w_i, while this normalized index has a more constant variance as a function of w_i.

[9]One could also define a heterophily index, which would be $\frac{\frac{s_i}{s_i+d_i}-w_i}{-w_i}$, reflecting the extent to which a group is outgoing. It would be 0 at baseline homophily and 1 if a group only formed different-type friendships.

[10]The measures H_i and IH_i have slight biases in small samples. For example, suppose that there was no bias in the friendship formation process so that we are in a "baseline" society. Then the expected fraction of same type meetings of type i is $\frac{N_i-1}{N-1}$. Thus, the expected value of $H_i - w_i$ in a baseline society is $-\frac{N-N_i}{N(N-1)}$, which vanishes as N becomes large. The expected value of IH_i is then $-\frac{1}{N-1}$, which is independent of i, vanishing in N, and slightly negative. Given that we are finding significantly positive values of IH_i, and the N's vary from 30 to 5000, this is not a problem in our analysis.

[11]The data from National Longitudinal Survey of Adolescent Health (commonly referred to as "Add Health") were collected over several years starting in 1994 from a carefully stratified sample of high schools and middle schools (to vary by size, location, include public and private, vary racial composition, and socio-

	Ethnicity of Students			
Percent of Friends	White	Black	Hispanic	Others
by Ethnicity:	n=131	n=96	n= 13	n=15
	%=51	%=38	%=5	%=6
White	85	7	47	74
Black	4	85	46	11
Hispanic	4	6	2	4
Others	7	2	5	11

Table 1: Percentage of Links Across Ethnicities in an American School; from Add Health 1994 Data.

Table 1 records the percentage of friendships by race. Cell ij indicates the percentage of all agents of type j's friendships that are with members of race i. Columns sum to 100 percent. If friendships were formed purely at random, we would observe a first row centered around 51%, a second centered around 38%, and so forth. The numbers on the main diagonal record the percentage of total friendships that are of same-type.[12] These values are substantially higher than the relative group's size for the two larger groups (whites and blacks), and lower for the smallest group (hispanics), reflecting inbreeding homophily for both whites and blacks, and some heterophily for hispanics. The IH index of inbreeding homophily is 0.69 for whites (whose relative population is 51%), 0.76 for blacks (relative population 38%) and -0.11 for Hispanics (2% of population). This non-monotonic trend of the IH index is discussed in Sections 7 and 8.

The network of friendships by race in this high school is pictured in Figure 1.

Figure 1 illustrates the tendency of white and black students to form distinct communi-

economic backgrounds). There are 112 schools in the data set with behavioral and demographic data, and here the data are from 84 schools for which extensive network information was obtained. The data are based on student interviews. The friendship data were based on reports of friendships by each student. Students were shown a list of all the other students in the school and permitted to name up to five friends of each sex. Only 3 percent nominated ten friends, and only 24 percent hit the constraint on one of the sexes, and so the constraints do not seem to be a substantial measurement issue (see Moody (2001) for more discussion). The data include information about how much interaction there is between individuals, which we do not make use of as it does not add much to our analysis. Here a tie is present if either student mentioned the other as a friend. Students could also identify other students with whom they had sentimental relations, which are not reported among friendships. The attribution of race is based on the self-reported classification.

[12]Table 1 provides a view similar to Table 3 in Baerveldt et al. (2004), who analyze inter-racial friendships in a Dutch high school, obtaining a similar structure.
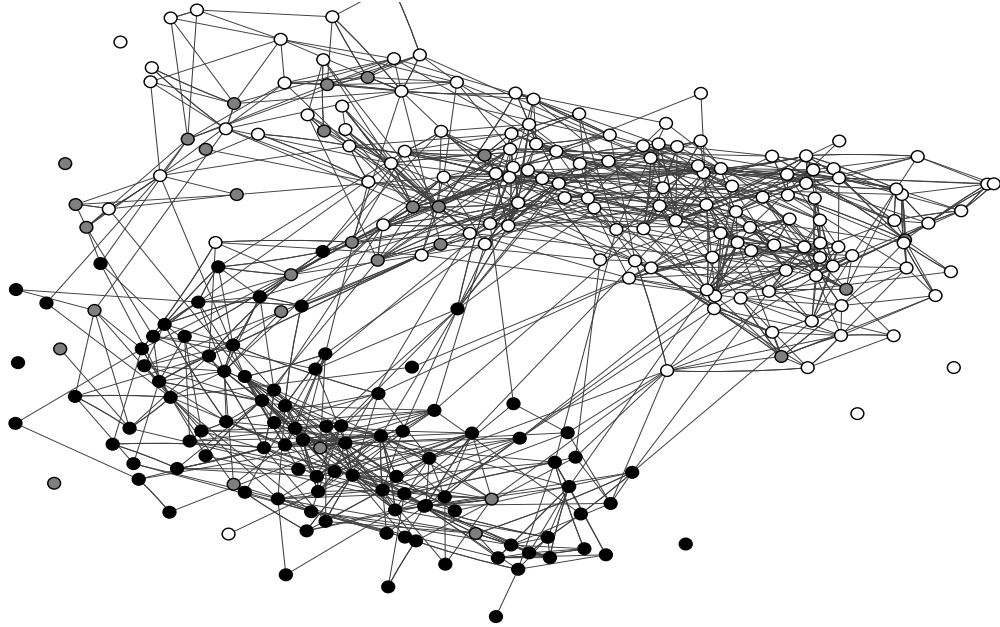
Figure 1: Friendships network in a U.S. School. Colors identify races: white=Whites; grey=Blacks; light–grey=Hispanics; black=Others..

ties (high levels of inbreeding homophily), and the different behavior of Hispanic students (less numerous), who integrate more with other races and fail to form an independent community.[13]

We now turn to examining the three observations mentioned in the introduction.[14]

3.1 Relative Homophily

In each of the 84 high schools there are 4 different racial categories, leading to 336 potential observations. There are 31 cases where there are no students of a given racial category, and so we have 305 total observations. Each observation is then a race within a particular school.

Figure 2 shows that relative homophily is pervasive in the Add Health schools. The horizontal axis lists the fraction w_i of a group i in its school and the vertical axis lists the (non-normalized) homophily index H_i. The 45 degree line provides the baseline homophily

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[13]The figure was drawn with the Pajek program, which uses an algorithm to cluster nodes together that are more densely connected.

[14]See Currarini, Jackson and Pin (2008) for more detailed analysis of these data.
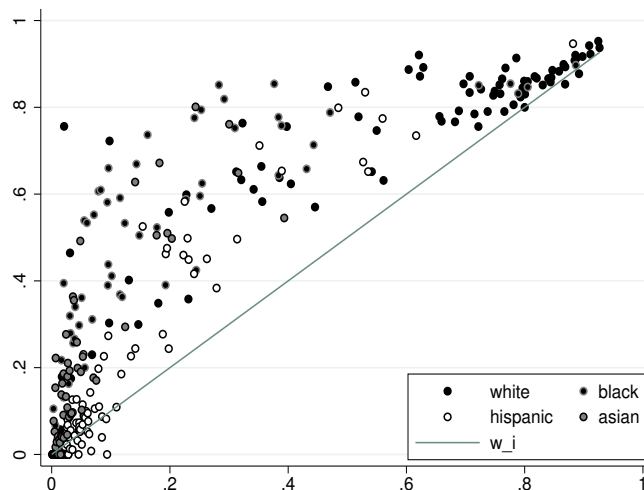
benchmark.

Figure 2: The homophily index.

Beyond the clear positive tendency of relative homophily,[15] we also note that the baseline homophily line acts as a lower envelope of the data, which is evidence of pervasive inbreeding homophily, as analyzed below.[16]

3.2   Group Size and the Total Number of Friends

Next, we examine how relative group size relates to the average number of friendships formed by group members.

In Figure 3, there is a clear positive relationship between the size of racial groups and the total number of per capita friends that they form.[17]

This aspect of friendship formation does not appear in Blau's (1977) analysis of opportunity driven homophily, nor in much of the literature that follows. Marsden (1987) does remark on such a pattern in his empirical study of the "discussion network" based on a

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[15]The slope of a regression line through the data is .99 with a t-statistic of 31. We fit a nonlinear curve below.

[16]Figure 2 is very similar to Figure 6 in Echenique and Fryer (2007) who focus on defining segregation measures and who develop a measure based on the spectral decomposition of the friendship matrix and test it on the same data. Thus, the same pattern exists for other measures as well.

[17]The coefficient of a regressed line is 1.9, with a t-statistic of 5.0, and the intercept is 5.9 with a t-statistic of 40.
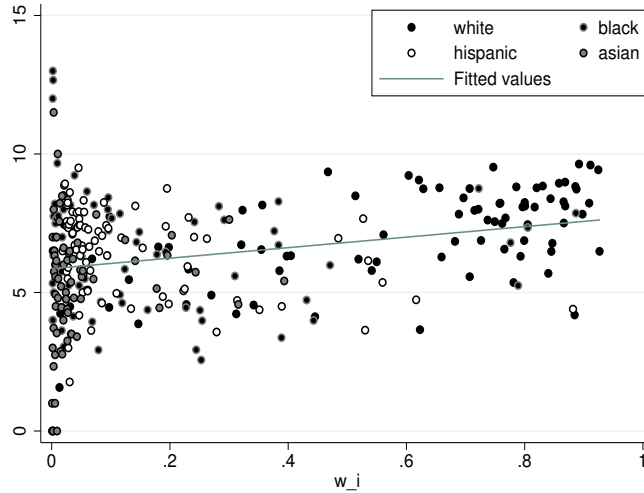
Figure 3: Larger groups make more friends.

national U.S. survey, but does not discuss a connection between such a pattern and other aspects of friendship formation.

3.3 Inbreeding Homophily

As mentioned above, another feature of the data is that there is inbreeding homophily for most groups. Moreover, there is a significant and distinctive pattern to IH_i as a function of relative group size. Figure 4 illustrates inbreeding homophily as a function of group size (and also as a function of race).[18]

Figure 4 indicates two clear patterns. First, there is inbreeding homophily for most groups, with some exceptions for the smallest groups. Second, there is a nonlinear and non monotone relationship, as inbreeding homophily is near 0 at the extremes, and significantly positive (reaching a maximum of almost .8) near the middle.

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[18]In a regression of IH_i versus w_i and w_i^2, we find a coefficient of 2.1 on w_i and a coefficient of -2.2 on w_i^2, with t-statistics of 16 and -15, respectively. The intercept term is .04 and insignificant. Running a regression of the inbreeding homophily index on the variable $w_i(1 - w_i)$ and forcing a zero intercept, we obtain a coefficient of 2.1 (standard error 0.07), with a t-value of 29.
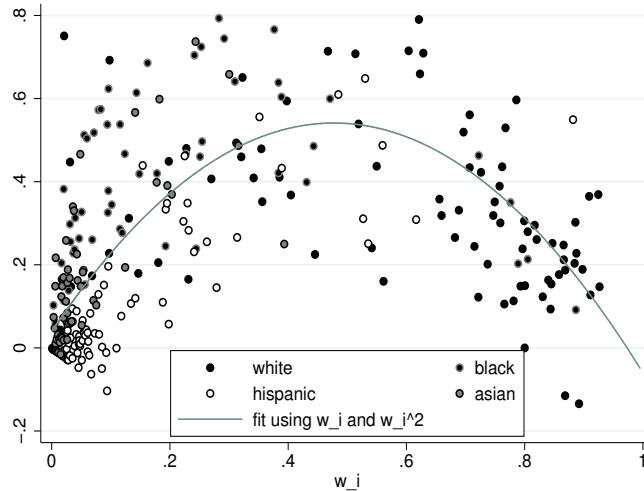
Figure 4: Patterns of Inbreeding Homophily by Relative Group Size and Race.

4 A Nested Set of Search-Based Models of Friendship Formation

We now examine a search/matching-based model of friendship formation.[19]

4.1 Agents, Types and Payoffs

Agents come in a finite variety of types $T = \{1, \ldots, K\}$. These might correspond to ethnicities, religious affiliations, professions, age, etc., or some combinations of traits.

The utility of an agent depends on the numbers of his or her friends[20] who are of his or her same type and who are of different types. For the purposes of this model, the agent only distinguishes between "same" and "different", and does not distinguish in any finer way among differences.[21]

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[19]Such search/matching models have been used in a variety of contexts (e.g., labor markets as in Mortensen and Pissarides (1994)). The model introduced here is distinct in that agents form many friendships.

[20]The word "friend" is used throughout, but this might also correspond to some other sort of relationship, depending on the application.

[21]This is consistent with empirical evidence. For example Marsden (1988) does not find any significant distinction between friendships and races after accounting for homophily. As McPherson, Smith-Lovin and Cook (2001, footnote 7) point out, "The key distinction appears to be same–different, not any more elaborated form of stratification."

The total utility to an agent i who has s_i same-type friends and d_i different-type friends is $U(s_i, d_i)$.

The function U is not indexed by the type of the agent. Thus agents' base preferences are the same across types, but their resulting outcomes may differ depending on the composition of the society that they face.

For convenience, we allow agents to form fractional friendships. This allows us to treat U as strictly increasing, continuous, with continuous first- and second-order partial derivatives.[22] Let $U_s(s_i, d_i)$ denote the partial derivative of U with respect to s evaluated at (s_i, d_i), and use similarly standard notation for other partial derivatives. U is strictly concave in s_i and is strictly concave in d_i.

We maintain an assumption that the utility function U exhibits *overall diminishing returns to friendships*. That is,

$$U(as, ad) < aU(s, d),$$

for all (s, d) and $a > 1$. Diminishing returns is a standard decreasing returns to scale condition, here applied to preferences for friendship. Geometrically, it requires that the marginal utility of friendships decreases along any ray going out of the origin in the space (s, d).[23] This condition is equivalent to the following condition on the total derivative

$$sU_s(as, ad) + dU_d(as, ad) < sU_s(s, d) + dU_d(s, d) \tag{1}$$

for all (s, d) and $a > 1$.

Diminishing returns to friendships is satisfied if s and d are substitutes in agents' preferences, given the strict concavity of U in each of s and d. It is also satisfied when s and d are complements, as, for instance, in Cobb-Douglas utility functions in which the sum of powers is less than one.

Finally, to ensure that agents will not form an infinite number of friendships, we assume that marginal utilities vanish as s and d go to infinity. More precisely, for all $\varepsilon > 0$ and s', there exist d' such that $U_s(s, d) < \varepsilon$ and $U_d(s, d) < \varepsilon$ for all $s > s'$ and $d > d'$ (and similarly for all $\varepsilon > 0$ and d').

Before moving on, let us mention a few illustrative examples that could provide a basis for various formulations of the utility functions $U(\cdot, \cdot)$.

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[22]The case in which agents have satiated preferences is also of interest in various applications and is treated in the Supplementary Material to this paper.

[23]This property is weaker than overall concavity of U. It implies concavity of U if U is quasiconcave and homothetic (see Friedman, 1973).

- Information Networks: Individuals receive information from their friends. For instance, the information they learn from a same-type individual could be more correlated with their own information than that of different types. However, a same-type individual could be easier to communicate with.[24]

- Professional Networks: Here, same-type individuals are easier to communicate with, but offer less creative synergy than different-type individuals.[25]

- Purely Social Networks: One shares more interests with same-type individuals than with different type individuals.[26]

- Risk Sharing Networks: A same-type individual's income is more highly correlated with own income, which offers less potential for risk-sharing than different-type individuals. However, same types are closer geographically or socially than different types, and thus easier to arrange a viable (and self-enforcing) risk-sharing agreement with.[27]

4.2 A Matching Process

We examine a continuous-time matching process with a new inflow of agents of mass N_i of type i per unit of time.

For each unit of time in the matching process, regardless of the outcome, an agent incurs an opportunity cost $c > 0$.[28] Thus the cost of an agent of type i who stays in the matching process for t_i units of time is $t_i c$.

If agents of type i stay for t_i units of time in the matching process, then the stock of type i's involved in the matching process in a steady-state will be $M_i = t_i N_i$. Thus the relative stock of a given type of agent increases by either increasing their inflow or by increasing how long they stay in the matching process.

Agents accumulate friendships over time in the matching process. Given that their utility is increasing in friends of each type, it is a dominant strategy for them to accept friendships with any agents whom they meet. Thus, their main decision is simply when to exit the matching process.

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[24]For instance, this could concern information such as job market information, as in the literature surveyed by Ioannides and Datcher Loury (2004).

[25]For a discussion see Page (2007).

[26]For an example of social or status based preferences, see Watts (2007).

[27]See Bramoullé and Kranton (2007) for more discussion of such trade-offs in a risk-sharing environment.

[28]Adding discounting complicates the expressions without adding any insight.

The mix of friendships that agents form is determined by the stocks of various types in the matching process. For any given agent of type i who searches for a new friend, the agent meets another agent of type i with probability q_i and an agent of different type with a probability $1 - q_i$. These probabilities are endogenous to equilibrium, determined by the steady-state stocks of agents in the matching process.

Generally, the matching process can be described by an $n \times n$ matrix $\mathbf{q} \in [0,1]^{n \times n}$, where q_{ij} is the fraction of i's meetings per unit of time that are with type j and the matrix is row stochastic so that $\sum_j q_{ij} = 1$. These relative meeting probabilities depend on the stocks of agents in the society and so a matching process is described by a function $F : \mathbb{R}^n_+ \to [0,1]^{n \times n}$, where $\mathbf{q} = F(M_1, \ldots, M_n)$ is the matching that occurs as a function of the relative stocks of agents in the society.

To be well-defined, the matching process needs to balance, so that the number of meetings where an i meets a j is the same as those where a j meets an i. A matching process F is *balanced* at a given M if $\mathbf{q} = F(M_1, \ldots, M_n)$ is such that $M_i q_{ij} = q_{ji} M_j$ for all i and j.

The canonical matching process is one where agents meet each other in proportion to their relative stocks. We call that the *unbiased matching process*, and it is such that $q_{ij} = \frac{M_j}{M}$, where $M = \sum_k M_k$.

Given that agent's preferences only depend on own an other types, we let $q_i = q_{ii}$ and then $1 - q_i = \sum_{j \neq i} q_{ij}$. Thus, if an agent of type i stays in the matching for a time t_i, then he or she forms a total of t_i friendships, with proportions $q_i t_i$ of them being of same type and $(1 - q_i) t_i$ being of different type.

We remark that in this model the homophily and segregation of types that results, never involves an individual being rejected as a friend by another individual.[29] Instead, it is governed by the relative sizes of populations, agents' preferences and the implications for their matching decisions, and feedbacks in the matching process which govern the mix of friendships that individuals are faced with forming.

Two simplifications that we have made in the model are the continuous time matching and the fact that the mix of types that agents meet is deterministic. These are both made for convenience. In Appendix A we provide a derivation of a random matching process with a continuum of agents, and show how taking limits as time between matchings goes to 0 can justify our formulation, at least for the case of an unbiased matching process.

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[29]Such rejection can result with satiated preferences, as detailed in the Supplementary Material to this paper.

4.3 An Agent's Decision Problem

Given the matching probabilities, q_i and $1 - q_i$, an agent of type i solves the following problem:

$$\max_{t_i} U(q_i t_i, (1 - q_i)t_i) - ct_i. \tag{2}$$

The following lemma states a straightforward but useful necessary condition for an optimum.

LEMMA **1** *An optimal choice of the time spent in the matching t_i for an agent of type i given matching probabilities q_i and $1 - q_i$ satisfies the following condition:*

$$q_i U_s(q_i t_i, (1 - q_i)t_i) + (1 - q_i)U_d(q_i t_i, (1 - q_i)t_i) = c. \tag{3}$$

In what follows we maintain the assumption that $U_s(0,0) > c$ and $U_d(0,0) > c$, so that agents will at least form some minimal number of friendships regardless of the matching process that they face.

Note that even though agents' strategies boil down to choosing a length of time to spend in the matching process, their choices will depend both on their preferences and on the mix of stocks of agents in the matching process. *Most importantly, the stocks of agents in the matching process will depend on the preferences, and so there can be a feedback effect that determines the equilibrium.* This is critical to understanding the model, since for example, if there is a bias in agents' preferences towards same-type friendships, then a larger group will face a more attractive matching, leading them to stay in the matching process longer, leading to an even more biased matching, and so forth. This endogeneity is captured in the definition of steady-state equilibrium.

4.4 Steady-State Equilibrium

A steady-state equilibrium accounts for the relation between players' strategies and the resulting matching process.

A steady-state equilibrium of the system for a given set of inflows N_i's and a utility function U is a specification of strategies for each type such that the resulting stocks of agents at each point in their search process lead to the relative meeting probabilities that justify the strategies, and such that the outflow of agents is the same as the inflow. More specifically, a *steady-state equilibrium* is a specification of a triple (t_i, M_i, q_i) for each i, representing, respectively, the time spent matching for the type, the stock of the type in

the matching, and relative probability of meeting the type in the matching, that satisfy the following conditions. There is at least one i such that $t_i > 0$, and:

(i) agents optimize given meeting probabilities: t_i solves (2) given q_i for each i,

(ii) strategies determine stocks: $M_i = N_i t_i$ for each i,

(iii) stocks determine meeting probabilities: $\mathbf{q} = F(M_1, \ldots, M_n)$, and

(iv) the matching process is balanced so that $q_{ij} M_i = q_{ji} M_j$ for each i and j.

The restriction that $t_i > 0$ for at least one i means that a "steady-state equilibrium" is a nontrivial equilibrium. The trivial situation where no agents enter the matching process expecting that no other agents will enter could potentially be thought of as an equilibrium if one allows for an empty matching process, but it would be unstable since all agents strictly prefer to enter the matching if there are any other agents entering the matching. Steady-state equilibrium under the above definition precludes such degenerate situations directly.

In the case of an unbiased matching process, where $q_{ij} = M_j/M$, it is clear that (iv) is satisfied, and so for such processes, one needs only then verify that (i) and (ii) are satisfied.

We do not include an explicit requirement that the inflows match the outflows as this is implied by (ii). Since the flow of agents of type i who are exiting at any given time is M_i/t_i, the exiting amount will be N_i under (ii).

Note also that in equilibrium q_i coincides with the homophily index H_i defined in Section 2, since $s_i = t_i q_i$ and $s_i + d_i = t_i q_i + t_i(1 - q_i) = t_i$, and so $H_i = s_i/(s_i + d_i) = t_i q_i/t_i = q_i$.

The following example illustrates the definition and shows the feedback effects of strategies affecting stocks and stocks affecting strategies.

4.5 An Example of a Steady-State Equilibrium

Consider preferences represented by

$$U(s, d) = (s + \gamma d)^\alpha,$$

where $0 < \alpha < 1$ captures diminishing returns, and the parameter $\gamma \geq 0$ determines the relative bias in preferences between same and different types. If $\gamma < 1$ then an agent sees higher marginal benefits from same-type friendships than different-type friendships, and if $\gamma > 1$ then the reverse is true.

It is then easy to verify that an agent's optimal time in the matching is

$$t_i = \left(\frac{\alpha}{c}\right)^{\frac{1}{1-\alpha}} \left(\gamma + (1-\gamma)q_i\right)^{\frac{\alpha}{1-\alpha}}.$$

For the case of $\alpha = \frac{1}{2}$ and an unbiased matching process we compute the steady-state equilibrium. Condition (iii) of steady state equilibrium implies that

$$\frac{q_i}{q_j} = \frac{w_i t_i}{w_j t_j} \quad.$$

Substituting,

$$\frac{q_i\left((1-\gamma)(1-q_i)+\gamma\right)}{(1-q_i)\left((1-\gamma)q_i+\gamma\right)} = \frac{w_i}{1-w_i} \quad.$$

Given that $w_i \in [0,1]$, there is a unique solution for $q_i \in [0,1]$, which is[30]

$$q_i = \frac{(1-2\gamma)\frac{w_i}{1-w_i} - 1 + \sqrt{\left[\frac{w_i}{1-w_i} - (1-2\gamma^2)\right]^2 + 4\gamma^2\left(1-\gamma^2\right)}}{2\left(\frac{w_i}{1-w_i} - 1\right)(1-\gamma)} \quad. \tag{4}$$

Figure 5 shows how q_i depends on w_i, when $\gamma < 1$ and $\gamma > 1$.
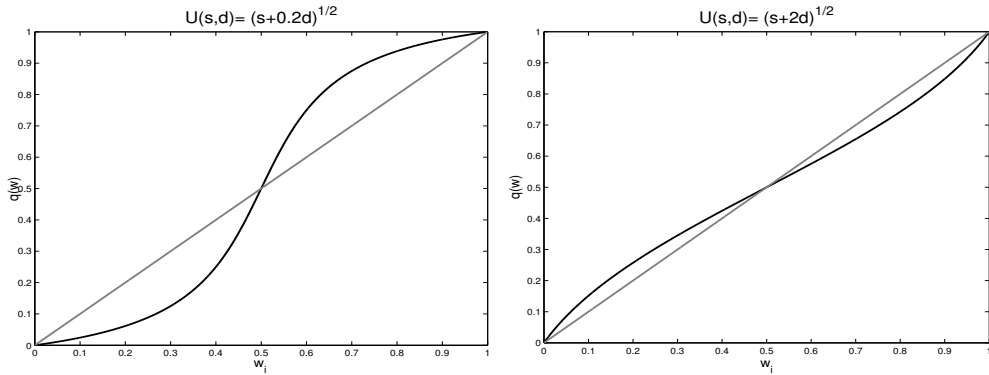
Figure 5: The behavior of q_i as a function of w_i when $U(s_i, d_i) = \sqrt{s_i + \gamma d_i}$. On the left hand side $\gamma = \frac{1}{5}$, an on the right hand side $\gamma = 2$. The straight line is the identity.

To see a full solution, we can specialize to let $\gamma = 1/2$, $w_1 = 2/3$ and $w_2 = 1/3$. Then

$$t_i = \frac{1}{8c^2}\left(1 + q_i\right).$$

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[30]Equation (4) is defined, at the limit, for $w_i = 1/2$ and $w_i = 1$, and the result are respectively $q_i = 1/2$ and $q = 1$, as expected.

It follows that $q_1 = \frac{2t_1}{2t_1+t_2}$ and $q_2 = \frac{t_2}{2t_1+t_2}$. Solving simultaneously for t_1 and t_2, leads to the unique solution of

$$t_1 = \frac{\sqrt{3}}{8c^2} \quad \text{and} \quad t_2 = \frac{3-\sqrt{3}}{8c^2}$$

and

$$q_1 = \sqrt{3} - 1 \quad \text{and} \quad q_2 = 2 - \sqrt{3}.$$

Here we see that $q_1 > w_1$ and $q_2 < w_2$ and so the relative stocks are biased towards the larger group and the stocks differ from the flows. This reflects the feedback effect: the larger group finds the matching more attractive and spends more time matching, and illustrates the feedback between strategies and the determination of the stocks in the matching.

5   Some Benchmark Results

We begin with some benchmark results.

First, we show that if preferences are type-neutral, then all equilibria involve all types forming the same total number of friendships. That is, if preferences are not sensitive to the ratio of same type friendships to different types, then all types form the same number of friendships.

We say that U is *type neutral* if $U(s,d) = U(s',d')$ whenever $s + d = s' + d'$. When preferences are not type-neutral we say that they are *type-sensitive*.

PROPOSITION 1 *Let U be type neutral. Then for any matching process and any steady-state equilibrium, all types of agents form the same total number of friendships.*

Proofs of this and other propositions are in the Appendix.

Note that an implication of Proposition 1 is that if the matching process is unbiased then there will be baseline homophily, since all types spend the same time in the matching process and then form friendships in proportion to their relative inflows.

Next we show that without some bias in the matching process, it is impossible to have all groups be homophilous at once. Simply by balance, if one group is homophilous, then another group must be heterophilous.

PROPOSITION 2 *If the matching process is unbiased, then for any preferences and any steady state equilibrium, if some type is homophilous so that $IH_i > 0$ for some i, then there exists at least one other type j that is heterophilous so that $IH_j < 0$.*

These two results show that in the context of our model, it is impossible to generate differences in the number of friendships formed without some type-sensitivity to preferences, and it is impossible to generate homophily for all groups without some bias in the matching. Thus, matching the observed patterns in the data requires type-sensitivity in preferences and a bias in the matching process. In Section 7.3 we discuss how these results depend on the constraints imposed by our model, and how alternative models are also consistent with the same data.

6 Unbiased Matching with Two Types

Given Propositions 1 and 2, it follows that type-sensitive preferences and biased matching are needed in our model to generate equilibria consistent with the data. It is informative to introduce these one at a time, since each one can explain specific patterns of the data independently of the other. We start by examining type-sensitive of preferences in the context of an unbiased matching.

In this section we focus on the case in which there are only two types. The case of two types provides much of the basic intuitions and insights, making the analysis more transparent. There are some complications when we move to more types that we explore in Section 8.

6.1 Relative Homophily and Equilibrium Conditions

We begin by showing that a steady-state equilibrium exists, and that equilibrium conditions alone imply relative homophily.

PROPOSITION **3 (Relative Homophily)** *There exists a steady-state equilibrium, and in all steady-state equilibria both types are active. Moreover, if $N_i > N_j$ then there exists an equilibrium where $q_i > 1/2$ and relative homophily is satisfied (regardless of the type-sensitivity of preferences).*

Thus, relative homophily is implied directly by equilibrium conditions, independently of how agents' preferences depend on the mix of friends; at least in equilibria where the relative stocks are ordered as the relative inflows (which always exist and in a general class of settings are unique, as discussed below). Simple accounting for friendships in equilibrium implies that cross-group friendships must add up, and so $N_i d_i = N_j d_j$ and so if $N_i > N_j$ then $d_i < d_j$. This does not directly imply that relative homophily is satisfied, as it is conceivable

that s_i is proportionally even smaller than s_j, but this is ruled out by diminishing returns to scale, as then the group j would be facing lower overall returns to forming friendships (as they would be forming more of each type), but are facing the same costs.

This idea that relative group sizes are really at the heart of relative homophily, without any biases in meetings or preferences (beyond relative population proportions), is relatively straightforward and has been noted before. For example, Blau (1977) pointed out how relative group sizes affect the opportunities of members of various groups to form ties. In a case of two groups, Blau observes that, due to the reciprocal nature of ties, there will be more cross-type friendships per capita for the smaller group than on the larger group. One does need some added structure (here diminishing returns to scale) in order to tie down same-type friendships and show that this implies relative homophily, but the basic accounting of cross-type friendships is a main driver.

While Proposition 3 states existence of equilibrium and that relative homophily is exhibited by all equilibria, it leaves open the possibility that there can be multiple equilibria.[31] While there are examples of multiple equilibria for some special preference specifications, for many reasonable classes of preferences, equilibrium is unique. In particular, if preferences are such that $t'(q) < \frac{t(q)}{1-q}$ when $q < 1$, then there is a unique steady-state equilibrium.[32]

CLAIM 1 *If preferences are such that $t(q)(1 - q)$ is decreasing in q, then there is a unique steady-state equilibrium.*

Claim 1 is shown in the Appendix.

Note that $t(q)(1 - q)$ is decreasing in q for all preferences in the class in Section 4.5, provided that $\alpha < \gamma$. The existence of multiple equilibria can occur when preferences are biased enough towards same-type so that one can start at one equilibrium, increase the stock of a given type, and have that type respond by increasing its friendship formation enough (and the other type decreasing enough) so that we generate the higher stock of that type in the matching. The above condition rules this possibility out.

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[31]Multiple equilibria occur in many search and matching models, given the externalities present. For example, see Diamond (1982) and the literature that followed for more discussion of such issues.

[32]This presumes that t has a derivative, which is true for many preference specifications. More generally, all that is needed is that $t(q)(1 - q)$ is a decreasing function of q, which is equivalent to the stated condition when t is differentiable.

6.2 Same-Type Bias, Numbers of Friends and Homophilous Behavior

Propositions 1 and 2 imply that although equilibrium conditions imply relative homophily, the empirical observations of inbreeding homophily and increasing numbers of friends with group size require additional structure. Moreover, the relative homophily observed in Section 3.1 was such that larger groups formed fewer different-type and more same-type friendships than smaller groups. To derive the full empirical evidence of Sections 3.1-3.3 the model must include some bias in preferences, as we see from Proposition 1. We now turn to the implications of preferences that are biased to have higher returns from mixes that are heavier in same-type.

DEFINITION **7** *Preferences exhibit a **same-type bias** if*

$$\frac{\partial}{\partial t}U\left(qt,(1-q)t\right) > \frac{\partial}{\partial t}U\left((1-q)t,qt\right) \tag{5}$$

for $q > 1/2$.

This states that the return to additional friendships (in proportion to the current mix) is larger when the mixture is biased towards same type, which is reminiscent of conditions on technological change in production functions. Note that this condition does not require that agents always prefer to substitute away from different types and towards same types. It simply requires that faced with a given mix or its reciprocal, agents get a higher marginal return from the one with more same-types. With such a same-type bias in preferences, we can then derive a result linking larger groups (as a percentage of population) to greater numbers of friendships.

PROPOSITION **4** *(**Increasing Numbers of Friends and Friendship Mixes with Biased Preferences**) Let preferences exhibit same-type bias and there be two groups i and j, such that $N_i > N_j$. Then there exists an equilibrium such that the larger group forms more total friendships per capita than the smaller group.*[33] *Moreover, in equilibrium the larger group forms more same type friendships and fewer different type friendships per capita than the smaller group; and the larger group exhibits inbreeding homophily while the smaller group exhibits inbreeding heterophily.*

The proof is based on the fact that, under same-type bias, facing more same types in the matching (a higher q_i) implies higher marginal returns from friendships. This implies

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[33]Again, when preferences are such that $t(q)(1-q)$ is decreasing, then this is unique.

that the larger group will spend more time in the matching process and form more total friendships per capita. This biases the stock of the larger group in the matching process to be larger than their relative flow, so that $q_i > w_i$ for the larger group. This leads to inbreeding homophily for the larger group and heterophily for the smaller group.

Note that while Proposition 4 shows that the larger group in an equilibrium will form a larger number of friendships, it does not state that if we look across equilibria then the friendships will be ordered by group size. For example, the Proposition implies that in a population that is 70 percent type 1's and 30 percent type 2's the type 1's will form more friendships in equilibrium than the type 2's. It does not state how those numbers of friendships will compare with another society that is 80 percent types 1's and 20 percent type 2's. The data have those extra comparisons, where we see not only that a 70 percent group forms more friendships on average than a 30 percent group, but also that this trend is generally increasing by group size, even when comparing across different populations (that is, different schools). To draw this extra conclusion, we need additional requirements, which amount to the condition that ensured uniqueness of equilibrium plus a strengthening of the same type bias condition. These conditions ensure a monotone relationship between the equilibrium q_i and w_i, and a monotone relationship between t_i and q_i, which coupled together ensure comparability of the total number of friendships across equilibria.

DEFINITION 8 *Preferences exhibit a **strong same-type bias** if $\frac{\partial}{\partial t} U(qt, (1-q)t)$ is increasing in q.*

This strong same-type bias condition states that the marginal returns from scaling up the number of total friends that an individual has, is increasing as the mix becomes more biased towards own type. This condition ensures that $t(q)$ is an increasing function of q.

PROPOSITION 5 *If preferences exhibit a strong same-type bias and are such that $t(q)(1 - q)$ is decreasing in q, then the unique steady-state equilibrium number of friendships formed per capita by a group i, t_i, is increasing in the group's relative size, w_i.*

Both of these conditions are satisfied by the preferences that appeared in Section 4.5, provided that $\gamma < 1$ to ensure strong same-type bias and $\alpha < \gamma$ to ensure that $t(q)(1-q)$ is decreasing in q.

7 Inbreeding Homophily and the Matching Process

Proposition 4 shows that the model with preference bias leads to the observed relation between group size and total number of friends, and exhibits inbreeding homophily for the

larger group. However, it also predicts that the smaller group *necessarily* should be heterophilous, which is not consistent with the empirical evidence discussed in Section 3, where we saw inbreeding homophily for most groups, and moreover largest inbreeding homophily for middle-sized groups.[34]

This was shown more generally in Proposition 2, but to get a fuller understanding of this, consider the inbreeding homophily index

$$IH_i = \frac{q_i - w_i}{1 - w_i} = \frac{\frac{s_i}{t_i} - w_i}{1 - w_i}.$$

Since the relative stocks must add up in equilibrium with an unbiased matching process (from (iii) and (iv) in the definition of equilibrium), so that $q_1 = 1 - q_2$ (when there are two groups), it follows that

$$IH_2 = \frac{1 - q_1 - (1 - w_1)}{1 - (1 - w_1)} = \frac{w_1 - q_1}{w_1}. \tag{6}$$

Thus, $IH_2 < 0$ if and only if $IH_1 > 0$. In order to have one group have inbreeding homophily, the other group must be heterophilous.

Proposition 2 and the above discussion make clear that some form of bias in the matching process is needed to generate the full spectrum of observables. In the next section we consider a specific form of such a matching bias (allowing all types to meet their own types at a frequency above their proportion in the matching process) that can generate the observed inbreeding homophily patterns.

7.1 A Biased Matching Process

We examine a bias in the matching process which works in favor of same-type matches. Individuals who have preferences with a same-type bias would tend to gain by biasing their search so that it yields higher ratio of same type compared to their stock in the matching process.[35] This could be done in various ways, including meeting friends through friends, as well as joining clubs or taking part in activities that are biased towards own type.

Let us emphasize that the unbiased matching process considered in the previous sections could still exhibit a bias in steady-state equilibrium away from relative population frequencies, in that the q_i's could differ significantly from the w_i's. What is different about a bias in the matching process is that the interplay between the matchings of different types is no

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[34]When there are only two types, and they are such that $N_i = N_j$, then our model exhibits, by symmetry, $q_i = q_j = 1/2 = w_i = w_j$ and so baseline homophily for both groups.

[35]Such biases in matching processes have appeared in the search literature under the title "directed search", as in Acemoglu and Shimer (1999).

longer a simple linear relationship. Now one can have situations where all types are meeting same types at a rate faster than their relative stocks in the matching process. So, there can be an inherent same-type bias in the matching process beyond the feedback and equilibrium effects.

To get an impression of how this works, consider a tractable but very flexible example of a matching process F which is specified for the case of two types and implicitly defined so that[36]

$$q_1^{\beta} + q_2^{\beta} = 1. \tag{7}$$

When $\beta = 1$, this is the unbiased (uniformly random) meeting technology that we examined above.[37] However, when $\beta > 1$, this leads to higher q_i than under uniform meeting, and actually leads to the highest impact for the smallest groups. Note that these groups have the most incentive to meet their own types, as they are not meeting them naturally.

Note that even though Figure 4 shows the greatest homophily for middle-sized groups, that does not mean that the bias in meeting has to be greatest for them. In fact, what we saw from the model without any meeting bias was that small groups would end up with negative homophily measures. Thus, in order to match the observed data small groups need to have the largest bias in meetings as the equilibrium balance conditions without any bias would force them to have quite negative homophily measures. This is in line with the β exponent in (7).

With such a matching process, existence can be established via a similar argument to that of the proof of Proposition 3, as can a version of Proposition 4, as we show in the appendix (see Proposition 9). In terms of Proposition 2, if we examine preferences that are type independent, then it is true that each type will form the same number of friendships. However, baseline homophily is no longer implied. For example, even if $w_1 = w_2 = 1/2$ it is possible that $q_i > w_i$ for each i as the meeting process can now be biased to same type, and inbreeding homophily can be satisfied for all types.

To get a feel for how the extended model works, we fit a special case of it to the Add Health data.

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[36]Note that in the case of two types, $q_{ij} = 1 - q_{ii}$ and so (iv) becomes $M_i(1 - q_i) = M_j(1 - q_j)$. F is then defined so q_1 and q_2 simultaneously satisfy the two equations (7) and $M_i(1 - q_i) = M_j(1 - q_j)$.

[37]The formal derivation appearing in Appendix A, which justifies our model as a limit of well defined meeting processes, is based on a completely uniform random mixing process as in Alós-Ferrer (1999). Extending that justification for the processes here requires extending the Alós-Ferrer (1999) results to a more general set of processes where meeting probabilities are biased. This appears to work, but working that out is a project in itself that would take us well beyond the scope of this paper. For the purposes of this section, it is enough to verify that it works for the examples in question, which one can do directly.

7.2 Fitting the Model to the Data

Reconsider the utility function from the example in Section 4.5:

$$U(s, d) = (s + \gamma d)^\alpha = (tq + \gamma t(1 - q))^\alpha.$$

Let the matching process be governed by (7).

We fit the data by numerically estimating the equilibrium for each case on a grid of values for α, β and γ. The grid was such that α, γ and $1/\beta$ were on a grid between 0 and 1 with increments of .1, and c was on a grid between 0 and .1 with increments of .01. We then examine the generated curves from each profile of parameters in the grid and compare them to the best-fit curves in the data. We choose the parameter values that lead to the minimum distance between the equilibrium curves and best-fit curves.[38] We find best-fit parameters of $\alpha = .2, \beta = 2, \gamma = .8$ and $c = .04$.

The fitted curves are pictured in Figure 6. The picture shows the behavior of the inbreeding homophily index $IH_i = \frac{q(w_i) - w_i}{1 - w_i}$, whose shape is not far from the scatter.
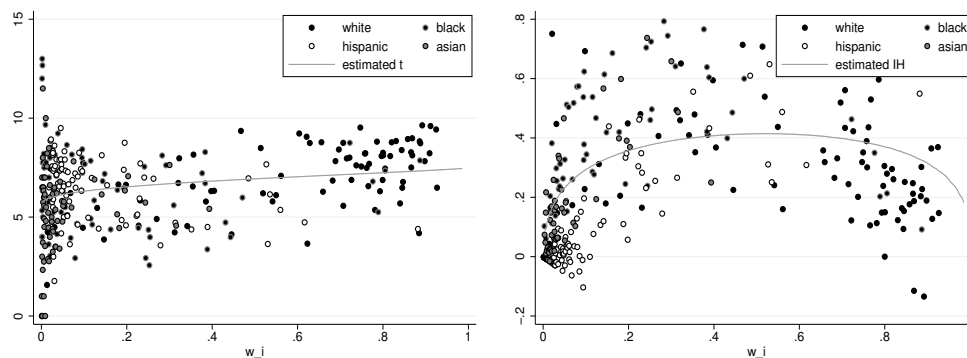

Figure 6: Fitted Model and Data. Left: Inbreeding Homophily by Relative Group Size, Right: Number of Friends by Relative Group Size. Parameter values: $\alpha = .2$, $\beta = 2$ and $\gamma = .8$, $c = .04$.

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[38]Distance was computed numerically in Euclidean distance (square root of squared distances), using the best fit curves from footnotes 17 and 18. The distance between curves was measured as the average distance on a grid of 200 values of w and was summed across the two curves.

7.3 On the Role of Preferences and Opportunities

Let us discuss the implications of our findings regarding the roles of preferences and opportunities in generating homophily in social networks.[39] As we pointed out in Propositions 1 and 2, a bias in the matching process (in favor of one's own type) is needed to generate inbreeding homophily for all groups, and having type-sensitive preferences (such that agents care about more than just total number of friends) is necessary, in the context of our model, to derive a relationship between group size and numbers of friends.

Figure 7 illustrates these properties relative to the fitting exercise from Section 7.2, where $\gamma = 1$ means that preferences are type-neutral, while $\beta = 1$ is the case of unbiased matching. We see that a bias in meetings generates the observed inbreeding homophily patterns regardless of the type-sensitivity of preferences, but does not generate the observed friendship relationship without type sensitivity in preferences, while type sensitivity of preferences generates the observed friendship relationship regardless of the bias in matching and does not generate the observed inbreeding homophily without a bias in matching.
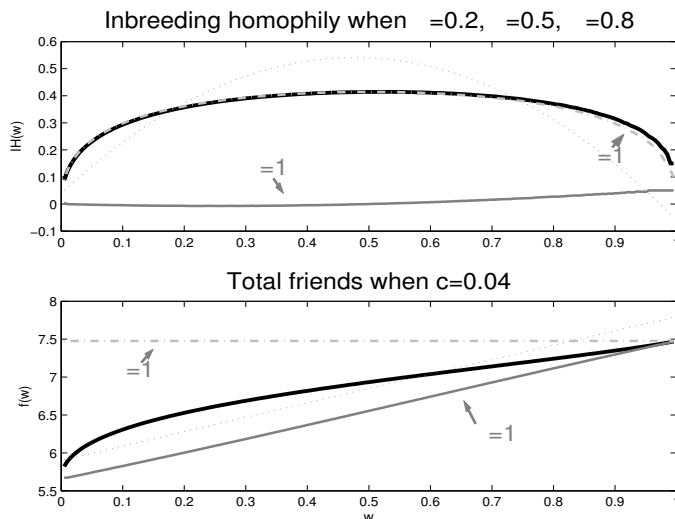

Figure 7: All cases: $\alpha = .2$ and $c = .04$. Top: Inbreeding Homophily. Bottom: Numbers of Friendships. Biased preferences only (grey solid): $\beta = 1$ and $\gamma = .8$. Biased matchings only (grey dashed): $\beta = 2$ and $\gamma = 1$. Full bias (black): $\beta = 2$ and $\gamma = .8$.

While our model's results point to a role of both preferences and the matching process in generating the observed friendship patterns in American high schools, there are alternative

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[39]For more background discussion on sources of homophily, see the survey by McPherson, Smith-Lovin and Cook (2001).

ways of explaining the same evidence that hinge only on matching opportunities. "Increasing returns" in the matching process, so that larger groups face lower costs/higher returns from search (e.g., similar to search models initiated by Diamond (1982) and used extensively in the literature on job market search) could lead to similar predictions as the type sensitivity of preferences. The concept of increasing returns to matching is different from the bias in matching that we have examined, which leads to a change in the mix of types met: increasing returns leads to faster overall rates of meeting for larger groups, while our bias affects only the relative mix of types that an agent meets, not the rate at which an agent is matched. To see how increasing returns to matching could generate differences in relative number of friendships formed even with type-neutral preferences, let τ_i denote the expected number of friends met by type i per unit of search. The necessary condition for optimality, (i) , in a case of type-neutral preferences becomes $\tau_i U'(t_i) = c$, from which it is clear that different types could then have different numbers of friendships formed per capita (and note also that the same could be obtained by assuming heterogeneous costs).[40] There are instances where such differences in rates of encounters are natural; e.g., in a case with cultural or linguistic barriers in communication. However, many factors which lead to differences in encounter rates (e.g., linguistic or cultural barriers) also tend to enter into the benefits of a friendship, and an explanation for the data built entirely on increasing returns would require the meeting process to be sensitive to type while the agents in the society are type-blind. Nevertheless, understanding the relative roles for increasing returns to matching and type-sensitivity in preferences is an important question for future research. To more fully identify the role the potential role of increasing returns to matching requires having information about the meeting process, which is not in the data considered here.

8   Many types

In this section we discuss the extension of the results from two types to many types. For these points, we return to the model with meetings in proportion to stocks. The same-type bias condition that was used to establish the positive relation between total number of friends and group size and the positive relation between inbreeding homophily and group size is strengthened to the strong same-type bias condition. The same-type bias condition can only compare the preferences for two groups that face matchings where their relative stocks sum

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[40]As suggested by an anonymous referee, a positive relation between group size and rate of encounters can be generated by assuming that friends of the same type are met at higher overall rather than relative rates than friends of different types.

to 1. With more groups, the same-type bias condition does not make comparisons across groups, while the stronger version does. This provides an analog to Proposition 4 for the case of more than two types.

PROPOSITION 6 *Let there be more than two types and U satisfy strong same-type bias. Then there exists a steady-state equilibrium such that $N_i > N_j$ implies $t_i > t_j$ and $q_i > q_j$ (larger groups make more friends). The largest group displays inbreeding homophily, while the smallest group displays inbreeding heterophily. Moreover, if $N_i > N_j$ then $IH_i > IH_j$ (larger groups display larger inbreeding homophily).*

Proposition 6 can be explained as follows. Diminishing overall returns to friendship and strong same-type bias imply that larger groups form more friends in equilibrium. By staying longer in the search process, members of larger groups are found more often than members of smaller groups, which implies a larger index of inbreeding homophily. To fully understand this, it is useful to outline the proof. The fact that in equilibrium a higher q implies a higher total number of friends, $t(q)$, follows the same steps as in the proof of Proposition 4, using strong same-type bias. The steady-state condition (iii) then implies that the term $\frac{tw}{q}$ is constant across types. Therefore, the terms $t(q)$ and $\frac{q}{w}$ move together across types. Interestingly, in equilibrium the effect of group size on total number of friends and on inbreeding homophily balance across groups. The fixed point argument used in the proof shows that there exist a steady-state equilibrium in which w and q are positively related, from which the conclusion that larger groups display larger inbreeding homophily follows.

For the many type case, we do not have a simple sufficient condition that guarantees uniqueness. We do, however, provide sufficient conditions to ensure that all equilibria will exhibit the properties of Proposition 6 (see Appendix C).

The issue of whether or not we can order the equilibrium t_i's across different equilibria cannot be established in the same manner as it was in the case of two types (e.g., Proposition 5). To understand this, consider a group that is 60 percent of the population. If it is in a society with two groups that are each 20 percent of the population, that can be very different than being in a population with one other group that comprises 40 percent of the population. In particular, it might have a larger relative stock when in the population with two smaller groups (who find their matchings less attractive and exit relatively earlier under a strong same-type bias) than when with one larger group. This can then mean that in a society of three groups with relative inflowing weights of $w = (.60, 20, 20)$ could lead to a higher stock and more friends for the largest group than in a society that has only two groups with relative inflows of $w = (.61, .39)$ (which is then similar to a three group society with a very tiny third group).

As for the case of two types, the result of Proposition 6 does not fit the observed patterns of inbreeding homophily: the smallest group still needs to have inbreeding heterophily, and inbreeding homophily is monotone in relative group size. Again, an additional bias in the matching can reverse this.

9   Welfare

The model we have developed can be used to explore welfare issues, since preferences are explicit in the model. This contrasts with much of the literature on this topic has implicitly or explicitly adopted the view that an even mix or maximal diversity is desirable per se (see, for instance, the influential paper by Moody (2001)). This is not to say that such a view cannot be justified, but more simply that such a justification should be based on some foundation and that a model such as ours can begin to provide the foundation for a welfare analysis.

At the same time as emphasizing a need for a foundational model for a welfare analysis, we also emphasize that our model is too simplistic to provide a welfare analysis that we could rely upon. In particular, our model misses many important aspects of the benefits of diversity, especially in a learning environment, as mixtures only enter preferences in direct friendships. We also emphasize that it does not directly account for the behavioral influences, diffusion, and other implications of social structure.

The reason for examining a welfare analysis in the context of the model is rather to derive some simple comparative statics across groups, and then also to point out that the equilibrium effects are such that small changes in preference specifications can lead to large differences in welfare conclusions. This suggests that more exacting analyses are needed to fully understand optimal policies in programs that might affect, for instance, racial mixes in schools or the extent to which there is tracking and other aspects within school that constrain the meeting process.

For the purposes of the welfare analysis, we examine a class of preferences that satisfy the following condition (and, again, this is satisfied by the preferences in Section 4.5).

We say that U is homogeneous of degree $\alpha \in (0,1)$ if $U(ks, kd) = k^\alpha U(s,d)$ for all (s,d).

Note that homogeneity of degree α does not preclude having a same-type bias in preferences. It is simply an assumption about how utility varies as the scale of friends is varied: the term "homogeneity" does not refer to comparisons of types but to a standard mathematical property.

PROPOSITION **7** *If U is homogeneous of degree $\alpha \in (0,1)$, then the steady state equilibrium average utility of group i, $U(s_i, d_i) = U(q_i t_i, (1 - q_i)t_i)$, is proportional to $t_i(q_i)$, the optimal total number of friends of type i. Thus, $\arg\max_q t(q) = \arg\max_q U(qt, (1 - q)t)$.*

Proposition 7 shows that in the case of homogeneous preferences, a given type's welfare is directly proportional to the number of friends it forms per capita. This has an immediate implication that larger groups will have higher equilibrium welfare (under the conditions of our early propositions having strong same-type bias) than will smaller groups.

In order to get some feeling what this implies for total welfare, we explore the case where $\beta = 1$, and then return to discuss changes for $\beta > 1$.

PROPOSITION **8** *If U is homogeneous of degree $\alpha \in (0,1)$ and $\beta = 1$, then, average total welfare $\sum_i w_i U_i$ equates $\frac{w_i}{q_i} U_i$ for all $i = 1..., K$. Therefore, maximizing average welfare corresponds to maximizing $\frac{w_1}{q_1} U_1$ which is equal to $\frac{c}{\alpha} t_1 \cdot \frac{w_1}{q_1}$.*

Propositions 7 and 8 taken together then show that aggregate welfare is proportional to the number of friends t_i formed in equilibrium divided by the term q_i/w_i. Both the optimal total number of friends t_i and q_i/w_i increase with the group size w_i (Proposition 6).[41] The change in aggregate welfare triggered by a change in relative populations is the result of these two effects. If the population of type i has increased its weight, the increase in total friends formed by i-type agents has a positive effect, while the increase of homophily of i-type agents has the effect of inhibiting the total friendships made by the other types, with a negative effect on welfare. The term $t_i \cdot \frac{w_i}{q_i}$ highlights the trade-off between these effects. In a society with two groups, i and j, the effect of increasing the size of group i (locally) depends on the relative magnitudes of the derivatives of t_i and of $\frac{q_i}{w_i}$ with respect to w_i. If the increase in the number of friends outweighs the increase in homophilous behavior, the total welfare increases, and not otherwise.

Another way of seeing the welfare effect of changing relative populations is to study the change in $w_1 U_1 + w_2 U_2$. This change is proportional to $w_1 t(q_1(w_1)) + (1 - w_1)t(1 - q_1(w_1))$, where $q(w_i)$ an equilibrium function of w_i. Taking the derivative with respect to w_1 yields

$$t(q_1(w_1)) - t(1 - q_1(w_1)) + w_1 t'(q_1(w_1))q' - (1 - w_1)t'(1 - q_1(w_1))q' \ .$$

So there is a direct effect of lowering the number of agents at a lower utility for those at the higher one, but then also indirect effects of affecting both of the utility levels.

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[41]Proposition 6 is stated in terms of IH_i rather than q_i/w_i, but the results are proven for both.

When we change to $\beta > 1$, we will still have a similar trade-off, but the specifics of the trade-off can be affected.

The utility function studied in Section 4.5 can be used to illustrate how simple variations in preferences can lead large changes in welfare conclusions. For simplicity, we consider the case where $\beta = 1$ but this also holds for $\beta > 1$. In particular, when preferences are homophilous ($\gamma < 1$) we obtain complete segregation, while heterophilous preferences ($\gamma > 1$) yield an optimal solution with two equally numerous groups, and when preferences are type-neutral ($\gamma = 1$) the mix is irrelevant.

What drives these extreme conclusions is the fact that same-type and different-type friends are substitutes, just perhaps not in a one-to-one ratio. Thus, these preferences lack a specific taste for diversity, and a constant marginal rate of substitution drives us to corner solutions.

If instead, there is some preference for diversity, the welfare conclusions change. Consider, for instance, $U(s, d) = \sqrt{s} + \gamma\sqrt{d}$. (We examine this form since it allows an explicit calculation of equilibrium.) Now an individual has an explicit taste to have friendships of both types. Here the optimal t is:

$$t(q) = \frac{(1 - \gamma^2)q + \gamma^2 + 2\gamma\sqrt{q(1 - q)}}{4c^2}.$$

It can be checked that $t(q)$ has a maximum at $q = \frac{1}{1+\gamma^2}$. If $0 < \gamma < 1$ then $\frac{1}{2} < \frac{1}{1+\gamma^2} < 1$. Figure 8 shows (numerical approximation) results for total welfare for the case of two types. Even when same-type bias is present, but not too strong (for values of γ not below $\frac{1}{2}$) an equal split of the population gives the optimal welfare. For higher homophily (lower γ) the optimal welfare is reached with an unequal distribution, which, however, never reaches full segregation.[42]

In Appendix D we illustrate the problem of organizing more than two groups for the simple case of log–utilities. Although those preferences are quite special, they allow for explicit analytic solutions in the more complex case of many types, and provide some insight on the trade-offs in that case.

10   Concluding remarks

In this paper we have started from some basic observation regarding friendship patterns in a

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[42]We checked numerically that the same qualitative behavior happens for a biased matching as the one discussed in Section 7. What happens is that, as β increases above 1, the γ that fixes a threshold between an optimal split which is equal or unequal also decreases below $\frac{1}{2}$ (e.g., for $\beta = 2$, this threshold is at $\gamma \simeq .379$).
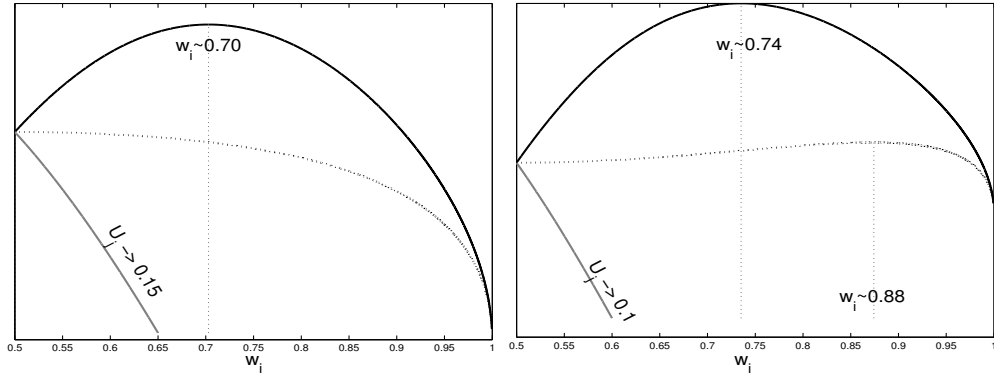
Figure 8: The two plots show the welfare of the two types i (higher curve) and j (lower curve), and the aggregate welfare (middle curve), versus w_i (plotted only from 0.5 to 1, by symmetry), when $U(s,d) = \sqrt{s} + \gamma\sqrt{d}$. The left hand frame is for $\gamma^2 = 0.3$, where the optimal aggregate welfare is for $w_i = 0.5$. The right hand frame is for $\gamma^2 = 0.2$, here the optimal aggregate welfare is for $w_i \simeq 0.88$.

selected sample of American high schools: larger racial groups form more friends per capita, while all groups display inbreeding homophily, with highest levels for middle sized groups. We have then studied a search-based model of friendship formation, trying to identify the role of opportunity and preferences in generating the observed data. We have shown that if all types meet the same number of other agents per unit of time, then generating differences in per capita friendships in our model requires having preferences depend on more than just total number of friends. So, without differences in meeting rates across type, to generate observed data preferences need to be sensitive to types. On the other hand, we find that the observed inbreeding homophily patterns can only be generated (in the context of our model) with some bias in the meeting process in favor of own type. Thus, according to our model's results, both type-sensitive preferences and biased opportunities play a role in friendship formation.

As we discussed in Section 7.3, these conclusions, and in particular the role played by preferences, depend on the assumption that all types meet the same number of friends per unit of time. If increasing returns to matching are introduced, so that members of larger groups meet more people in total per unit of time (rather than just meeting more of their own type per unit of time) than members of smaller groups, then the structure of the matching process alone can lead to patterns of friendship formation similar to those observed here. Even though there may not be obvious explanations for why the matching process should be

based on type when agents ignore type, such potential alternative explanations point to the importance of obtaining more detailed data on the matching process that leads to friendship formation, and developing richer models of friendship formation to help disentangle the relative roles of preferences and matching, important subjects for further research.

Finally, our analysis has also shown the sensitivity of overall welfare to details of the setting. Generally, this suggests that more attention should be paid to modeling the homophily and the patterns of social ties that emerge from variations on matching processes, preferences, and equilibrium conditions, especially given the importance of social structure in many applications. Finally, our analysis suggests that the presence of more than two groups in a society or community opens the way to equilibrium phenomena which do not arise when there are only two groups.

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Appendix A   The Matching Process

In this appendix we provide a foundation for the matching process which justifies our definition of steady-state equilibrium with a continuum of agents. We proceed as follows. First, we show that if agents were to stay an integer number of periods, then there exists a discrete-time random matching process over the continuum so that the process would be in steady-state, and the outgoing agents would have a distribution of realized meetings so that the number of same-types (and hence different-types) that they meet would be governed by a binomial distribution. Next, by taking a limit as the number of periods becomes large, we have a process where the outgoing distribution will converge (in the sense of weak convergence of measures, or convergence in distribution) to a Dirac measure with weight 1 on the expected proportion of meetings with own type, p_i. Finally, we take this limit of large periods by

subdividing the matches so that each period an agent makes ε of a new friend, where $\varepsilon \to 0$, and so many matches are needed to reach t_i total friends.

Consider a discrete-time matching process, with a measure of new types of i entering in each period of N_i and staying an integer T_i number of periods.

This leads to masses of $M_i(T_i) = N_i T_i$ of agents of type i in the matching at any given period and a total mass of $M(T) = \sum_i M_i(T_i)$ of agents in the matching, where $T = (T_1, \ldots, T_n)$. Let $p_i(T) = M_i(T_i)/M(T)$ be the proportion of type i's in the matching.

Agents are labeled from $[0, M(T)]$, with a label a.

The matching-type of an agent a currently in the matching is a vector $\theta_a = (i_a, t_a, p_a)$ which indicates, the type of the agent i_a, how many periods the agent has been in the matching t_a, and the proportion of same-types that the agent has been matched with to date p_a. Given T, there is a finite set of possible θ's, denoted by $\Theta(T)$.[43]

A matching is a Borel measurable bijection $\sigma : M(T) \to M(T)$, with the property that $\sigma^2(a) = a \neq \sigma(a)$.

A random matching scheme is a probability space with a countably additive probability measure on the space of matchings.

Let μ be a Lebesgue measure and let $I_\theta(a)$ be an indicator function which states whether agent a is of type θ.

Let us begin a process with a fraction of $\nu(\theta)$ of type θ's.

LEMMA **2** *[Alós-Ferrer (1999), Proposition 4.4][44] There exists a matching scheme with probability measure P such that*

P1. *$\mu(\sigma(E)) = \mu(E)$ for all Borel E, P-almost surely.*

P2. *For all a, $P(\sigma(a)$ is of type $\theta) = \nu(\theta)$.*

P3. *For all θ and θ',*

$$\int_a I_\theta(a) I_{\theta'}(\sigma(a)) d\mu = \nu(\theta)\nu(\theta'), \ \ P\text{-almost surely.}$$

P4. *The matching is atomless, so that $P(\sigma(a) = b) = 0$ for all a and b.*

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[43]Lemma 4.1 in Alós-Ferrer (1999), given that we start the process with a measurable mapping from agents to matching-types, allows us to relabel agents in a measurable way, so that we can effectively simply partition the interval into subintervals collecting agents of given matching-types together.

[44]See also Duffie and Sun (2007).

Lemma 2 provides a random matching which is measure-preserving (P1), atomless (which can be seen as a minimal form of anonymity) (P4), has the distribution over matches for each agent that is proportional to the relative fraction of matching types in the population (P2), and such that a conclusion equivalent to that of the Law of Large Numbers holds, so that the measure of type i who mix with type j's is proportional to the product of the proportions in the population (P3).

Thus, Lemma 2 allows for a matching which operates *as if* the Law of Large Numbers holds for the continuum.

Now, given T, set $\nu((i,t,p)) = \frac{M_i(T)}{T_i} B_{p_i(T),t}(p)$, where $B_{p_i(T),t}(p)$ is the probability of having a fraction of p same types out of t draws with a probability of $p_i(T)$ on each draw, when the matches follow a binomial distribution. This gives the binomial distribution over possible sequences of realized matches for those of type i who are in their t-th period of matching, who make up a fraction of $\frac{M_i(T)}{T_i}$ of the overall set of type i's.

Given that we start the types in proportion to what their realized frequency should be under the binomial distribution, it then follows from Lemma 2 (especially P3) that we can find a random matching that gives back the same proportions over types as their new matches and resulting new matching-types will have in a realization that is governed by the binomial distribution.

LEMMA **3** *There exists a random-matching process satisfying (P1)-(P4), such that if ν is as described above, then we are in steady-state so that ν is the outflowing distribution of matching-types.*

What we have shown is that we can find a discrete-time random matching process, so that if each type stayed for an integer number of periods, then we could find a steady-state equilibrium where the outflow of agents of a given type would have a distribution of realized fractions of matchings with same types that matches a binomial distribution.

Next, we take a limit, letting the number of periods become large for each i. We do this as follows. Instead of making a unit of friendship in each period, consider a setting where from each match agents get a ε of friendship, where $\varepsilon > 0$. Then let

$$T_i(\varepsilon) \in \operatorname{argmin}_T |T\varepsilon - t_i|$$

be the number of periods that an agent of type i would have to stay in order to accumulate a total amount t_i of friendships, where t_i is the desired number from the text. For each ε, we have a well-defined $T(\varepsilon)$, and a resulting steady-state matching process where in each period there is an inflow and outflow of a mass of $(N_1, \ldots, N_n)$ of the various types of agents,

and where the outgoing agents of type i have a distribution over the fraction of same types that they met which is governed by a binomial distribution on draws with $T_i(\varepsilon)$ draws with probability $p_i(T(\varepsilon))$ of meeting own type on each draw. Note that as $\varepsilon \to 0$, it follows that each $T_i(\varepsilon)\varepsilon \to t_i$ and that the above described distribution over realized meetings converges (in the sense of weak convergence of measures) to the Dirac measure with mass 1 on p_i. This follows from standard results concerning the limit of a sequence of Binomial distributions (e.g., see Billingsley, 1958).

Appendix B   Proofs

Proof of Proposition 1: If preferences are type-neutral, then (i) in the definition of a steady state equilibrium implies that for each group i :

$$U'(t_i) = c.$$

Then strict concavity of U implies that $t_i = t_j$ for all groups i and j, for any matching process in our model. ∎

Proof of Proposition 2: If the matching process is unbiased, then for all types the probability of meeting type i is equal to q_i. This implies that $\sum_i q_i = \sum_i w_i = 1$, so that if $q_i > w_i$ for some type i, then $q_j < w_j$ for at least one type j. ∎

Proof of Proposition 3: Without loss of generality, let $N_i \geq N_j$. An equilibrium must specify q_i, M_i, and a t_i for each type such that (i)-(iii) are satisfied (noting that (iv) is satisfied automatically with an unbiased meeting process). Let $t(q)$ be the function that assigns the unique optimal t to any $q \in (0,1)$, as in Lemma 1.[45]

As argued following the proposition, it is easy to see that steady-state conditions (ii)-(iii) require that $N_j d_j = N_i d_i$ or $N_j t_j(q_j)(1 - q_j) = N_i t_i(q_i)(1 - q_i)$. Given that $q_j = 1 - q_i$, it follows that a necessary condition for an equilibrium is

$$\frac{N_j}{N_i} = \frac{t(q_i)(1 - q_i)}{t(1 - q_i)q_i}. \tag{8}$$

We claim that if we find a q_i satisfying (8), then the specification of $t(q_i), M_i = N_i t(q_i), q_i$ together with $t(1 - q_i), M_j = N_j t(1 - q_i), q_j = 1 - q_i$ form an equilibrium. The fact that (i)

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[45]The uniqueness follows from the conditions on preferences that imply that $q_i U_s(q_i t_i, (1 - q_i)t_i) + (1 - q_i)U_d(q_i t_i, (1 - q_i)t_i)$ is decreasing in t_i.

and (ii) are satisfied follow directly from the definition of t and the way in which the M's are defined. So let us check that (iii) is also satisfied. Let us verify that $q_i = M_i/(M_i + M_j)$ for each i so that (iii) holds. It is enough to check this for i, given that $q_j = 1 - q_i$. By (8) it follows that

$$N_j t(1 - q_i)q_i = N_i t(q_i)(1 - q_i),$$

or

$$\left(N_j t(1 - q_i) + N_i t(q_i)\right) q_i = N_i t(q_i).$$

Then

$$\left(M_j + M_i\right) q_i = M_i,$$

which implies the desired conclusion.

So, to establish existence of an equilibrium we show that there exists a q_i satisfying (8). When $q_i = \frac{1}{2}$ it follows that $\frac{t(q_i)}{t(1-q_i)} = 1$. Thus, if $N_i = N_j$, then $q_i = \frac{1}{2}$ is an equilibrium. Next, note that as q_i increases, the continuous function $\frac{t(q_i)(1-q_i)}{t(1-q_i)q_i}$ converges to 0 as $q_i \to 1$.[46] Thus, when $N_i > N_j$, there always exists a solution to (8) such that $1 > q_i > 1/2$. ∎

Proof of Claim 1: If $t(q_i)(1-q_i)$ is a decreasing function, then $t(1-q_i)q_i$ is an increasing function. Thus, the expression on the right hand side of (8) is a decreasing function and the equality can hold at at most one value of q. ∎

Proof of Proposition 4: Consider the equilibrium whose existence is proven in Proposition 3. This equilibrium satisfies relative homophily and is such that $q_i > 1/2$. The same-type biased preference condition (5) implies that

$$\frac{\partial}{\partial t}U(q_i t_j, (1 - q_i)t_j) > \frac{\partial}{\partial t}U((1 - q_i)t_j, q_i t_j) = \frac{\partial}{\partial t}U(q_j t_j, (1 - q_j)t_j) = c. \tag{9}$$

and then optimization by j implies that

$$\frac{\partial}{\partial t}U((1 - q_i)t_j, q_i t_j) = \frac{\partial}{\partial t}U(q_j t_j, (1 - q_j)t_j) = c. \tag{10}$$

(9) and (10) then imply that

$$\frac{\partial}{\partial t}U(q_i t_j, (1 - q_i)t_j) > c. \tag{11}$$

The condition of overall diminishing returns to friendship then implies that $s_i + d_i = t_i > s_j + d_j = t_j$ (given that the left-hand side of (11) is equal to c at the optimal choice t_i for type i). Therefore

$$t_i = \frac{d_i}{1 - q_i} > t_j = \frac{d_j}{q_i}, \tag{12}$$

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[46]Note that t is bounded above and is bounded away from 0 under our preference conditions.

which implies that

$$\frac{q_i}{1 - q_i} > \frac{d_j}{d_i} = \frac{N_i}{N_j} = \frac{w_i}{w_j}. \tag{13}$$

Since $q_i + q_j = 1$ and $w_i + w_j = 1$, (13) implies that $\frac{q_i}{q_j} > \frac{w_i}{w_j}$, $q_i > w_i$ (inbreeding homophily) and $q_j < w_j$ (inbreeding heterophily). The fact that $s_i > s_j$ is implied by the equilibrium condition $N_i d_i = N_j d_j$ (which implies $d_i < d_j$) and by the fact that $t_i > t_j$. ∎

Proof of Proposition 5: We first show that if U exhibits strong same–type bias, then $t(q)$ is increasing in q. By strong same-type bias, it follows that

$$\frac{\partial}{\partial t} U\left(\widehat{q} t(q), (1 - \widehat{q}) t(q)\right) > \frac{\partial}{\partial t} U\left(q t(q), (1 - q) t(q)\right) = c \tag{14}$$

when $\widehat{q} > q$. The assumption of overall diminishing returns to friendship of U implies that $t(\widehat{q}) > t(q)$.

Next, the fact that $(1 - q)t(q)$ is decreasing implies that the expression on the right hand side of (8) is decreasing in q. Thus, since as w_i increases, the unique solution of q_i to (8) increases, since the left hand side of (8) is simply $(1 - w_i)/w_i$. This in turn implies that the steady-state equilibrium value of t_i increases, as claimed. ∎

Proof of Proposition 6: Note that by the proof of Proposition 5, we know that $t(q)$ is increasing in q.

Let $(N_1, N_2, ..., N_K)$ be ordered in nondecreasing order.

We show the existence of an equilibrium satisfying the statement of the proposition by showing the existence of an interior fixed point of an appropriately constructed function F mapping the $(K - 1)$-dimensional simplex into itself. F is constructed as follows.

Let F_1 denote the function that maps a vector $\vec{q} = (q_1, q_2, ..., q_K)$ of the simplex to the vector $F^1(q) = (t(q_1), t(q_2), ..., t(q_K))$.

Let F^2 be the continuous function that orders any K-vector in nondecreasing order.

Let F^3 be the continuous function defined by

$$F^3(x_1, x_2, ..., x_K) = \left(\frac{N_1 x_1}{\sum N_i x_i}, \frac{N_2 x_2}{\sum N_i x_i}, ..., \frac{N_K x_K}{\sum N_i x_i}\right). \tag{15}$$

Note that for any $\vec{q} = (q_1, q_2, ..., q_K)$, $F(\vec{q}) \equiv F^3 \circ F^2 \circ F^1(\vec{q})$ is ordered in nondecreasing order and the sum of its elements is 1.

F is a continuous function from the simplex to itself, and therefore possesses a fixed point by Brower's Theorem. Note that, at the fixed point, F^2 preserves the ordering of the vector $\vec{q}$ because $t(q)$ is increasing, so that F^2 applied to the fixed point is just the identity

function.

We conclude the proof by showing that such a fixed point is interior. Suppose not, so that there is at least one q_i such that $q_i = 0$. By the assumption that $U_s(0,0) > c$ and $U_d(0,0) > c$, $F^3 \circ F^2 \circ F^1$ has all positive values (because F^1 has all positive values), which implies that $q_i = 0$ cannot be an element of the fixed point.

Finally, note that given the steady state condition $\frac{t_i w_i}{q_i} = \frac{t_j w_j}{q_j}$, the vector $\left(\frac{q_1}{w_1}, \frac{q_2}{w_2}, ..., \frac{q_K}{w_K}\right)$ must be ordered in nondecreasing order (given that the vector $(t_1, t_2, ..., t_K)$ is nondecreasing, as shown above). Given that $\sum q_i = 1$ and $\sum w_i = 1$, we conclude that $\frac{q_K}{w_K} > 1$ while $\frac{q_1}{w_1} < 1$. Also, since $w_i > w_j$ implies that $q_i/w_i > q_j/w_j$, it also follows easily that $\frac{q_i - w_i}{1 - w_i} > \frac{q_j - w_j}{1 - w_j}$ (simply noting that this can be rewritten as $\frac{\frac{q_i}{w_i} - 1}{\frac{1}{w_i} - 1} > \frac{\frac{q_j}{w_j} - 1}{\frac{1}{w_j} - 1}$). ∎

Proof of Proposition 7: Homogeneity of degree α, together with the condition for the optimal stopping imply that

$$c = \frac{\partial}{\partial t} U(qt, (1-q)t).$$

This implies that

$$t(q) = \alpha \frac{U(qt, (1-q)t)}{c},$$

which in turn implies for each type i,

$$U_i(q_i t(q_i), (1 - q_i)t(q_i)) = \frac{ct(q_i)}{\alpha}, \tag{16}$$

so that the optimal number of friends given q_i is proportional to the maximized level of utility up to the constant $\frac{\alpha}{c}$. ∎

Proof of Proposition 8: Equations (ii) and (iii) for steady state equilibrium imply that

$$\overline{M} = \frac{N_i t_i}{q_i}$$

for each i.

This implies that

$$\frac{N_i t_i}{q_i} = \frac{N_j t_j}{q_j}$$

for every i and j.

Therefore, in equilibrium

$$\frac{w_i U_i}{q_i} = \frac{w_j U_j}{q_j}.$$

Maximizing the aggregate (or average) utility in society, then amounts to maximizing

$$\sum_i w_i U_i = w_1 U_1 \left(1 + \sum_{j \geq 2} \frac{q_j}{q_1} \right) = \frac{w_1}{q_1} U_1 = \frac{c w_1}{\alpha q_1} t_1,$$

which establishes the claim. ∎

PROPOSITION **9** *Consider a biased matching process in a society with two types described by* $q_2 = f(q_1)$, *where* f *is continuous and decreasing. There exists an equilibrium and one such that* $q_1 > q_2$. *Moreover, if preferences satisfy strong same type bias then there exists an equilibrium where larger groups make more total friends.*

Proof of Proposition 9: In equilibrium it follows that

$$N_2 t(q_2)(1 - q_2) = N_1 (t(q_1)(1 - q_1) \tag{17}$$

or

$$\frac{N_2}{N_1} = \frac{t(q_1)(1 - q_1)}{t(q_2)(1 - q_2)}. \tag{18}$$

Let the bias be $q_2 = f(q_1)$, with $f(1) \to 0$ and $f(0) \to 1$. Then we rewrite (18) as:

$$\frac{N_2}{N_1} = \frac{t(q_1)(1 - q_1)}{t(f(q_1))(1 - f(q_1))}. \tag{19}$$

Let $\hat{q}$ be a fixed point of f. Then $q_1 = q_2 = \hat{q}$ is an equilibrium for $N_1 = N_2$. If we let q_i increase and tend to 1 starting from $\hat{q}$, the term on the RHS of (19) tends to zero, since $t(q_1)$ is bounded and $t(0) > 0$ under our assumptions. So there exists an equilibrium with $q_1 > \hat{q}$ when $N_1 > N_2$.

Since f is decreasing, it has a unique fixed point $\hat{q}$, and $q_1 > \hat{q}$ implies that $q_2 = f(q_1) < q_1$.

To conclude the theorem, apply the first part of the proof of proposition 5, showing that strong same type bias implies that $t(q)$ is increasing, to the equilibrium where $q_1 > q_2$. ∎

Appendix C   Ruling out non well-ordered equilibria

In this section we establish conditions under which the result of Proposition 6 apply to all steady state equilibria. Recall that U is homogeneous of degree $\alpha \in (0, 1)$ if $U(ks, kd) = k^\alpha U(s, d)$ for all (s, d).

PROPOSITION **10** *Let $U(s,d)$ be homogeneous of degree $\alpha \in \left(0, \frac{1}{2}\right]$. In every steady state equilibrium, $N_i > N_j$ implies $q_i > q_j$. If preferences also exhibit same type bias, then $t_i = s_i + d_i > t_j = s_j + d_j$ whenever $N_i > N_j$, and i exhibits more inbreeding homophily than j ($IH_i > IH_j$).*[47]

Proof of Proposition 10: From condition (iii) of equilibrium it follows that if $N_i > N_j$, then $\frac{t_i}{q_i} < \frac{t_j}{q_j}$. If $\frac{\partial \frac{t}{q}}{\partial q} < 0$ it follows that $q_i > q_j$.

$\frac{\partial \frac{t}{q}}{\partial q} = \frac{\frac{\partial t}{\partial q} q - t}{q^2} < 0$ if and only if $\frac{\partial t}{\partial q} \cdot \frac{q}{t} < 1$.

(3) and the fact that U is homogeneous of degree α imply that

$$t(q) = \alpha \frac{U(qt, (1-q)t)}{c} \quad . \tag{20}$$

We take the derivative of (20) with respect to q, obtaining

$$\frac{\partial t}{\partial q} = \alpha \frac{t\Big(U_s(qt, (1-q)t) - U_d(qt, (1-q)t)\Big) + \frac{\partial t}{\partial q}\Big(\frac{\partial}{\partial t}U(qt(1-q)t)\Big)}{c} \quad , \tag{21}$$

from which, since $\frac{\partial}{\partial t}U(qt, (1-q)t) = c$, it follows that

$$\frac{\partial t}{\partial q} = \frac{\alpha}{1-\alpha} \frac{t\Big(U_s(qt, (1-q)t) - U_d(qt, (1-q)t)\Big)}{c} \quad . \tag{22}$$

¿From (22) it follows that

$$\begin{aligned}
\frac{\partial t}{\partial q} \cdot \frac{q}{t} &= \frac{\alpha}{1-\alpha} \frac{q\Big(U_s(qt, (1-q)t) - U_d(qt, (1-q)t)\Big)}{c} \\
&< \frac{\alpha}{1-\alpha} \frac{qU_s(qt, (1-q)t) + (1-q)U_d(qt, (1-q)t)}{c} \\
&= \frac{\alpha}{1-\alpha} \leq 1 \ ,
\end{aligned}$$

where the first inequality comes from the fact that $U_d > 0$, while the second comes from the fact that $\alpha \leq \frac{1}{2}$.

The last part of the proof follows from Proposition 6. ∎

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[47]The next example shows that, even with two types, if U is homogeneous of degree $\alpha > \frac{1}{2}$, we can have equilibria with $N_i > N_j$ but $q_i < q_j$ and $t_i < t_j$. Consider $U(s,d) = s^{0.9} + \frac{1}{2}d^{0.9}$, $2N_1 = N_2$ and $c = 0.9$. Given homogeneity it is possible to express explicitly

$$t(q) = \left(q^{0.9} + (1-q)^{0.9}\right)^{10} \quad .$$

We find numerically two steady state equilibria: 1) a non well-ordered equilibrium in which $q_1 \simeq 0.62$, $q_2 \simeq 0.38$, $t_1 \simeq 0.23$ and $t_2 \simeq 0.07$; 2) a well-ordered equilibrium in which $q_1 \simeq 0.001$, $q_2 \simeq 0.999$, $t_1 \simeq 0.0001$ and $t_2 \simeq 1.00$.

Appendix D Welfare analysis with many types and logarithmic preferences

We use a logarithmic form of preferences to investigate alternative policies when there are more than two groups. Let $U(s,d) = a \log s + b \log(s+d) + g \log d$, where a, b and g are nonnegative. Note that in this case $U(s,d)$ is not homogeneous.[48]

Consider the case in which $a = 1$, $b > 0$ and $g = 0$. The utility of type i agents is then:

$$U_i(w_i) = \log\left(w_i \frac{1+b}{c}\right) + b \log\left(\frac{1+b}{c}\right) - 1 - b \ ,$$

The utility of type i agents is indifferent to a split of type j in more types, as long as w_i is kept constant. There is no externality on other types if a type splits in more sub–groups. The type i utility is always increasing in w_i. If the number of types is $K = 2$, then the expected average welfare $w_i U(w_i) + (1 - w_i) U(1 - w_i)$ is:

$$w_i \log\left(w_i \frac{1+b}{c}\right) + b \log\left(\frac{1+b}{c}\right) + (1-w_i) \log\left((1-w_i)\frac{1+b}{c}\right) - 1 - b.$$

which is convex, achieves always a minimum at $w_i = \frac{1}{2}$ and a maximum in the completely segregated case where $w_i = 0$ or $w_i = 1$.

Consider a case in which $a = 1$, $b = 0$ and $g > 0$. The utility function of type i agents depends on w_i and is

$$U_i(w_i) = \log\left(w_i \frac{1+g}{c}\right) + g \log\left((1-w_i)\frac{1+g}{c}\right) - 1 - g \ .$$

There is still no externality if another type is split into more types. It is easy to check that $U_i(w_i)$ is increasing in w_i up to the point where $w_i = \frac{1}{1+g}$, and then is decreasing. If g is a positive integer, then the social optimum would be to split equally $g + 1$ types (if available) in the matching process.

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[48]The *log* function can be considered a case of homogeneity α at the limit where $\alpha \to 0$. It is, however, easy to check that not all our results concerning homogeneous functions hold in the logarithmic case. Equation (3), implies that for any type i, the optimal number of friends is $t_i = \frac{a+b+g}{c}$, independently of q_i. Condition (iii) of steady-state equilibrium implies that for any type i:

$$q_i = \frac{N_i t_i}{\sum_j^K N_j t_j} = \frac{N_i}{\sum_j^K N_j} = w_i \ .$$

The steady state equilibrium is then easy to compute. For any type i: $s_i = q_i t_i = w_i \frac{a+b+g}{c}$ and $d_i = (1 - q_i) t_i = (1 - w_i)\frac{a+b+g}{c}$. $U(s,d)$ trivially exhibits baseline homophily ($q_i = w_i$ for any type i).

If we consider only $K = 2$ types, then the expected average welfare is

$$\left(w_i + g(1 - w_i)\right) \log \left(w_i \frac{1+g}{c}\right) + \left(w_i g + (1 - w_i)\right) \log \left((1 - w_i) \frac{1+g}{c}\right) - 1 - g \quad .$$

The previous function has always a critical point in $w_i = \frac{1}{2}$. Its second derivative with respect to w_i is $\frac{-g+(1+g)w_i(1-w_i)}{w_i^2(1-w_i)^2}$, which is negative for any $w_i \in (0, 1)$ only if $g > \frac{1}{3}$. If $g \geq \frac{1}{3}$ then the average welfare obtains its maximum for $w_i = \frac{1}{2}$.

If instead $g < \frac{1}{3}$, then $w_i = \frac{1}{2}$ is a local minimum for the aggregate welfare, whose functional form is symmetric and bimodal, as in Figure 8.
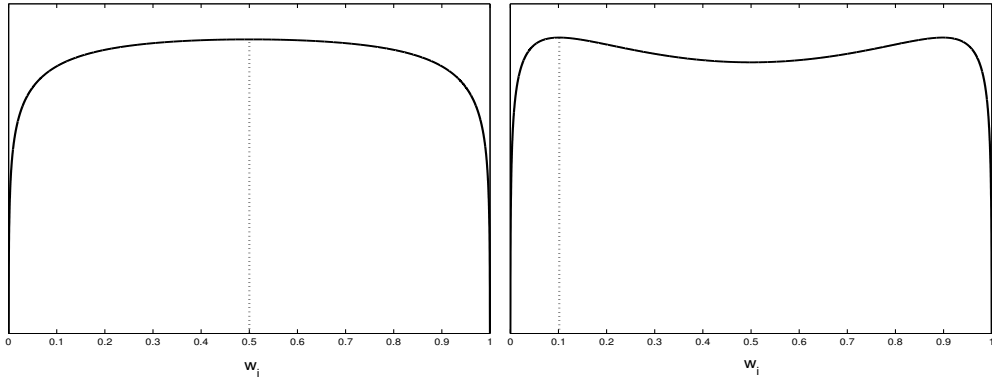
Figure 9 shows the two cases.

Figure 9: Average aggregate welfare when $U(s, d) = \log s + g \log d$. Left hand side is for $g = \frac{1}{2}$, right hand side is for $g = \frac{1}{5}$.