Information Lags Induce Cycles in Congestion Games

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Abstract

This paper studies the emergence of a cycling phenomenon as a result of information lags in congestion games with binary choices. A population of agents chooses between two alternatives, with the objective being to land on the less congested choice. However, agents *cannot* obtain real-time congestion information. This paper calls attention to the role of information lags in strategic interactions, a subject not well examined at the micro level in previous economic literature. Lagged information fails to take into account other agents' responses between information-updating intervals. Responding sensitively to such information in congestion games, i.e., choosing the reportedly less congested choice with a high probability, leads to overreaction in the next period. This constitutes the first half of a two-period cycle of aggregate congestions on the two choices. The oscillating phenomenon can be observed in examples such as fluctuating traffic congestions as well as price cycles in commodity markets. In a heterogeneous population, agents may also hold diverse attitudes and respond differently to the same information; whether the system dynamic diverges to cycles depends on the information lags and on the *average* of response sensitivities of the heterogeneous population. More interestingly, when agents can endogenously choose how sensitively they respond to lagged information, this paper presents the irreconcilability between agents' incentive to avoid oscillation and the incentive to pick the less congested choice, the origin of which can be traced to the amount of randomness that agents introduce into their behavioral responses. Considering both incentives at the same time gives rise to the observation of a meta-cycle of two-period cycles. Lastly, the possibility of a prediction with perfect foresight is considered.

1 Introduction

This paper is motivated by the author's experience on his regular trips to Chicago via I-90, a two-lane highway in the state of Wisconsin. In the late afternoons, the highway traffic is usually heavy, but not bumper-to-bumper. Interestingly, rather than both lanes moving smoothly at the same speed, one lane is periodically more congested than the other, even on a long stretch of highway with no entry or exit. From the driver's seat, the author first noticed that cars in the neighboring lane, be it the left lane or the right, were moving faster. Tempted by less congestion, he decided to switch lanes. However, he failed to take into account that, in the meantime, other drivers in the same lane, having been through a similar thought process, were switching lanes just as he was. As a result, the once less-congested neighboring lane slowed down, and the author found himself stuck in the slower lane once again. The same process then repeated itself. Developing the insight from the above observation, this paper studies how information lags give rise to the cycling phenomena in a population game with two congested choices.

In congestion games, an agent's utility is negatively related to the number of agents sharing the same strategy. As an example, consider daily commuters choosing between two ex ante identical routes, both of which lead to a common destination. The conventional rationalistic analysis supposes that each morning an agent forms some expectation regarding which route will be less congested that day. However, if everyone holds the same expectation, which may take all previous history into consideration, such a common expectation is immediately falsified when everyone chooses the same route.[1] This interesting property of congestion games may be considered a kind of "self-defeating prophecy", as opposed to a self-fulfilling prophecy. In a supermarket, when it is announced that "Aisle 7 is now open," Aisle 7 often becomes the longest queue.[2] Once the commonality of expectation breaks down, making a rational decision becomes difficult from the point of view of an agent, as he does not know what to expect of the behavior of the many other agents, and vice versa.

In this paper, an agent's behavior is modeled with a general *response function* $p(x)$, which is a reduced form from the disutility functions and specifies the probability of switching to the alternative if the current choice is congested with proportion x of agents. The response function allows for a wide range of functional forms, including, but not limited to, the *logit* choice rule, thus enabling it to cover significant ground in modeling agents' behavior. For instance, an agent who is *payoff sensitive* switches to the less congested option with a high probability, which is characterized by an increasing response function with a steep slope;[3] a *payoff insensitive* agent is described by a response function with a flat slope. We can also accommodate agents with *contrarian* views, who are less likely to switch away from a more congested option, which is represented by a decreasing $p(x)$.

An equilibrium in a congestion game with binary choices refers to the state in which both choices are equally congested. There has been a number of studies in the literature of learning that show how an equilibrium emerges from various underlying adaptive mechanisms. To cite a few representative ones, Cominetti et al. (2008) analyze congestions games with n choices with smooth fictitious play: agents form perceptions based on prior experiences, follow a random choice rule such as logit, and then use the realized payoff to update the perceptions. With this learning process, the average of

[1] For details on this scenario, see the proof in Young (2004, p. 51).

[2] This example is originally found in Mitzenmacher (2000).

[3] The most sensitive agent is one who always perfectly best responds, characterized by a step function with the jump from 0 to 1 at the point where the disutilities from the two options are equal.

all agents' mixed actions is shown to converge asymptotically towards an equilibrium. Duffy and Hopkins (2005) study market entry games[4] with reinforcement learning and predict an equilibrium with sorting: agents play a pure strategy equilibrium, with some agents permanently in the market, and some permanently out. However, the conclusions from the learning literature–of converging to an equilibrium–do not explain the author's observation of periodic oscillations of congestions between the two lanes on I-90. Moreover, such an observation is corroborated by the laboratory experiments in Selten et al. (2007) with a two-route choice scenario, in which fluctuations persist until the end of the sessions (Figure 1).

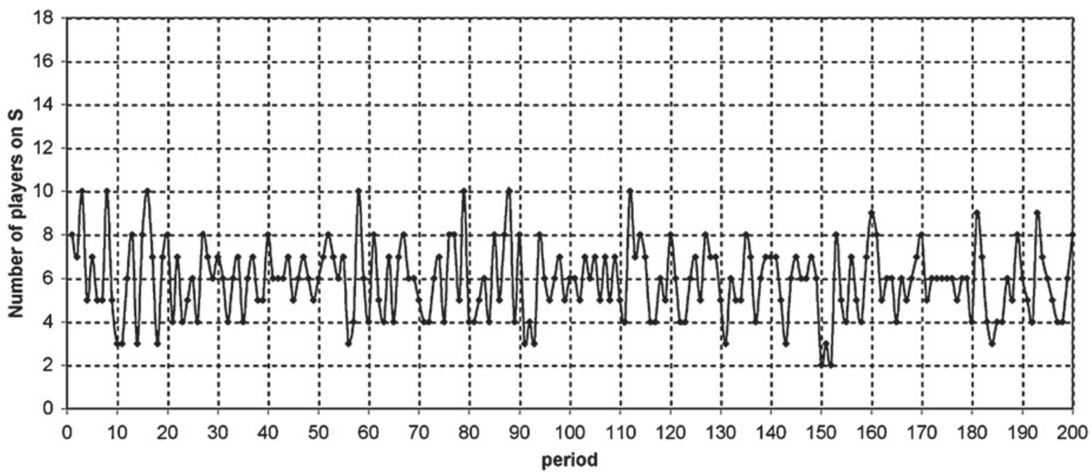


Figure 1: Selten et al. (2007): fluctuations persist after 200 periods.

If the learning literature helps us to understand the emergence of an *equilibrium* in congestion games, this paper explains the observed *oscillating phenomenon* due to information lags. To distinguish the two perspectives, both Cominetti et al. (2008) and Duffy and Hopkins (2005) utilize techniques from Benaim and Hirsch (1999) to approximate *discrete* stochastic process with an associated *continuous* deterministic dynamic in the asymptotic limit. The role of information lag is de-emphasized when taking the continuous limit. This paper, on the other hand, places its focus on the effect of information lags and keeps the time structure discrete. As a matter of fact, if

[4] In a weak sense, market entry games can be regarded as congestion games with two asymmetric options, as the payoff from the option of staying out is weakly decreasing, or constant in the number of users.

the information lag takes the limit to 0, the model of this paper also concludes convergence to an equilibrium, linking the two perspectives.

This paper highlights the consequence of *information lags*[5] in congestion games, which is the time between information gathering and decision-making. Information lags may stem from different sources. To begin with, it takes time to gather and relay information to all agents. In the route choice example, morning commuters listen to the radio traffic report, which updates congestion information every fifteen minutes or so. Moreover, the time from a decision to its implementation may not be immediate, which also contributes to the information lag. A commuter needs to make the decision of which highway to take well before he gets onto the ramp of that highway. The time between receiving the radio message and arriving at the highway becomes a significant part of the information lag. Combining both, the congestion experienced by the commuters may have changed considerably from that described in the radio report.

Other fields also recognize that outdated information may disrupt convergence to an equilibrium. In transportation research, Ben-Akiva et al. (1991) point out that if a substantial fraction of agents receive the same message and react uniformly, it only "causes congestion to transfer from one road to another." They mention that information lags "may also generate oscillations in road usage." Emmerink et al. (1995) demonstrate with simulations that old traffic information provided by the Advanced Traveler Information System (ATIS) may actually increase travel time if agents overreact to the same message. Wahle et al. (2000) employ simulations based on cellular automata to show that information lags lead to undesirable oscillations. In computer science, constantly keeping an eye on the levels of server loads consumes computing resource, thus for practical reasons, the load information is broadcast periodically rather than realtime. Mitzenmacher (2000) recognizes that if the load information is out of date, simply assigning new tasks to the shortest queue easily leads to overloading of servers; he suggests that a strategy with randomness improves performance. Fischer and Vöcking (2009) refer to information with lag as "stale information" and present sufficient conditions to avoid performance oscillations. Cycling phenomena due to information lags have also been documented in agricultural economics[6] and in the labor markets of lawyers or engineers.[7]

[5] The presence of information lags in decisions has been studied in macroeconomic theory, termed as "sticky information". Mankiw and Reis (2002, 2006, 2007) and Reis (2006) assume that in each period only a fraction of consumers, workers, or firms obtain up-to-date information, which provides an alternative source of stickiness in place of the "sticky price" assumption. In that literature, agents do not frequently update their information because of the high costs of processing the information. In comparison, we are interested in the scenarios where information is not outdated *by choice*, but is *inherent* in the time structure of events.

[6] Rosen et al. (1994) attribute the consistent beef cattle price cycles in the U.S. market to the information lag from the breeding cycle of cattle.

[7] Both vocations demand years of training before entering the labor market, see Freeman (1975, 1976).

Most of the above-mentioned works, however, are either observational or simulation results. Adding to this literature, this paper builds an analytical model with parameters characterizing the information lags and the agents' response sensitivities. The analytical framework not only provides a better understanding of the oscillating dynamic, but can also be used to explore more interesting population compositions and agents' adaptive behavior. In addition, it serves as a framework for calibrations in future empirical studies.

In our basic model, agents decide simultaneously between two options at discrete time periods and they want to avoid congestion. Lagged information fails to take into account other agents' responses between information-updating intervals. Responding sensitively to such information in congestion games, i.e., choosing the reportedly less congested choice with a high probability, leads to overreaction in the next period. This constitutes the first half of a two-period cycle of the aggregate congestions on the two choices. Section 2 establishes the basic model and concludes that the dynamic converges to a rest point only if information updates frequently and agents are not too aggressive in switching choices. Otherwise, it is shown to bifurcate to a stable two-period cycle. The conclusion also applies to games with *asymmetrically* congested choices. In Section 3, agents may also hold diverse attitudes and respond differently to the same information. In a heterogeneous population, whether the system dynamic diverges to cycles depends on the information lag and the *average* of response sensitivities of the heterogeneous population. The presence of contrarian agents decreases the average response sensitivity of the population. In Section 4, allowing agents to endogenously choose how sensitively they respond to lagged information, this paper presents the irreconcilability between agents' incentive to avoid oscillation and the incentive to pick the less congested choice. Considering both incentives at the same time gives rise to the observation of a meta-cycle of two-period cycles. We also discuss in details the implication of different response functions, and trace the origin of the irreconcilability back to the amount of randomness that agents introduce into their responses. Finally, we introduce a Rational Expectation (RE) predictor with the ability to forecast correctly the next period of congestion with perfect foresight. If agents follow the RE predictor with a probability greater than 1/2, the rest point at the equilibrium is stable and irreconcilability disappears; however, a strong assumption for the RE predictor to be available is complete knowledge of agents' behavior. All proofs are presented in the Appendix.

2 The Basic Model

2.1 Model Setup

The basic setting begins with a unit mass of homogeneous agents choosing between two options: left and right, respectively. The model adopts a discrete time structure. At the beginning of period $t+1$, a proportion θ of agents observe last period's congestions status x_t, the proportion of agents using the left in period t[8], and revise their choices. Thus, the most up-to-date information available to agents carries a lag of one period. Parameter θ, the proportion of agents who revise their choices during each period, also quantifies information lags: how much time in actual that one normalized period stands for. A large θ indicates a long lag in the information, e.g., one hour, such that many changes can take place in one period; a small θ means that the time between each period is short, e.g., one minute, and that there are only a few revisions before the information is updated. Consider the two limiting cases of this parameter: $\theta \to 0$ corresponds to continuous information updating; $\theta = 1$ represents completely out-dated information after one period, because when the information arrives, every agent has revised his choice.

To take x_t as the prediction for the next period is naive, as it does not take into account how the other agents respond to the commonly-observed x_t. This model uses response functions to describe agents' behavior in response to the lagged information. When proportion x of agents use the left and proportion $1-x$ choose the right, let $U_l = -l(x)$ and $U_r = -r(1-x)$ be the disutilities from the congestion on the left and right, respectively. The agents' response functions take a reduced form from the disutility functions: $p(x) = p(U_l, U_r) = p(-l(x), -r(1-x))$; it specifies the probability that the agent switches to the right if the left was congested by proportion x in last period. Similarly, $q(1-x) = q(U_r, U_l) = q(-r(1-x), -l(x))$ represents the probability of switching from right to left given the congestion status $(x, 1-x)$. With the reduction, changes in disutility functions or in agents' attitude toward the disutilities are all reflected in the corresponding changes of the response functions. If an agent responds to more congestion by a higher probability of switching away, $p(x)$ is an increasing function. Alternatively, some agents may think one step further and believe that a message of low congestion will attract many others, and therefore the option will actually be crowded in the next period. Borrowing the term from Selten et al. (2007), we refer to them as the *contrarian* agents, for whom $p(x)$ is a *decreasing* function.

[8] Since there are only two options, the proportion of agents on the right is always $1 - x_t$.

One particular functional form of $p(x)$ is the logit choice rule: [9]

$$p(x) = \frac{e^{\beta l(x)}}{e^{\beta l(x)} + e^{\beta r(1-x)}}.$$

With the logit choice rule, $q(1-x) = 1 - p(x)$.

Agents following the logit choice rule are best-responding in a perturbed manner. Parameter β captures agents' responsiveness to payoff differences. A *payoff-sensitive* agent has a high β. An agent with $\beta \to \infty$ perfectly best responds, i.e., switches to the alternative even it is reported to be slightly better than the current choice. Agents with β less than 0, thus $p'(x) < 0$, are the contrarian type.

Most of the subsequent conclusions apply to any increasing functional form of $p(x)$. Those that apply only with the logit form will be explicitly noted.

2.2 The Rest Point and Its Stability

In the homogeneous population, the change of usage on the left option between period t and $t+1$ is

$$x_{t+1} - x_t = -\theta x_t p(x_t) + \theta(1 - x_t) q(1 - x_t)$$

where the first term on the right represents the proportion leaving left for right in the next period, and the second term describes the flow into the left option.

We first discuss the case in which the two options are symmetric: if both options are occupied by the the same amount of agents, the resulting disutilities are also the same on both, i.e., $l(x) = r(x)$ and $p(x) = q(x)$. With symmetry, the disutilities of congestion from two options are equal at $x = \frac{1}{2}$, which is a rest point of the above dynamic equation. Normalize around $\frac{1}{2}$ with $z = x - \frac{1}{2}$, and we have the one period dynamic equation $z_{t+1} = f(z_t)$, which maps z_t of period t to the next period realization of z_{t+1}:

$$z_{t+1} = f(z_t) = z_t - \theta\left(\tfrac{1}{2} + z_t\right) p\left(\tfrac{1}{2} + z_t\right) + \theta\left(\tfrac{1}{2} - z_t\right) p\left(\tfrac{1}{2} - z_t\right). \tag{1}$$

To ensure that the rest point $z = 0$ is locally stable, the slope of $f(z)$ must be less than 1 in absolute value. Proposition 10 identifies the following two attributes as the determinants in the

[9] See Anderson et al. (1992) and Hofbauer and Sandholm (2002) for the dual representations of the logit choice rule: It can be derived from additive random utilities with the perturbation terms extreme-value distributed. Alternatively, suppose that the decision to switch with probability p involves a control cost that is the entropy function of p, the maximization with the deterministic perturbation also yields the logit choice rule.

local stability condition at $z = 0$: information lag and agent's response sensitivity.

Proposition 1 *In a homogeneous population of agents with an increasing response function* $p(x)$*, the rest point* $z = 0$ *is locally stable if and only if* $\frac{2}{\theta} > 2p\left(\frac{1}{2}\right) + p'\left(\frac{1}{2}\right)$.

Corollary 2 *In a homogeneous population of agents following the logit choice rule with sensitivity* β*, the rest point* $z = 0$ *is locally stable if and only if* $\frac{2}{\theta} > \frac{1}{2}\beta l'\left(\frac{1}{2}\right) + 1$.

Corollary 2 states that both θ and β need to be small to ensure convergence to the rest point.[10] In other words, fixing a particular value of β, there is a threshold $\theta_0(\beta)$ of θ, above which the rest point becomes unstable because the information lag is too long. Similarly, fixing a particular value of θ, there is a threshold $\beta_0(\theta)$ of β, above which the agents' responses are so sensitive that they render the rest point unstable. Corollary 2 also implies that if agents want to avoid oscillation, the reasonable level of response sensitivity β is inversely related to the information lag θ. This implication will be brought up again in Section 4, where we further discuss the response functions.

2.3 Two-Period Cycle

When the rest point $z = 0$ is not stable, there can potentially be a wide range of possibilities for dynamic patterns. In particular, Brock and Hommes (1997) present a model of supply-demand in which producers choose between two predictors: with the naive predictor, the price forecast is the old price from the last period (the information lag is one period); the rational expectation predictor has perfect foresight and forecasts the price accurately with a cost. Brock and Hommes (1997) show that a high sensitivity to payoff difference of the two predictors will not only make the price dynamics unstable, but will also lead to "a rational route to randomness", showing *chaotic* price fluctuations. Nonetheless, we are going to show that the dynamic in our model will not go as far on the route to chaos.

The *Period Doubling Bifurcation* Theorem in Robinson (1995) considers a general map with one parameter $x_{t+1} = f(x_t, \theta)$, where $x \in \mathbb{R}$ and $\theta \in \mathbb{R}$. The theorem shows that, with a set of nondegenerate conditions, the period doubling bifurcation or *flip bifurcation* occurs as θ passes through θ_0, where the slope of $f(x_t, \theta_0)$ is equal to -1; the original rest point becomes repelling and a two-period cycle branches off from the rest point. Moreover, the stability of the two-period cycle depends on the second-iterate map $f^2_{\theta_0}$. Applied to our model as described in Equation (1), it

[10] Fischer and Vöcking (2009) reach the same conclusion for the convergence condition.

can be verified that with any logit response function, the nondegenerate conditions are met at rest point $z_0 = 0$ with $\theta_0 = \frac{2}{2p\left(\frac{1}{2}\right)+p'\left(\frac{1}{2}\right)}$, and that the two-period cycle after bifurcation is attracting.

Notice that the Period Doubling Bifurcation Theorem deals with any general map of one variable with one parameter, both of which have domains in $\mathbb{R}$. With the specific form of $f(z_t)$ in Equation (1), we can also show the occurrence of period doubling bifurcation and that the two-period cycle after bifurcation is attracting for any *non-Logit* response function that is increasing.

Proposition 3 *In a homogeneous population of agents with* any *increasing response function $p(x)$, if $\theta > \theta_0$, the rest point at* 0 *is unstable and bifurcates to an attracting two-period cycle, the uniqueness of which depends on the functional form of $p(x)$. In the case of logit choice rule, the two-period cycle is unique and globally stable.*

The proof of Proposition 3 first establishes that $f(z_t)$ is bounded between $-\frac{1}{2}$ and $\frac{1}{2}$ and that the slope $f(z_t)$ does not exceed 1 for all $z \in [-\frac{1}{2}, \frac{1}{2}]$. The former bound gives us the existence of a two-period cycle, while the latter condition on the slope ensures that if the point-symmetric map of $f(z_t)$ takes a tent-shape, the tent roof is not too steep, which guarantees that the dynamic goes through period doubling bifurcation once and only once, and that the two-period cycle is attracting. We thus are able to exclude other dynamic possibilities than the attracting two-period cycle. Figure 2 illustrates the mappings.

With the logit choice rule, $f(z)$ crosses the negative 45° line only once between 0 and $\frac{1}{2}$. When we allow for a more general functional form of $p(x)$, it is, in theory, possible that $f(z)$ intersects the negative 45° line more than once in the positive open half-plane. When this happens, since the bounded condition of $f(z)$ requires that $f\left(\frac{1}{2}\right) > -\frac{1}{2}$, the addition of intersections must come in pairs, in which one of the two intersections corresponds to a stable two-period cycle; the other one is unstable and separates the basins of attractions from the neighboring two-period cycle.

We can also derive the relation between information lag θ and the magnitude of the cycle.

Proposition 4 *In a homogeneous population of agents with an increasing response function $p(x)$, the magnitude of the two-period cycle increases with information lag θ.*

Proposition 4 states that a greater information lag leads to a larger magnitude of oscillation. Proposition 4, together with Proposition 1, depicts how the dynamic changes continuously with information lag θ: when θ is small enough with respect to agents' response sensitivity, the symmetric equilibrium is stable; as θ passes the threshold θ_0, the rest point $z = 0$ becomes unstable and

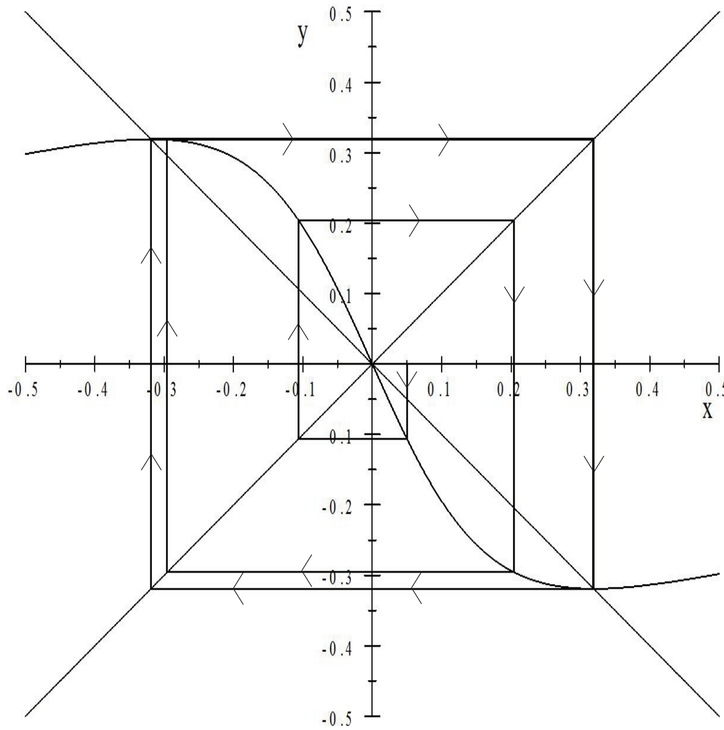


Figure 2: An unstable rest point bifurcates to a stable two-period cycle

bifurcates to a two-period cycle. The magnitude of the cycle increases continuously with θ from 0 to a positive value.

2.4 Asymmetrically Congested Options

When the symmetry in the two choices is relaxed, the model can extend more generally to *any* games in which an agent's payoff is negatively related to the number of agents sharing the same strategy. Such scenarios may arise as commuters choose between a highway or a local road leading to a same destination, or when a number of competing firms decide on supplying one of two markets, with profit decreasing with the number of suppliers, or when a population of farmers choose to grow corn or beans, or even among shoppers choosing between two styles, retro or avant garde. Suppose the two available options, conveniently referred to as the *left* and the *right*, now represent any pair of binary choices in a game with congestion effects. In the asymmetric case, the disutilities $U_l = -l(x)$ and $U_r = -r(1-x)$ take different functional forms, $l'(x) > 0$ and $r'(x) > 0$. The agents' response functions in reduced forms are generally different: $p(x) = p(U_l, U_r) = p(-l(x), -r(1-x))$, and

$q(1-x) = q(U_r, U_l) = q(-r(1-x), -l(x))$. The dynamic equation takes the form:

$$x_{t+1} = f(x_t) = x_t - \theta x_t p(x_t) + \theta(1-x_t) q(1-x_t). \tag{2}$$

Following the same vein of thought as in the previous section, a direct application of Robinson (1995) Period Doubling Bifurcation Theorem on Equation (2) concludes that for logit response functions, when the rest point is unstable, it bifurcates to an attracting two-period cycle. The conclusion also extends to all *non-logit* response functions with a proof in the spirit of that for Proposition 3.

In summary, with a population of homogeneous agents facing two possibly asymmetric options in a congestion game, if the agents' switch aggressively to the less congested option in response to information with information lag, two-period cycle results, and it is shown to be attracting. That all agents respond to a message in exactly the same way is a strong assumption. An alternative interpretation may be that the switching probability $p(x)$ describes a population's aggregate behavior, instead of identical individual responses. In Section 3, we introduce heterogeneity to individual behavior rules and show that most of the conclusions in this section extend to a heterogeneous population.

3 Heterogeneous Populations

3.1 Model with Two Types of Agents

There are a number of motivations for us to explore a heterogeneous population. First of all, we want to make sure that conclusions in the previous section are not dependent on the homogeneity assumption, and that they are robust when extended to agents' heterogeneous responses. With this extension, agents may also hold diverse attitudes and respond differently to the same information; the framework is able to model much more complex and interesting population compositions, including the possibility of contrarian agents. More importantly, not like in the learning literature, the different responses do not automatically smooth themselves out and converge to an equilibrium as the diversity increases; rather, whether the system dynamic diverges to cycles depends on the information lags and on the *average* of response sensitivities of the heterogeneous population.

The agents still face two choices, but there are now two types of agents, A and B. They are differentiated by their behavior rules, respectively $p^A(x)$ and $p^B(x)$. The proportions of type A

and type B agents in the total population are fixed at α and $1-\alpha$. Let $x_t^i, i=A,B$ represents the proportion of type i agents who use the left option in period t. The total usage of the left option in period t becomes $x_t = \alpha x_t^A + (1-\alpha)\, x_t^B$. As in the basic setting, θ represents the proportion of agents who revise their choices during each period and provides as a measure of information lag.

3.2 The Rest Point and Its Stability

With two types of agents, $\left(\frac{1}{2},\frac{1}{2}\right)$ is a rest point if the two options are symmetric. At a rest point, the inflow balances the outflow for both types. The rest point $\left(\frac{1}{2},\frac{1}{2}\right)$ is also shown to be unique.

Proposition 5 *In a heterogeneous population of two types of agents with increasing $p^A(x)$ and $p^B(x)$, $\left(\frac{1}{2},\frac{1}{2}\right)$ is the unique rest point.*

After Proposition 5, we normalize $z_t^i = x_t^i - \frac{1}{2}, i=A,B$ to bring the rest point to the origin $(0,0)$, and the system of dynamic equations becomes:

$$\begin{aligned}
z_{t+1}^A &= f^A\left(z_t^A, z_t^B\right) = z_t^A - \theta\left(\frac{1}{2}+z_t^A\right)p^A\left(\frac{1}{2}+z_t\right)+\theta\left(\frac{1}{2}-z_t^A\right)p^A\left(\frac{1}{2}-z_t\right)\\
z_{t+1}^B &= f^B\left(z_t^B, z_t^A\right) = z_t^B - \theta\left(\frac{1}{2}+z_t^B\right)p^B\left(\frac{1}{2}+z_t\right)+\theta\left(\frac{1}{2}-z_t^B\right)p^B\left(\frac{1}{2}-z_t\right).
\end{aligned}$$

The interaction between the two types takes place through $z_t = \alpha z_t^A + (1-\alpha)\, z_t^B$, which both types experience and contribute to.

In a heterogeneous population, the stability condition of the rest point depends on the information lag and the *average* response sensitivity in very much the same way as that in a homogeneous population.

Proposition 6 *In a heterogeneous population of two types of agents with $p^i\left(\frac{1}{2}\right)=\frac{1}{2}, i=A,B$, the rest point $(0,0)$ is locally stable if and only if $\frac{2}{\theta} > 1+\alpha p^{A\prime}\left(\frac{1}{2}\right)+(1-\alpha)\,p^{B\prime}\left(\frac{1}{2}\right)$.*

Corollary 7 *In a heterogeneous population of two types of agents following logit choice rules with sensitivities β^A and β^B, the rest point $(0,0)$ is locally stable if and only if $\frac{2}{\theta} > 1+\frac{1}{2}\alpha\beta^A l'\left(\frac{1}{2}\right)+\frac{1}{2}(1-\alpha)\,\beta^B l'\left(\frac{1}{2}\right)$.*

The rest point $(0,0)$ loses stability when one of the eigenvalues of its Jacobian matrix $D(f)_{(0,0)}$ exceeds 1 in absolute values. As θ crosses the threshold $\theta_0 = \frac{2}{1+\alpha p^{A\prime}\left(\frac{1}{2}\right)+(1-\alpha)p^{B\prime}\left(\frac{1}{2}\right)}$, the eigenvalue drops below -1, where a period doubling bifurcation occurs. The assumption $p^i\left(\frac{1}{2}\right)=\frac{1}{2}, i=A,B$

is useful to obtain tractable eigenvalues. It means that when the two options are reported to be equally congested, agents of both types will be indifferent between the two and choose either with equal probability, which is the case with any logit choice rule.

We can also relate the stability condition of a heterogeneous population to the stability conditions of the constituent types' homogeneous populations.

Corollary 8 *Assume $p^i\left(\frac{1}{2}\right) = \frac{1}{2}, i = A, B$. If $z^A = 0$ is locally stable in a homogeneous population with only type A agents, and $z^B = 0$ is likewise locally stable in a population with only type B agents, then for a mixed population with type A and type B agents, the rest point $(0, 0)$ is also locally stable.*

The local stability condition for a mixed population in Proposition 6 can be derived from the stability conditions for type A and type B homogeneous populations by way of a weighted sum of the two inequalities from Proposition 1. However, Corollary 8 is not an if-and-only-if statement because the reverse may not be true: one can easily construct a population that consists of a majority of insensitive agents mixed with a tiny proportion of highly sensitive agents, in which $(0, 0)$ is still locally stable. Nonetheless, those sensitive agents, when left alone in a homogeneous population of their own, may well diverge from the rest point $z = 0$.

3.3 Two-Period Cycle

Having established the stability condition of the rest point, we next present the key proposition that, when the rest point $(0, 0)$ becomes unstable, the system dynamic diverges to a two-period cycle that is stable.

Proposition 9 *In a heterogeneous population of two types of agents with increasing $p^A(x)$ and $p^B(x)$, when the rest point $(0, 0)$ is unstable, it bifurcates to a two-period cycle characterized by $\left(z^{A+}, z^{B+}\right)$ and $\left(z^{A-}, z^{B-}\right)$ from any initial point except those on the straight line $z_0^B = -\frac{\alpha}{1-\alpha} z_0^A$, which approach the rest point along that line in an unstable way.*

Points on the straight line $z_0^B = -\frac{\alpha}{1-\alpha} z_0^A$ are on a saddle line:they approach the rest point along that line as long as there is no perturbation. With a small deviation, the dynamic will no longer head to the unstable rest point $(0, 0)$, but will be attracted to the two-period cycle characterized by $\left(z^{A+}, z^{B+}\right)$ and $\left(z^{A-}, z^{B-}\right)$.

A number of additional difficulties are brought about by the heterogeneity: First, the dynamic equation $z_{t+1}^A = f\left(z_t^A, z_t^B\right)$ is no longer an odd function in z_t^A, i.e., $f\left(-z_t^A, z_t^B\right) \neq -f\left(z_t^A, z_t^B\right)$, which prevents us from using symmetry to identify potential two-period cycles. Secondly, the interaction between the two types means that it is difficult to draw conclusions regarding the system dynamic by studying either type's dynamic equation separately. Since both z_t^A and z_t^B are changing during each period, for the study of two-period cycles it is more convenient to work with the second-iterate dynamic equations, which map from z_t^A and z_t^B to z_{t+2}^A and z_{t+2}^B:

$$z_{t+2}^A = \widetilde{f^A}\left(z_t^A, z_t^B\right) = f^A\left(f^A\left(z_t^A, z_t^B\right), f^B\left(z_t^A, z_t^B\right)\right) \tag{3}$$

$$z_{t+2}^B = \widetilde{f^B}\left(z_t^A, z_t^B\right) = f^B\left(f^A\left(z_t^A, z_t^B\right), f^B\left(z_t^A, z_t^B\right)\right). \tag{4}$$

At first sight, it is not straightforward how to analyze the dynamic picture in the four-dimensional space $\left(z_t^A, z_t^B\right) \times \left(z_{t+2}^A, z_{t+2}^B\right)$. Our solution is to first disassemble the high-dimensional space into cross sections of lower dimensions, examine the cross sections, and finally map and summarize the results from the cross sections in a single two-dimensional plane, on which we get the conclusion of the two-period cycle.

With the details of the proof in the Appendix, here are the steps in our plan:

1. Consider Equation (3) as a three-dimensional space of $\left(z_t^A, z_t^B\right) \times z_{t+2}^A$, and cut a cross section at a given $z^B \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, which has the dimension $z_t^A \times z_{t+2}^A$.

2. Show that, in the cross section of z^B, when the rest point is not stable, after two periods $z_{t+2}^A = \widetilde{f}\left(z_t^A, z^B\right)$ moves toward either z^{A+}, if z_t^A is more than the unstable rest point z^{A*}, or z^{A-}, if z_t^A is less than z^{A*}.

3. After executing step 2 for every cross section of z^B between $-\frac{1}{2}$ and $\frac{1}{2}$, collect all such points z^{A+}, z^{A*}, z^{A-} to form continuous curves $z^{A+}\left(z^B\right), z^{A*}\left(z^B\right), z^{A-}\left(z^B\right)$ in the $z_t^A \times z_t^B$ plane (Figure 3). Notice that after two periods, z_{t+2}^A is attracted toward either $z^{A+}\left(z^B\right)$ or $z^{A-}\left(z^B\right)$, with $z^{A*}\left(z^B\right)$ separating the two basins of attraction.

4. As with type A agents, perform Step 1 to 3 with type B agents. Replace z_{t+2}^A with z_{t+2}^B on the z-axis to obtain the three-dimensional space $\left(z_t^A, z_t^B\right) \times z_{t+2}^B$. The same argument shows that, in any cross-section plane of a given z^A, when the rest point is not stable, after two periods z_{t+2}^B moves toward either $z^{B+}\left(z^A\right)$ or $z^{B-}\left(z^A\right)$, depending on whether z_t^B lies to the right or the left of $z^{B*}\left(z^A\right)$.

5. Altogether, we have curves $z^{A+}\left(z^B\right), z^{A*}\left(z^B\right), z^{A-}\left(z^B\right)$ and $z^{B+}\left(z^A\right)$, $z^{B*}\left(z^A\right)$, $z^{B-}\left(z^A\right)$

in the same plane of $z_t^A \times z_t^B$ (Figure 3). We show that the straight line $z^B = -\frac{\alpha}{1-\alpha} z^A$ always stands in between $z^{B*}\left(z^A\right)$ and $z^{A*}\left(z^B\right)$, except at $(0,0)$ where the two intersect. Since $z^{A+}\left(z^B\right) \geq z^{A*}\left(z^B\right) \geq z^{A-}\left(z^B\right)$ and $z^{B+}\left(z^A\right) \geq z^{B*}\left(z^A\right) \geq z^{B-}\left(z^A\right)$, we conclude that there can be only two more intersections of those curves other than the origin: at $\left(z^{A+}, z^{B+}\right)$ from $z^{A+}\left(z^B\right)$ crossing with $z^{B+}\left(z^A\right)$, and at $\left(z^{A-}, z^{B-}\right)$ from $z^{A-}\left(z^B\right)$ crossing with $z^{B-}\left(z^A\right)$. The point $(0,0)$ is the unstable rest point, and the other two intersection points, $\left(z^{A+}, z^{B+}\right)$ and $\left(z^{A-}, z^{B-}\right)$, form a two-period cycle that is attracting.

Figure 3 illustrates the assembly of results as in Step 5[11].

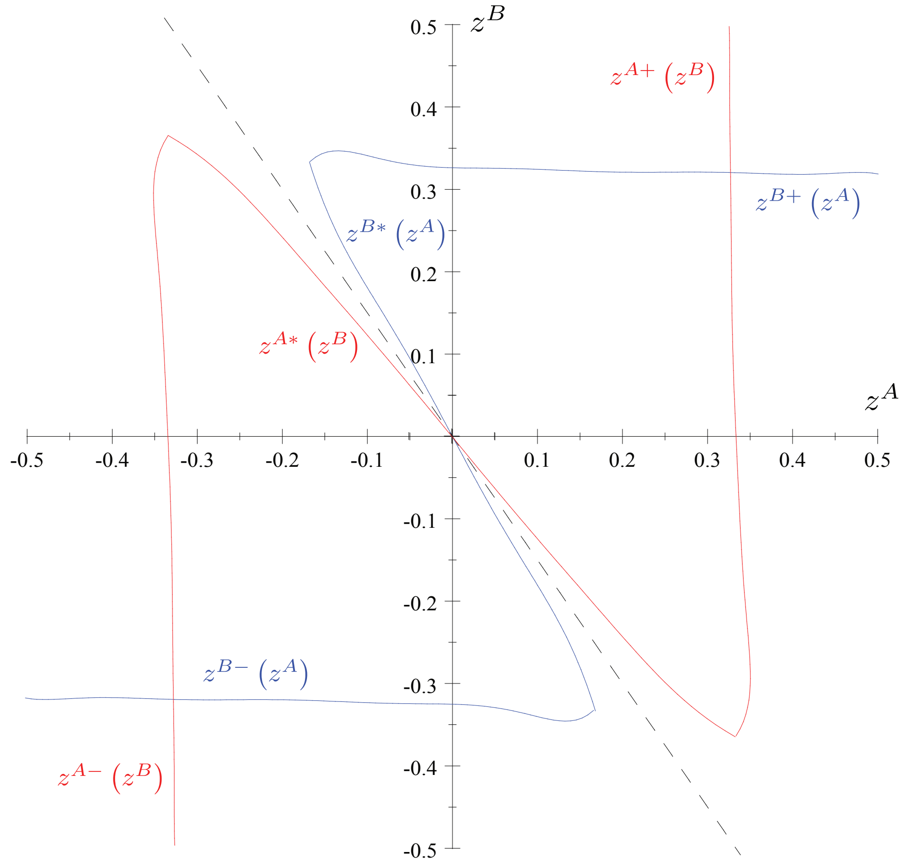


Figure 3: Illustration of Step 5.

[11] In the example illustrated, the population is comprised of 60% type A agents following the Logit rule with $\beta = 16$, and 40% type B agents following the Logit rule with $\beta = 10$. The information lag θ is set to be 0.8.

3.4 Multiple Heterogeneous Types and the Contrarians

Suppose the population consists of n types, each with a different response function $p^i(x), i = 1 \ldots n$. The proportion of types i agents in the whole population is exogenous and fixed at a^i. The dynamic equation of type i is

$$x_{t+1}^i = x_t^i - \theta x_t^i p^i \left(\sum_{j=1}^n a^j x_t^j \right) + \theta \left(1 - x_t^i\right) p^i \left(1 - \sum_{j=1}^n a^j x_t^j \right), i = 1 \ldots n.$$

The local stability condition requires that the eigenvalues of the Jacobian matrix at the rest point be less than 1 in absolute values. Without contrarian agents, the stability, similar as before, depends on the information lag and the *weighted average* of response sensitivities of all n types.

Proposition 10 *Assume $p^i\left(\frac{1}{2}\right) = \frac{1}{2}$ and $p^{i\prime}(x) > 0$ for all $i = 1 \ldots n$. The rest point $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ is locally stable if and only if $\frac{2}{\theta} > 1 + \sum_{i=1}^n a^i p^{i\prime}\left(\frac{1}{2}\right)$.*

The eigenvalues of the Jacobian matrix are found to be

$$\begin{aligned} \lambda_1 &= 1 - \theta - \theta \sum_{i=1}^n a^i p^{i\prime}\left(\tfrac{1}{2}\right) \\ \lambda_2 &= 1 - \theta. \end{aligned}$$

It is straightforward that λ_2 is always between $(0, 1)$. For non-contrarian agents, $p^{i\prime}(x) > 0, i = 1 \ldots n$ implies that $\lambda_1 < 1$, and the inequality in Proposition 10 ensures that $\lambda_1 > -1$. If the agents follow the the logit choice rule, the stability condition becomes

$$\frac{2}{\theta} > 1 + \frac{1}{2} l'\left(\tfrac{1}{2}\right) \sum_{i=1}^n a^i \beta^i. \tag{5}$$

The stability condition for a population of contrarian agents is quite different.

Proposition 11 *Assume $p^i\left(\frac{1}{2}\right) = \frac{1}{2}$ and $p^{i\prime}(x) < 0$ for all $i = 1 \ldots n$. The rest point $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ is locally stable if and only if $1 > -\sum_{i=1}^n a^i p^{i\prime}\left(\frac{1}{2}\right)$.*

If a population consists of all contrarian agents, $p^{i\prime}(x) < 0, i = 1 \ldots n$ implies that $\lambda_1 > 0$. The inequality in Proposition 11 is necessary and sufficient for $\lambda_1 < 1$. If the contrarian agents also follow the Logit choice rule, the stability condition becomes

$$1 > -\frac{1}{2} l' \left(\tfrac{1}{2}\right) \sum_{i=1}^{n} a^i \beta^i. \tag{6}$$

When comparing Equation (5) with Equation (6), we notice that, with the contrarians, the stability condition *does not* involve the information lag θ. The intuition is as follows: with a population of non-contrarian agents, the cycle occurs as a result of *overreaction*, which necessarily requires the information lag to be long enough to occur. With a population that consists of all contrarian agents, any deviation from the rest point is met with the agents' responses choosing the *more congested* option, augmenting the deviation. This happens even with a very small value of θ, as long as the absolute values of the β^i's are high. Since both the inflows to and the outflows from an option are proportional with θ, this parameter only affects how fast the augmentation is and plays no role in the stability condition.

When the population includes both non-contrarian and contrarian agents, if the weighted average of response sensitivities $\bar{\beta} = \sum_{i=1}^{n} a^i \beta^i > 0$, the population is *non-contrarian in general* and its dynamic can be described in a similar way as that of a non-contrarian population. The presence of those contrarian agents lowers the population's average response sensitivity $\bar{\beta}$, and mitigates or completely eliminates the oscillation that may otherwise occur. On the other hand, a population is *contrarian in general* if the weighted average of the responsiveness is less than 0, in which case the dynamic behaves just like the contrarian population described above, and its stability condition is independent of the information lag.

Nonetheless, the fact that the weighted average of response sensitivities determines the stability of the rest point yields the following implication: provided that proportion of agents with low β is small, they will be forced into a two-period cycle by a majority of high β agents. Injecting a small proportion of very patient drivers into an aggressive traffic scene may not stop the oscillating pattern of route use. However, this is not bad news for the less aggressive drivers, as they fair better in terms of avoiding congestion: if the route is currently more congested, a low β agent is more likely to wait until the congestion drops in the next period; while a high β agent hastily switches to the alternative, only to find it has become the more congested route. Agents would prefer a low β in an oscillating scenario, which will be further discussed in Section 4.

4 Endogenous Type Choice and Discussion on Response Functions

If the choice of an agent's response sensitivity becomes endogenous, we recognize that he may have incentives that steer his choice in opposite directions: an agent prefers low sensitivity of the perturbed best response so as to avoid oscillation. However, a high sensitivity is advantageous in choosing the less congested option, if the two options are asymmetric.

In the subsequent discussion, we use an asymmetric example in which the left option is four times less prone to congestion than the right: $l(x) = d(x/4)$ and $r(x) = d(x)$. If agents follow the logit rule with $\beta = 32$, the rest point is $x = 0.69$ using the left option. What is interesting about this rest point is that at $x = 0.69$ the congestion experienced on the left is *not* equal to that on the right. The disutilities from these two asymmetric options are the same if proportion 0.8 of agents used the left and 0.2 used the right, which is referred to as a user equilibrium. The distance between the the rest point and user equilibrium is a consequence of agents' *perturbed* best-responses. The higher the β in the logit choice rule, the closer the rest point is to the user equilibrium 0.8.

4.1 Endogenous Agent Types

If an agent can choose a "better performing" type as he wishes, what sort of criteria does he have in mind when evaluating his type choice β? We examine the following as agents' incentives:

1. An agent may feel "regret" from his previous decision, after seeing the realization. Specifically, if the realization of congestion in period $t + 1$ turns out to be worse than that in the previous period, he wishes he had chosen a higher switching probability than $p^{\beta}(x_t)$.

2. An agent always wants to be on the less congested of the two options; we call this the "envy" factor. If the congestion experienced on the left route is more than that on the right, the agent wants to switch away from the left with as high a probability as possible.

The above two utility measures have very different implications on the motion of type distribution. The "regret" factor effectively prompts an agent to decrease β when the dynamic is in oscillation. A high β type switches more often, yet is more likely to experience regret than a low β type when in cycles. As the type distribution shifts lower, the magnitude of the cycle diminishes or the cycle vanishes completely, after which the "regret" factor has no impact.

In contrast, the "envy" factor may point an agent in different directions. If the dynamic is convergent to a stable rest point, or oscillates not as far-reaching as to 0.8, the "envy" factor pushes up the β to incite more oscillation towards 0.8. Otherwise, if currently the dynamic oscillates beyond 0.8, the "envy" factor drives down the β and shrinks the cycle magnitude towards 0.8. The "envy" factor always contributes to oscillation as long as the rest point differs from the user equilibrium with asymmetric options.

The dynamic becomes more interesting when agents consider both the regret and envy factors. For clarity, we include only two types in the following simulation, a low $\beta_1 = 2$, and a high $\beta_2 = 32$, as an illustrative example. With agents following a monotonic dynamic that considers a weighted sum of the above two incentive measures for their type choice, Figure 4 shows the resulting time path of x_t, the usage of option left, as well as the associated changes of the proportion of agents with more sensitive responses. The time paths may be described as a meta-cycle of two-period cycles.

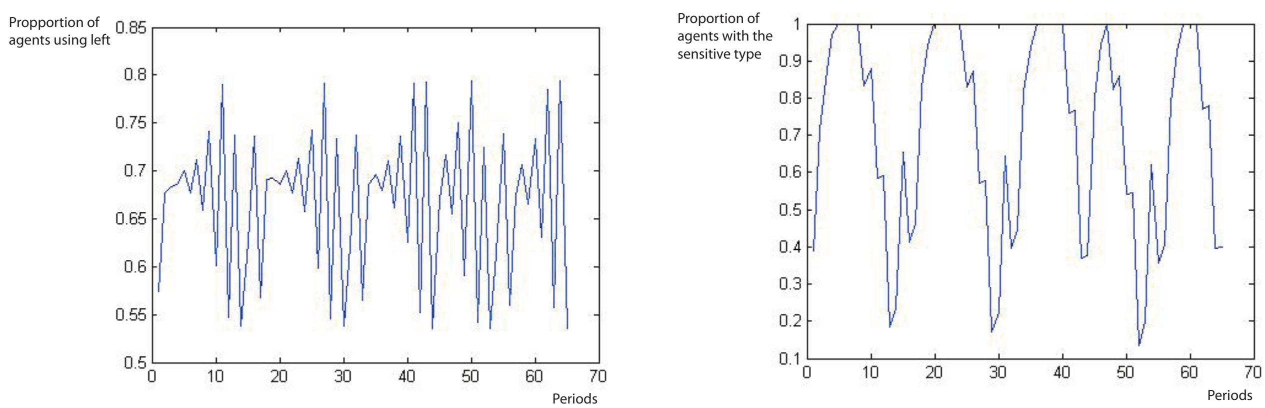


Figure 4: Time path of the proportion using option left, and that of the proportion of agents with more sensitive responses.

4.2 The Response Functions

The previous section presents a dilemma of choosing between stability and being close to the user equilibrium. This is also visible in Figure 5, which plots the rest points and the cycle magnitudes as the average type $\bar{\beta}$ increases. The "regret" factor is dormant until $\bar{\beta} = 17$, after which the rest point becomes unstable. Trying to approach the user equilibrium, the "envy" factor pushes up $\bar{\beta}$ and reaches for the cycle that involves 0.8, marked with diamonds in Figure 5. As the two-period cycle is initiated, the "regret" factor begins to take effect and brings the dynamic back to where the rest point is stable, which completes a full cycle of the meta-cycle.

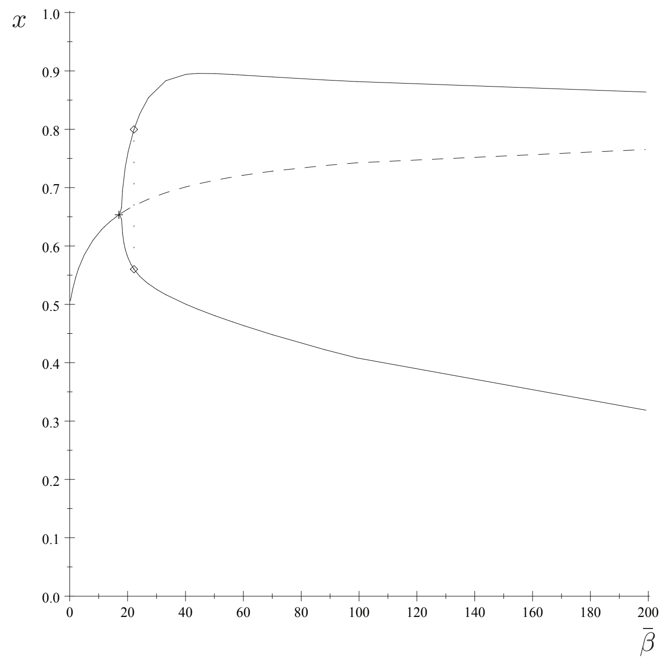


Figure 5: The rest point increases with $\bar{\beta}$ and approaches 0.8. It becomes unstable and bifurcates to two-period cycle as $\bar{\beta}$ passes the threshold $\bar{\beta} = 17$.

Rather than answering the question of which β is preferable, we examine the response functions, where the irreconcilability originates. Recall Equation (2) for the dynamic with binary asymmetric options:

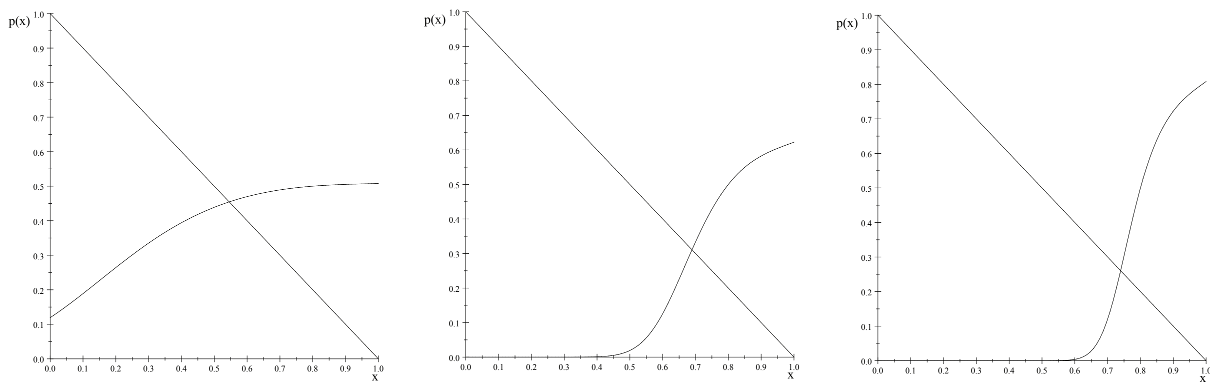
$$x_{t+1} = f(x_t) = x_t - \theta x_t p(x_t) + \theta(1 - x_t) q(1 - x_t).$$

With the condition $p(x) + q(1 - x) = 1$, the rest point is found at

$$p(x^*) = 1 - x^*.$$

The condition $p(x) + q(1 - x) = 1$ is a consistency requirement on the response functions: in response to congestion status $(x, 1 - x)$, the probability that an agent on the left switches to right is equal to the probability that a same agent, if he were put on the right, *remains* on the right, $p(x) = 1 - q(1 - x)$. In other words, faced with $(x, 1 - x)$, an agent is going to choose left with probability $q(1 - x)$ and choose right with probability $p(x)$ regardless of whether his current position is on the left or the right.

An intuitive interpretation of the equality $p(x^*) = 1 - x^*$ is: as an agent revises his options, the probability of switching to the right is always equal to the proportion of agents on the right at the rest point. Graphically, the rest point x^* lies at the intersection of the response function with the straight line $1-x$, which does not always coincide with the user equilibrium in the asymmetric case. Figures (6a) - (6c) show how the rest point approaches the user equilibrium 0.8 as $\bar{\beta}$ increases. As $\bar{\beta}$ climbs and the rest point gets closer and closer to 0.8, the slope at x^* also increases, which makes the dynamic prone to two-period cycle. On the other hand, as $\bar{\beta}$ descends and the rest point turns stable, x^* also departs the user equilibrium, and there rises the irreconcilability.

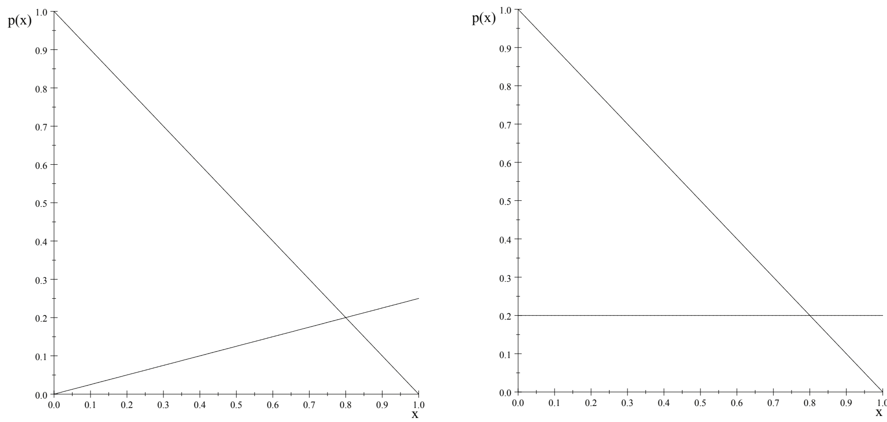


(a) $\bar{\beta} = 2, x^* = 0.55$ (b) $\bar{\beta} = 32, x^* = 0.69$ (c) $\bar{\beta} = 92, x^* = 0.74$

Figure 6: The intersections approach the user equilibrium 0.8 as $\bar{\beta}$ increases.

Nonetheless, there are response functions that do not suffer from the issue of irreconcilability, e.g., $p(x) = x/4$ or $p(x) = 0.2$. Both find the rest point at the user equilibrium 0.8 and the slope is moderate or flat, which keeps the rest point stable (Figure 7). The second response function, in particular, deserves special attention: if agents disregard (or do not receive) any information from the previous period, choosing the left with probability 0.8 and the right with probability 0.2 is a *Nash equilibrium* play for the whole population of agents in a congestion game with delays $l(x) = d(x/4)$ and $r(x) = d(x)$.

A completely flat response function like $p(x) = 0.2$ leads to a stable rest point that coincides with the user equilibrium, even with asymmetrically congested options. But do agents behave as this Nash equilibrium play instructs? Observations in various scenarios suggest otherwise: agents' responses *are* positively sensitive to even an outdated message. Farmers decide to increase the field area of one crop if its price last season was high (while the relevant price is determined by the supply and demand this season); on a crowded highway one is tempted to switch lanes if the flow in the neighboring lane has been moving faster for the last minute (while the fast flow of traffic that has passed matters less than how many drivers attempt to switch along the way); commuters listen to



(a) $p(x) = x/4$

(b) $p(x) = 0.2$

Figure 7: Two proposed response functions with stable rest points at the user equilibrium 0.8.

the radio traffic report half an hour before reaching the chosen route (while the experienced delay depends on fellow commuters' choices in the half hour after the radio report).

Before delving further into the discussion of agents' response functions, let us first look at a familiar situation in daily life. In a supermarket, there are two check-out aisles. The left aisle has four cash registers, and the right has only one. Suppose currently there are eight customers waiting on the left aisle and two customers on the right. For simplicity, assume that the processing time for each customer is a constant. Which aisle should the next customer choose? Since the waiting time for both aisles is the same, being indifferent, he may pick either or follow any probabilistic rule, e.g., 50% chance of both. If there are four customers on the left and four on the right, he should, of course, go for the left aisle with probability 1. In short, the next agent should best respond to what he observes. However, if instead of a single customer, he arrives in a group of 100 customers, all of whom observe that there are eight on the left and two on the right and decide simultaneously, how would his strategy be different? As a rational agent, he would realize that his strategy depends more on the choices of the remaining 99 in his group than the 12 already in the lines, and presume that something close to $(0.8, 0.2)$ may be a good choice, provided that the other 99 agents think alike.

In our model, parameter θ captures the (ir)relevance of the outdated information. Requiring all agents to stick to a flat response function is like asking the agents to discard the information after receiving it. A response function that is insensitive to the information may be sensible only when $\theta = 1$, i.e., the information arrives after the whole population has revised its choices. With

$\theta < 1$, the flat response function would not be a good fit as a description of agents' behavior. As seen in the previous supermarket example, if the customer arrives alone and finds that there are nine customers in the left aisle with four cash registers and only one in the right aisle with one cash register, he is not expected to choose left with probability 0.8; instead, his probability of choosing left is probably much lower.

Generally speaking, as θ gets closer to 0, the information is more relevant, and agents' responses should be more sensitive to it. However, there are also scenarios in which agents are not even clearly aware of the value of θ, in which case it is justifiable that their behavior is somewhat responsive to the message. For examples, farmers may not know precisely the proportion of fellow farmers who also change their crop area; a driver has little idea how many aggressive lane switchers there are ahead of him (usually he confidently assumes that his move precedes others'); commuters cannot perceive how many others are listening to the same radio traffic report. This is less of a problem with the group of consumers at the check-out aisles: their group size is obvious.[12]

4.3 The Rational Expectation Predictor

One of the observations in the last section implies that information lag may undermine the effectiveness of traffic information from ATIS devices: as the proportion of agents who respond to the ATIS traffic message (the market penetration rate of ATIS) approaches 1, a driver is supposed to disregard the ATIS information altogether. Partly for this reason, ATIS engineering literature differentiates *instantaneous* travel times (descriptive information) from *actual* travel times (prescriptive information, on the basis of ATIS-estimated travel times).[13] This section introduces a Rational Expectation (RE) predictor, as in Brock and Hommes (1997), which has perfect foresight and forecasts correctly the change in the coming period. If agents follow the RE predictor with probability α, the RE predictor is a solution to x_{t+1} in the equation:

$$x_{t+1} = x_t - \theta\left(1-\alpha\right) x_t p\left(x_t\right) + \theta\left(1-\alpha\right)\left(1-x_t\right) q\left(1-x_t\right) - \theta\alpha x_t p\left(x_{t+1}\right) + \theta\alpha\left(1-x_t\right) q\left(1-x_{t+1}\right).$$

Proposition 12 *With the consistency condition $p(x)+q(1-x)=1$, $\alpha > \frac{1}{2}$ is a sufficient condition for the rest point to be locally stable.*

[12] Some smart individuals in the group may move ahead of the rest, or switch aisles if the other aisle turns out to be shorter, but this changes the game then, and essentially makes $\theta \to 0$.

[13] See Bifulco et al. (2009).

When such a RE predictor is freely available through ATIS, it serves as a converging force toward the rest point. As long as the prediction is followed with a probability higher than $\frac{1}{2}$, the rest point is guaranteed to be stable, regardless of the response functions. On the other hand, a correct RE prediction requires the knowledge of the parameter θ as well as the response functions, which calls for empirical work to calibrate the responses of ATIS users.

5 Ongoing and Future Research

The ongoing research involves an empirical study of the fluctuations in road use and measuring its welfare implications. In the past decade, the use of loop detectors has collected lane-specific data on freeway traffic flows that make this study possible.

There are several potential directions that future research can go for. The model can be made more specific and be applied to scenarios of interest. As examples, the model can be made applicable to the choice between free lanes and toll lanes on High Occupancy Toll (HOT) lane facilities with dynamic toll pricing.[14] Other interesting congestion games in which the study of information lags is promising include market entry games in industrial organization, and how the lags in price information cause fluctuations in financial market of speculations.

I also intend to employ perspectives from behavioral and experimental economics to further our understanding of how a population of agents play in congestion games. This directly links to the discussion taking place in Section 4.2 on the response functions that describe agents' behavior facing information with lags. This model serves as a good framework in which we can calibrate the parameters of interests such as agents' response sensitivities.

A Appendix

Proof of Proposition 1.

$$\begin{aligned}
\frac{df}{dz} &= 1-\theta\left(p\left(\frac{1}{2}+z\right)+\left(\frac{1}{2}+z\right)p'\left(\frac{1}{2}+z\right)+p\left(\frac{1}{2}-z\right)+\left(\frac{1}{2}-z\right)p'\left(\frac{1}{2}-z\right)\right)\\
\left.\frac{df}{dz}\right|_{z=0} &= 1-2\theta p\left(\frac{1}{2}\right)-\theta p'\left(\frac{1}{2}\right)<1.
\end{aligned}$$

[14]On I-15 north of San Diego and I-394 in Minneapolis, tolls are updated every few minutes.

For $z = 0$ to be locally stable, we need

$$\begin{aligned}
& \left.\frac{df}{dz}\right|_{z=0} > -1 \\
\Leftrightarrow \quad & 1 - 2\theta p\left(\frac{1}{2}\right) - \theta p'\left(\frac{1}{2}\right) > -1 \\
\Leftrightarrow \quad & 2 > 2\theta p\left(\frac{1}{2}\right) + \theta p'\left(\frac{1}{2}\right) \\
\Leftrightarrow \quad & \frac{2}{\theta} > 2p\left(\frac{1}{2}\right) + p'\left(\frac{1}{2}\right).
\end{aligned}$$

■

Before proving Proposition 3, let us first establish the following useful Lemmas:

Lemma 13 $f(-z) = -f(z)$.

Lemma 14 $-\frac{1}{2} < f(z) < \frac{1}{2}$ *for all* $z \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Proof of Lemma 14. For all $z \in \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$\begin{aligned}
f(z) &= z - \theta\left(\frac{1}{2}+z\right)p\left(\frac{1}{2}+z\right) + \theta\left(\frac{1}{2}-z\right)p\left(\frac{1}{2}-z\right) \\
&< z + \theta\left(\frac{1}{2}-z\right)p\left(\frac{1}{2}-z\right) \\
&\leq z + \left(\frac{1}{2}-z\right) = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
f(z) &= z - \theta\left(\frac{1}{2}+z\right)p\left(\frac{1}{2}+z\right) + \theta\left(\frac{1}{2}-z\right)p\left(\frac{1}{2}-z\right) \\
&> z - \theta\left(\frac{1}{2}+z\right)p\left(\frac{1}{2}+z\right) \\
&\geq z - \left(\frac{1}{2}+z\right) = -\frac{1}{2}
\end{aligned}$$

Thus, $-\frac{1}{2} < f(z) < \frac{1}{2}$ for all $z \in [-\frac{1}{2}, \frac{1}{2}]$. ■

Lemma 15 $\dfrac{df}{dz} < 1$ *for all* $z \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Proof of Lemma 15. For all $z \in \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$\frac{df}{dz} = 1 - \theta\left(p\left(\frac{1}{2}+z\right) + p\left(\frac{1}{2}-z\right) + \left(\frac{1}{2}+z\right)p'\left(\frac{1}{2}+z\right) + \left(\frac{1}{2}-z\right)p'\left(\frac{1}{2}-z\right)\right)$$

since $\frac{1}{2} \pm z \geq 0$, $p\left(\frac{1}{2} \pm z\right) > 0$ and $p'\left(\frac{1}{2} \pm z\right) > 0$, $\dfrac{df}{dz} < 1$. ∎

Proof of Proposition 3. Lemma 14 shows that $f(z)$ is bounded between $-\frac{1}{2}$ and $\frac{1}{2}$, in particular, $f\left(\frac{1}{2}\right) > -\frac{1}{2}$ and $f\left(-\frac{1}{2}\right) < \frac{1}{2}$. If the rest point $z = 0$ is not stable, $\left.\dfrac{df}{dz}\right|_{z=0} < -1$, the curve $f(z)$ must cross the negative 45° line again at a pair of intersections, z^* and $-z^*$:

$$\begin{aligned} f(z^*) = z^* - \theta\left(\frac{1}{2} + z^*\right) p\left(\tfrac{1}{2} + z^*\right) + \theta\left(\frac{1}{2} - z^*\right) p\left(\tfrac{1}{2} - z^*\right) &= -z^* \\ \theta\left(\frac{1}{2} + z^*\right) p\left(\tfrac{1}{2} + z^*\right) - \theta\left(\frac{1}{2} - z^*\right) p\left(\tfrac{1}{2} - z^*\right) &= 2z^*. \end{aligned}$$

in the logit case, z^* is also unique and can be written out explicitly

$$z^* + \theta\left(\frac{e^{\beta l\left(\frac{1}{2} - z^*\right)}}{e^{\beta l\left(\frac{1}{2} + z^*\right)} + e^{\beta l\left(\frac{1}{2} - z^*\right)}} - \frac{1}{2} - z^*\right) = -z^*$$

$$\begin{aligned} z^* &= \frac{1}{\frac{4}{\theta} - 2} \frac{e^{\beta l\left(\frac{1}{2} + z^*\right)} - e^{\beta l\left(\frac{1}{2} - z^*\right)}}{e^{\beta l\left(\frac{1}{2} + z^*\right)} + e^{\beta l\left(\frac{1}{2} - z^*\right)}} \\ &= \tfrac{1}{\frac{4}{\theta} - 2} \tanh\left(\tfrac{\beta}{2}\left(l\left(\tfrac{1}{2} + z^*\right) - l\left(\tfrac{1}{2} - z^*\right)\right)\right) \end{aligned}$$

With Lemma 13, $f(z^*) = -z^*$ implies that $f(-z^*) = z^*$. This constitutes a symmetric two-period cycle that oscillates between z^* and $-z^*$. We will next show that this two-period cycle is also attracting and stable.

Due to symmetry, the proof only needs to focus on $z \in \left[0, \frac{1}{2}\right]$ and refer to absolute values.

1. If for all $z \in \left[0, \frac{1}{2}\right]$, $\dfrac{df}{dz} < 0$. In this case, for all $z \in [0, z^*)$, $f(z)$ is below the negative 45° line with a negative slope. Any $z \in [0, z^*)$ after one iteration becomes $f(z)$, whose absolute value is higher than the original z and approaches z^* monotonically from below. Any $z \in [0, z^*)$ initiates a dynamic of oscillating cycles with monotonically increasing magnitude until it reaches z^* and stays on the period 2 limit cycle $z^*, -z^*, z^*, -z^*, ...$ Similarly, for all $z \in (z^*, \frac{1}{2}]$, $f(z)$ is above the negative 45° line with a negative slope. Any $z \in (z^*, \frac{1}{2}]$ is mapped to a lower absolute value of $f(z)$ and approaches z^* monotonically from above. The oscillating cycles with monotonically decreasing magnitude also end at the period 2 limit cycle $z^*, -z^*, z^*, -z^*, ...$, which shows that the limit cycle is attracting and stable.

2. If for some $z \in (0, \frac{1}{2}]$, $\dfrac{df}{dz} > 0$. By continuity of $f(z)$, there must exist a turning point

z' such that $\left.\dfrac{df}{dz}\right|_{z=z'} = 0$ and $\forall z \in [0, \bar{z})$, $\dfrac{df}{dz} < 0$. In $[0, \bar{z})$, as in the first part of the proof, the oscillating cycles' magnitude increases monotonically and approaches z^* from below, until it either reaches z^* or steps into the region (z', z'') (with z'' being the next point with zero slope for some possible functional forms of $p(x)$, if there is one, otherwise $z'' = \frac{1}{2}$). For all $z \in (z', z'')$, $\dfrac{df}{dz} > 0$,. In addition, by Lemma 15, $0 < \dfrac{df}{dz} < 1$. What happens in (z', z'') looks just like a cobweb model with mappings between $f(z)$ and the negative 45° line. Lemma 15 guarantees that this is an attracting cobweb, and it will bring the dynamics to the intersection of $f(z)$ with the negative 45° line, which is just z^*. This time, however, instead of monotonically approaching from one direction, the magnitude is shrinking towards z^* from both directions. Again, the attracting limit is the period cycle $z^*, -z^*, z^*, -z^*, ...$

■

Proof of Proposition 4. We get $\frac{dz}{d\theta}$ from total differentiation:

$$2z - \theta\left(\frac{1}{2}+z\right)p\left(\frac{1}{2}+z\right) + \theta\left(\frac{1}{2}-z\right)p\left(\frac{1}{2}-z\right) = 0$$

$$\begin{aligned}
\frac{dz}{d\theta} &= \frac{\left(\frac{1}{2}+z\right)p\left(\frac{1}{2}+z\right) - \left(\frac{1}{2}-z\right)p\left(\frac{1}{2}-z\right)}{2 - \theta p\left(\frac{1}{2}+z\right) - \theta\left(\frac{1}{2}+z\right)p'\left(\frac{1}{2}+z\right) - \theta p\left(\frac{1}{2}-z\right) - \theta\left(\frac{1}{2}-z\right)p'\left(\frac{1}{2}-z\right)} \\
&= \frac{\frac{1}{2}\left(p\left(\frac{1}{2}+z\right) - p\left(\frac{1}{2}-z\right)\right) + z\left(p\left(\frac{1}{2}+z\right) + p\left(\frac{1}{2}-z\right)\right)}{2 - \theta\left(p\left(\frac{1}{2}+z\right) + \left(\frac{1}{2}+z\right)p'\left(\frac{1}{2}+z\right) + p\left(\frac{1}{2}-z\right) + \left(\frac{1}{2}-z\right)p'\left(\frac{1}{2}-z\right)\right)}
\end{aligned}$$

$\frac{1}{2}\left(p\left(\frac{1}{2}+z\right) - p\left(\frac{1}{2}-z\right)\right) + z\left(p\left(\frac{1}{2}+z\right) + p\left(\frac{1}{2}-z\right)\right) > 0$ by monotonicity of $p(.)$. We get the sign of the denominator from the condition $\left.\dfrac{df}{dz}\right|_{z} > -1$:

$$\begin{aligned}
\left.\frac{df}{dz}\right|_{z} &= 1 - \theta\left(p\left(\frac{1}{2}+z\right) + \left(\frac{1}{2}+z\right)p'\left(\frac{1}{2}+z\right) + p\left(\frac{1}{2}-z\right) + \left(\frac{1}{2}-z\right)p'\left(\frac{1}{2}-z\right)\right) > -1 \\
&\Rightarrow 2 - \theta\left(p\left(\frac{1}{2}+z\right) + \left(\frac{1}{2}+z\right)p'\left(\frac{1}{2}+z\right) + p\left(\frac{1}{2}-z\right) + \left(\frac{1}{2}-z\right)p'\left(\frac{1}{2}-z\right)\right) > 0
\end{aligned}$$

The above shows that the sign of the denominator is also positive. We thus are able to conclude that $\dfrac{\partial z}{\partial \theta} > 0$. ■

Proof of Proposition 5. The stationary conditions for type i agents $(i = A, B)$ is:

$$\begin{aligned} & x_{t+1}^i - x_t^i = 0 \\ \Leftrightarrow\ & \theta x_t^i p^i(x_t) = \theta\left(1 - x_t^i\right) p^i(1 - x_t) \\ \Leftrightarrow\ & x_t^i = \frac{p^i(1 - x_t)}{p^i(x_t) + p^i(1 - x_t)} \end{aligned}$$

The rest point $\left(x^{A*}, x^{B*}\right)$ lies at the intersection of the implicit functions $x^A\left(x^B\right)$ and $x^B\left(x^A\right)$. $(x = \alpha x^A + (1 - \alpha) x^B)$:

$$\begin{aligned} x^A\left(x^B\right) &= \frac{p^A(1 - x)}{p^A(x) + p^A(1 - x)} \\ x^B\left(x^A\right) &= \frac{p^B(1 - x)}{p^B(x) + p^B(1 - x)}. \end{aligned}$$

We next show that $\left|\frac{\partial x^B\left(x^A\right)}{\partial x^A}\right| < \frac{1}{\left|\frac{\partial x^A\left(x^B\right)}{\partial x^B}\right|}$, i.e. the slope of $x^B\left(x^A\right)$ is always less than that of $x^A\left(x^B\right)$, which implies that $x^B\left(x^A\right)$ and $x^A\left(x^B\right)$ cross with each other once and only once at the unique rest point $\left(\frac{1}{2}, \frac{1}{2}\right)$:

$$\begin{aligned} dx^A &= (-\alpha) \frac{p^{A\prime}(1 - x) p^A(x) + p^A(1 - x) p^{A\prime}(x)}{\left(p^A(x) + p^A(1 - x)\right)^2} dx^A \\ &\quad - (1 - \alpha) \frac{p^{A\prime}(1 - x) p^A(x) + p^A(1 - x) p^{A\prime}(x)}{\left(p^A(x) + p^A(1 - x)\right)^2} dx^B \end{aligned}$$

For shorthand, let $q^A(x) = \frac{p^{A\prime}(1-x)p^A(x)+p^A(1-x)p^{A\prime}(x)}{\left(p^A(x)+p^A(1-x)\right)^2}$, we have:

$$\begin{aligned} dx^A &= (-\alpha) q^A(x) dx^A - (1 - \alpha) q^A(x) dx^B \\ \frac{\partial x^A}{\partial x^B} &= -\frac{(1 - \alpha) q^A(x)}{1 + \alpha q^A(x)} \end{aligned}$$

$$\begin{aligned} dx^B &= -(1 - \alpha) \frac{p^{B\prime}(1 - x) p^B(x) + p^B(1 - x) p^{B\prime}(x)}{\left(p^B(x) + p^B(1 - x)\right)^2} dx^B \\ &\quad + (-\alpha) \frac{p^{B\prime}(1 - x) p^B(x) + p^B(1 - x) p^{B\prime}(x)}{\left(p^B(x) + p^B(1 - x)\right)^2} dx^A \end{aligned}$$

Also, let $q^B(x) = \frac{p^{B\prime}(1-x)p^B(x)+p^B(1-x)p^{B\prime}(x)}{\left(p^B(x)+p^B(1-x)\right)^2}$, we get:

$$\begin{aligned}
dx^B &= -(1-\alpha)\,q^B(x)\,dx^B + (-\alpha)\,q^B(x)\,dx^A \\
\frac{\partial x^B}{\partial x^A} &= -\frac{\alpha q^B(x)}{1+(1-\alpha)\,q^B(x)}.
\end{aligned}$$

Finally, if we compare the two slopes, we find that

$$\begin{aligned}
& \left|\frac{\partial x^B}{\partial x^A}\right| < \frac{1}{\left|\frac{\partial x^A}{\partial x^B}\right|} \\
\Leftrightarrow\ & \frac{\alpha q^B(x)}{1+(1-\alpha)\,q^B(x)} < \frac{1+\alpha q^A(x)}{(1-\alpha)\,q^A(x)} \\
\Leftrightarrow\ & \alpha(1-\alpha)\,q^A(x)\,q^B(x) < 1+\alpha q^A(x)+(1-\alpha)\,q^B(x)+\alpha(1-\alpha)\,q^B(x)\,q^A(x) \\
\Leftrightarrow\ & 0 < 1+\alpha q^A(x)+(1-\alpha)\,q^B(x)
\end{aligned}$$

The last inequality always holds because $q^A(x) > 0$ and $q^B(x) > 0$.

$\left|\frac{\partial x^B}{\partial x^A}\right| < \frac{1}{\left|\frac{\partial x^A}{\partial x^B}\right|}$ implies that the slope of $x^B\left(x^A\right)$ is always flatter than that of $x^A\left(x^B\right)$, so that once $x^B\left(x^A\right)$ intersects with $x^A\left(x^B\right)$ at $\left(x^{A*}, x^{B*}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$, the two curves never cross again. This makes $\left(\frac{1}{2}, \frac{1}{2}\right)$ the unique rest point of the dynamic. ■

Proof of Proposition 6. For the stability condition at $(0,0)$, we check if the eigenvalues of the Jacobian matrix are less than 1 in absolute value.

$$\begin{aligned}
\frac{\partial z^A_{t+1}}{\partial z^A_t} &= 1-\theta p^A\left(\tfrac{1}{2}+z_t\right)-\alpha\theta\left(\frac{1}{2}+z^A_t\right)p^{A\prime}\left(\tfrac{1}{2}+z_t\right)-\theta p^A\left(\tfrac{1}{2}-z_t\right)-\alpha\theta\left(\frac{1}{2}-z^A_t\right)p^{A\prime}\left(\tfrac{1}{2}-z_t\right) \\
\frac{\partial z^B_{t+1}}{\partial z^B_t} &= 1-\theta p^B\left(\tfrac{1}{2}+z_t\right)-(1-\alpha)\,\theta\left(\frac{1}{2}+z^B_t\right)p^{B\prime}\left(\tfrac{1}{2}+z_t\right) \\
&\quad -\theta p^B\left(\tfrac{1}{2}-z_t\right)-(1-\alpha)\,\theta\left(\frac{1}{2}-z^B_t\right)p^{B\prime}\left(\tfrac{1}{2}-z_t\right) \\
\frac{\partial z^A_{t+1}}{\partial z^B_t} &= -(1-\alpha)\,\theta\left(\left(\frac{1}{2}+z^A_t\right)p^{A\prime}\left(\tfrac{1}{2}+z_t\right)+\left(\frac{1}{2}-z^A_t\right)p^{A\prime}\left(\tfrac{1}{2}-z_t\right)\right) \\
\frac{\partial z^B_{t+1}}{\partial z^A_t} &= -\alpha\theta\left(\left(\frac{1}{2}+z^B_t\right)p^{B\prime}\left(\tfrac{1}{2}+z_t\right)+\left(\frac{1}{2}-z^B_t\right)p^{B\prime}\left(\tfrac{1}{2}-z_t\right)\right)
\end{aligned}$$

The Jacobian matrix at the rest point $(0,0)$ is:

$$\begin{bmatrix} \frac{\partial z_{t+1}^A}{\partial z_t^A} & \frac{\partial z_{t+1}^A}{\partial z_t^B} \\ \frac{\partial z_{t+1}^B}{\partial z_t^A} & \frac{\partial z_{t+1}^B}{\partial z_t^B} \end{bmatrix}_{\left(z_t^A, z_t^B\right)=(0,0)} = \begin{bmatrix} 1-2\theta p^A\left(\frac{1}{2}\right)-\alpha\theta p^{A\prime}\left(\frac{1}{2}\right) & -\left(1-\alpha\right)\theta p^{A\prime}\left(\frac{1}{2}\right) \\ -\alpha\theta p^{B\prime}\left(\frac{1}{2}\right) & 1-2\theta p^B\left(\frac{1}{2}\right)-\left(1-\alpha\right)\theta p^{B\prime}\left(\frac{1}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 1-\theta-\alpha\theta p^{A\prime}\left(\frac{1}{2}\right) & -\left(1-\alpha\right)\theta p^{A\prime}\left(\frac{1}{2}\right) \\ -\alpha\theta p^{B\prime}\left(\frac{1}{2}\right) & 1-\theta-\left(1-\alpha\right)\theta p^{B\prime}\left(\frac{1}{2}\right) \end{bmatrix}$$

The two eigenvalues of this matrix are $\lambda_1 = 1-p^{B\prime}\left(\frac{1}{2}\right)\theta\left(1-\alpha\right)-p^{A\prime}\left(\frac{1}{2}\right)\theta\alpha-\theta$ and $\lambda_2 = 1-\theta$.

Since $\theta \leq 1$, the eigenvalue $1-\theta$ is within $(-1,1)$. Next we check the other eigenvalue. Because $p^{A\prime}\left(\frac{1}{2}\right) > 0$ and $p^{B\prime}\left(\frac{1}{2}\right) > 0$, $\lambda_1 < 1$, for local stability, we need only one more condition, that $\lambda_1 > -1$. The rest point $(0,0)$ is locally stable if and only if

$$\begin{aligned} & 1-p^{B\prime}\left(\frac{1}{2}\right)\theta\left(1-\alpha\right)-p^{A\prime}\left(\frac{1}{2}\right)\theta\alpha-\theta > -1 \\ \Leftrightarrow\quad & \frac{2}{\theta} > \left(1-\alpha\right)p^{B\prime}\left(\frac{1}{2}\right)+\alpha p^{A\prime}\left(\frac{1}{2}\right)+1 \end{aligned}$$

■

Proof of Propostision 9. The proof follows the steps from our plan in Section 3.

As in Step 1, we take on

$$z_{t+2}^A = \widetilde{f^A}\left(z_t^A, z_t^B\right) = f^A\left(f^A\left(z_t^A, z_t^B\right), f^B\left(z_t^A, z_t^B\right)\right)$$

as a 3-dimensional space of $\left(z_t^A, z_t^B\right) \times z_{t+2}^A$ (Figure ??):

Cut a cross section with some $z_t^B = z^B \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ (Figure ??)to get a plane with dimension $z_t^A \times z_{t+2}^A$.

Proving Step 2, we first show that the dynamic equations are bounded between $-\frac{1}{2}$ and $\frac{1}{2}$:

$$\begin{aligned} f^A\left(z_t^A, z_t^B\right) &= z_t^A-\theta\left(\frac{1}{2}+z_t^A\right)p^A\left(\tfrac{1}{2}+\alpha z_t^A+\left(1-\alpha\right)z_t^B\right)+\theta\left(\frac{1}{2}-z_t^A\right)p^A\left(\tfrac{1}{2}-\alpha z_t^A-\left(1-\alpha\right)z_t^B\right) \\ &< z_t^A+\theta\left(\frac{1}{2}-z_t^A\right)p^A\left(\tfrac{1}{2}-\alpha z_t^A-\left(1-\alpha\right)z_t^B\right) \\ &< z_t^A+\left(\frac{1}{2}-z_t^A\right) = \frac{1}{2}. \end{aligned}$$

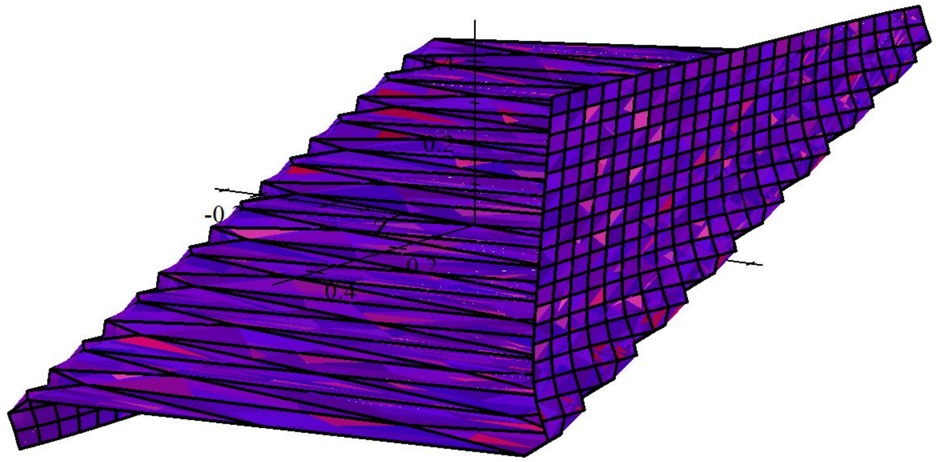


Figure 8: $z_{t+2}^A = \widetilde{f^A}\left(z_t^A, z_t^B\right)$.

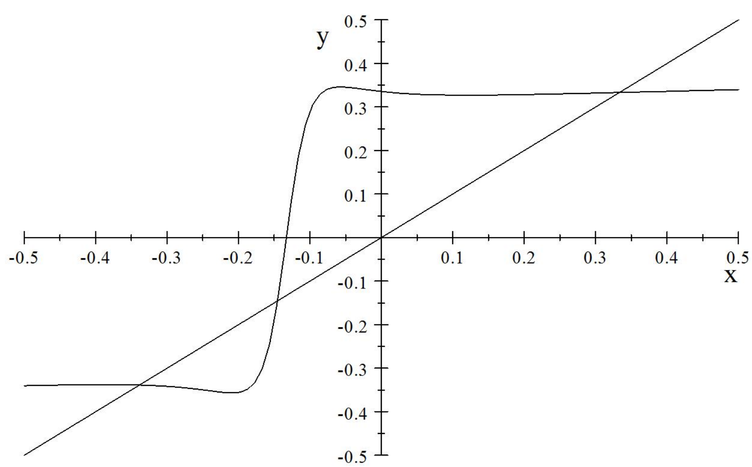


Figure 9: Cross section of $z_t^B = z^B \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ from $\widetilde{f^A}\left(z_t^A, z_t^B\right)$.

$$\begin{aligned}
f^A\left(z_t^A, z_t^B\right) &= z_t^A - \theta\left(\frac{1}{2} + z_t^A\right) p^A\left(\tfrac{1}{2} + \alpha z_t^A + (1-\alpha)\, z_t^B\right) + \theta\left(\frac{1}{2} - z_t^A\right) p^A\left(\tfrac{1}{2} - \alpha z_t^A - (1-\alpha)\, z_t^B\right) \\
&> z_t^A - \theta\left(\frac{1}{2} + z_t^A\right) p^A\left(\tfrac{1}{2} + \alpha z_t^A + (1-\alpha)\, z_t^B\right) \\
&> z_t^A - \left(\frac{1}{2} + z_t^A\right) = -\frac{1}{2}
\end{aligned}$$

The same can be said about $f^B\left(z_t^A, z_t^B\right)$. Summing the above, we have:

$$-\frac{1}{2} < f^A\left(z_t^A, z_t^B\right) < \frac{1}{2} \text{ and } -\frac{1}{2} < f^B\left(z_t^A, z_t^B\right) < \frac{1}{2}, \forall z_t^A, z_t^B \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Since $\widetilde{f^A}\left(z_t^A, z_t^B\right)$ and $\widetilde{f^B}\left(z_t^A, z_t^B\right)$ are both derived from the one-step dynamic equations, it is implied that $\widetilde{f^A}\left(z_t^A, z_t^B\right)$ and $\widetilde{f^B}\left(z_t^A, z_t^B\right)$ are bounded between $-\frac{1}{2}$ and $\frac{1}{2}$.

We are now ready to begin the proof for Step 2:

Proof for Step 2. In the cross-section plane of a given z^B, $\widetilde{f^A}\left(z_t^A, z^B\right)$ intersects the 45° line at point z^{A*}. If the dynamic is not stable at z^{A*}, $\left.\frac{d\widetilde{f^A}\left(z_t^A, z^B\right)}{dz_t^A}\right|_{z_t^A = z^{A*}} > 1$, i.e., the slope of $\widetilde{f^A}\left(z_t^A, z^B\right)$ at z^{A*} is greater than 1. By the boundedness of the dynamic equations, $\widetilde{f^A}\left(\frac{1}{2}, z^B\right) < \frac{1}{2}$ and $\widetilde{f^A}\left(-\frac{1}{2}, z^B\right) > -\frac{1}{2}$ This implies that the curve of $\widetilde{f^A}\left(z_t^A, z^B\right)$ must cross the 45° line at least once in $\left(z^{A*}, \frac{1}{2}\right)$ and once in $\left(-\frac{1}{2}, z^{A*}\right)$. Refer to the other two intersections as $z^{A+} \in \left(z^{A*}, \frac{1}{2}\right)$ and $z^{A-} \in \left(-\frac{1}{2}, z^{A*}\right)$, respectively. By the construction of the 2-period dynamic, $z^{A+}, z^{A-}, z^{A+}, z^{A-}, \ldots$ constitute a two-period cycle within the cross section of $z_t^B = z^B$. Furthermore, the fact that $\widetilde{f^A}\left(z_t^A, z^B\right)$ intersects the 45° line at z^{A+} (or z^{A-}) from above (or below) implies that $\left.\frac{d\widetilde{f}\left(z_t^A, z^B\right)}{dz_t^A}\right|_{z_t^A = z^{A+}} < 1$ (or $\left.\frac{d\widetilde{f}\left(z_t^A, z^B\right)}{dz_t^A}\right|_{z_t^A = z^{A-}} < 1$), so the two-period cycle is also stable. For all $z_t^A \in (z^{A*}, \frac{1}{2}]$, $z_{t+2}^A = \widetilde{f^A}\left(z_t^A, z^B\right)$ moves toward the point z^{A+}; for all $z_t^A \in [-\frac{1}{2}, z^{A*})$, z_{t+2}^A moves toward z^{A-}. ■

Although we will no longer be in the cross-section plane of z^B after just one period, the above proof is useful in the sense that it indicates the direction of movement from z_t^A to z_{t+2}^A after two periods, when we start with $z_t^B = z^B$. In Step 3, we apply Step 2 for every cross section of z^B from $-\frac{1}{2}$ to $\frac{1}{2}$, and collect all such points z^{A+}, z^{A*}, z^{A-} to form continuous curves $z^{A+}\left(z^B\right), z^{A*}\left(z^B\right), z^{A-}\left(z^B\right)$ in the $z_t^A \times z_t^B$ plane. We now know that z_{t+2}^A moves toward $z^{A+}\left(z^B\right)$ or $z^{A-}\left(z^B\right)$, with $z^{A*}\left(z^B\right)$ separating the two basins of attraction.

In Step 4, The same argument follows with type B agents: Starting with any given z^A, z_{t+2}^B moves toward either $z^{B+}\left(z^A\right)$ or $z^{B-}\left(z^A\right)$ after two periods, depending on whether z_t^B lies to the right or left of $z^{B*}\left(z^A\right)$.

In the final step (Step 5) of assembly, we put all the curves, $z^{A+}\left(z^B\right), z^{A*}\left(z^B\right), z^{A-}\left(z^B\right)$ and $z^{B+}\left(z^A\right)$,$z^{B*}\left(z^A\right), z^{B-}\left(z^A\right)$, together in the $z_t^A \times z_t^B$ plane. We then show that the straight line through the origin with slope $-\frac{\alpha}{1-\alpha}$ always stands in-between $z^{B*}\left(z^A\right)$ and $z^{A*}\left(z^B\right)$ except at $(0,0)$ where the two meet. The formal statement of that proposition is:

Provided that $p^A\left(\frac{1}{2}\right)=\frac{1}{2}$, and $p^B\left(\frac{1}{2}\right)=\frac{1}{2}$,

$$\begin{aligned} z^{B*}\left(z^A\right) &< -\frac{\alpha}{1-\alpha}z^A, \forall z^A>0, \text{ and } z^{B*}\left(z^A\right) > -\frac{\alpha}{1-\alpha}z^A, \forall z^A<0. \\ \text{Similarly, } z^{A*}\left(z^B\right) &< -\frac{1-\alpha}{\alpha}z^B, \forall z^B>0, \text{ and } z^{A*}\left(z^B\right) < -\frac{1-\alpha}{\alpha}z^B, \forall z^B<0. \end{aligned}$$

Proof for Step 5. Without loss of generality, we prove the proposition in a cross section of z^B from $\widetilde{f^A}\left(z_t^A, z^B\right)$, and let $z^B>0$. Define $\bar{z}^A=-\frac{1-\alpha}{\alpha}z^B$, which is on the straight line through the origin with slope $-\frac{\alpha}{1-\alpha}$, and we are interested to see how the dynamic moves after two periods from the point $\left(\bar{z}^A, z^B\right)$.

After one period,

$$\begin{aligned} f^A\left(\bar{z}^A, z^B\right) &= \bar{z}^A-\theta\left(\frac{1}{2}+\bar{z}^A\right)p^A\left(\tfrac{1}{2}+\alpha\bar{z}^A+(1-\alpha)z^B\right)+\theta\left(\frac{1}{2}-\bar{z}^A\right)p^A\left(\tfrac{1}{2}-\alpha\bar{z}^A-(1-\alpha)z^B\right) \\ &= \bar{z}^A-\theta\left(\frac{1}{2}+\bar{z}^A\right)p^A\left(\tfrac{1}{2}\right)+\theta\left(\frac{1}{2}-\bar{z}^A\right)p^A\left(\tfrac{1}{2}\right) \\ &= \bar{z}^A-\bar{z}^A\theta \\ &= \bar{z}^A(1-\theta) \end{aligned}$$

$$\begin{aligned} f^B\left(\bar{z}^A, z^B\right) &= z^B-\theta\left(\frac{1}{2}+z^B\right)p^B\left(\tfrac{1}{2}+\alpha\bar{z}^A+(1-\alpha)z^B\right)+\theta\left(\frac{1}{2}-z^B\right)p^B\left(\tfrac{1}{2}-\alpha\bar{z}^A-(1-\alpha)z^B\right) \\ &= z^B-\theta\left(\frac{1}{2}+z^B\right)p^B\left(\tfrac{1}{2}\right)+\theta\left(\frac{1}{2}-z^B\right)p^B\left(\tfrac{1}{2}\right) \\ &= z^B(1-\theta). \end{aligned}$$

This shows that $f^A\left(\bar{z}^A, z^B\right)=-\frac{1-\alpha}{\alpha}f^B\left(\bar{z}^A, z^B\right)$ if $\bar{z}^A=-\frac{1-\alpha}{\alpha}z^B$, and the point $\left(\bar{z}^A(1-\theta), z^B(1-\theta)\right)$ is still on the same line with slope $-\frac{\alpha}{1-\alpha}$ after one period.

After two periods,

$$\begin{aligned} z_{t+2}^A &= \widetilde{f^A}\left(z_t^A, z^B\right)=f^A\left(f^A\left(\bar{z}^A, z^B\right), f^B\left(\bar{z}^A, z^B\right)\right) \\ &= f^A\left(\bar{z}^A(1-\theta), z^B(1-\theta)\right) \\ &= \bar{z}^A(1-\theta)^2 \\ &> \bar{z}^A. \end{aligned}$$

Notice that both $\bar{z}^A$ and $\bar{z}^A(1-\theta)^2$ are less than zero. The fact that $\bar{z}^A(1-\theta)^2>\bar{z}^A$ means that

$\left(\bar{z}^A, \bar{z}^A (1-\theta)^2\right)$, a point on the $\widetilde{f^A}\left(z_t^A, z^B\right)$ curve, is above the 45° line in the lower left quadrant of $z_t^A \times z_{t+2}^A$ plane. Our point of interest is z^{A*}, which is the intersection of the $\widetilde{f^A}\left(z_t^A, z^B\right)$ curve with the 45° line. To show that $z^{A*} < \bar{z}^A = -\frac{1-\alpha}{\alpha} z^B$, we simply need to prove that the slope of $\widetilde{f^A}\left(z_t^A, z^B\right)$ is positive from $\bar{z}^A$ to z^{A*}.

We already have $\left.\frac{d\widetilde{f^A}\left(z_t^A, z^B\right)}{dz_t^A}\right|_{z_t^A = z^{A*}} > 1$ because the dynamic is assumed to be unstable; it remains for us to show that $\left.\frac{d\widetilde{f^A}\left(z_t^A, z^B\right)}{dz_t^A}\right|_{z_t^A = \bar{z}^A} > 0$:

$$
\begin{aligned}
\left.\frac{d\widetilde{f^A}\left(z_t^A, z^B\right)}{dz_t^A}\right|_{z_t^A = \bar{z}^A} &= f_1^A\left(\bar{z}^A (1-\theta), z^B (1-\theta)\right) * f_1^A\left(\bar{z}^A, z^B\right) + f_2^A\left(\bar{z}^A (1-\theta), z^B (1-\theta)\right) * f_1^B\left(\bar{z}^A, z^B\right) \\
&= \left(1-\theta-\alpha\theta p^{A\prime}\left(\tfrac{1}{2}\right)\right)^2 + \left(-(1-\alpha)\theta p^{A\prime}\left(\tfrac{1}{2}\right)\right)\left(-\alpha\theta p^{B\prime}\left(\frac{1}{2}\right)\right) \\
&= \left(1-\theta-\alpha\theta p^{A\prime}\left(\tfrac{1}{2}\right)\right)^2 + \alpha(1-\alpha)\theta^2 p^{A\prime}\left(\tfrac{1}{2}\right) p^{B\prime}\left(\frac{1}{2}\right) \\
&> 0
\end{aligned}
$$

This completes the Proof for Step 5. ■

With the above proof, we learn that the straight line through the origin with slope $-\frac{\alpha}{1-\alpha}$ separates $z^{B*}\left(z^A\right)$ and $z^{A*}\left(z^B\right)$. Since $z^{A+}\left(z^B\right) \geq z^{A*}\left(z^B\right) \geq z^{A-}\left(z^B\right)$ and $z^{B+}\left(z^A\right) \geq z^{B*}\left(z^A\right) \geq z^{B-}\left(z^A\right)$, we conclude that there can be only two other intersections: $\left(z^{A+}, z^{B+}\right)$ from $z^{A+}\left(z^B\right)$ crossing with $z^{B+}\left(z^A\right)$, and $\left(z^{A-}, z^{B-}\right)$ from $z^{A-}\left(z^B\right)$ crossing with $z^{B-}\left(z^A\right)$. The point $(0,0)$ is the unstable rest point, and the two other intersection points, $\left(z^{A+}, z^{B+}\right)$ and $\left(z^{A-}, z^{B-}\right)$, form a two-period cycle that is attracting, which finalizes the Proof of Proposition 9. ■

Proof of Proposition 10. Similar to the Proof of Proposition 6 for local stability conditions with two heterogenous types, we check if the eigenvalues of our Jacobian matrix with n types are less than 1 in absolute value.

$$
\begin{aligned}
\left.\frac{\partial z_{t+1}^i}{\partial z_t^i}\right|_{\left(z_t^1, \ldots, z_t^n\right) = (0, \ldots, 0)} &= 1-\theta-a^i\theta p^{i\prime}\left(\tfrac{1}{2}\right) \\
\left.\frac{\partial z_{t+1}^i}{\partial z_t^j}\right|_{\left(z_t^1, \ldots, z_t^n\right) = (0, \ldots, 0)} &= -a^j\theta p^{i\prime}\left(\tfrac{1}{2}\right)
\end{aligned}
$$

The Jacobian matrix is:

$$\begin{bmatrix} \frac{\partial z^1_{t+1}}{\partial z^1_t} & \frac{\partial z^1_{t+1}}{\partial z^2_t} & \cdots & \frac{\partial z^1_{t+1}}{\partial z^n_t} \\ \frac{\partial z^2_{t+1}}{\partial z^1_t} & \frac{\partial z^2_{t+1}}{\partial z^2_t} & \cdots & \frac{\partial z^2_{t+1}}{\partial z^n_t} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial z^n_{t+1}}{\partial z^1_t} & \frac{\partial z^n_{t+1}}{\partial z^2_t} & \cdots & \frac{\partial z^n_{t+1}}{\partial z^n_t} \end{bmatrix}_{\left(z^1_t,\ldots,z^n_t\right)=(0,\ldots,0)}$$

$$= \begin{bmatrix} 1-\theta-a^1\theta p^{1\prime}\left(\frac{1}{2}\right) & -a^2\theta p^{1\prime}\left(\frac{1}{2}\right) & \cdots & -a^n\theta p^{1\prime}\left(\frac{1}{2}\right) \\ -a^1\theta p^{2\prime}\left(\frac{1}{2}\right) & 1-\theta-a^2\theta p^{2\prime}\left(\frac{1}{2}\right) & \cdots & -a^n\theta p^{2\prime}\left(\frac{1}{2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ -a^1\theta p^{n\prime}\left(\frac{1}{2}\right) & -a^2\theta p^{n\prime}\left(\frac{1}{2}\right) & \cdots & 1-\theta-a^n\theta p^{n\prime}\left(\frac{1}{2}\right) \end{bmatrix}.$$

There are two eigenvalues. One is

$$\lambda_1 = 1-\theta-\theta\sum_{i=1}^{n} a^i p^{i\prime}\left(\frac{1}{2}\right).$$

which is associated with the eigenvector $\left\{\begin{matrix} \frac{p^{1\prime}}{p^{n\prime}} \\ \frac{p^{2\prime}}{p^{n\prime}} \\ \vdots \\ \frac{p^{n\prime}}{p^{n\prime}} \end{matrix}\right\}$.

And the other eigenvalue is

$$\lambda_2 = 1-\theta$$

which is associated with $n-1$ eigenvectors: $\left\{\begin{matrix} -\frac{a^2}{a^1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\}, \left\{\begin{matrix} -\frac{a^3}{a^1} \\ 0 \\ 1 \\ \vdots \\ 0 \end{matrix}\right\}, \ldots, \left\{\begin{matrix} -\frac{a^n}{a^1} \\ 0 \\ 0 \\ \vdots \\ 1 \end{matrix}\right\}$.

Since $\theta \leq 1$, we have $-1 < \lambda_2 = 1-\theta < 1$.

Considering the fact that $1-\theta-\theta\sum_{i=1}^{n} a^i p^{i\prime}\left(\frac{1}{2}\right) < 1$, the rest point $(0,0)$ is locally stable if and only if

$$
\begin{aligned}
& 1-\theta-\theta \sum_{i=1}^{n} a^{i} p^{i\prime}\left(\frac{1}{2}\right)>-1 \\
\Leftrightarrow \quad & \frac{2}{\theta}>1+\sum_{i=1}^{n} a^{i} p^{i\prime}\left(\frac{1}{2}\right) .
\end{aligned}
$$

■

Proof of Proposition 12.

$$
\begin{aligned}
x_{t+1} &= x_t-\theta(1-\alpha) x_t p\left(x_t\right)+\theta(1-\alpha)\left(1-x_t\right) q\left(1-x_t\right)-\theta \alpha x_t p\left(x_{t+1}\right)+\theta \alpha\left(1-x_t\right) q\left(1-x_{t+1}\right) \\
x_{t+1} &= x_t-\theta(1-\alpha) x_t p\left(x_t\right)+\theta(1-\alpha)\left(1-x_t\right)\left(1-p\left(x_t\right)\right)-\theta \alpha x_t p\left(x_{t+1}\right)+\theta \alpha\left(1-x_t\right)\left(1-p\left(x_{t+1}\right)\right) \\
d x_{t+1} &= d x_t-\theta(1-\alpha) d x_t\left(1+p^{\prime}\left(x_t\right)\right)-\theta \alpha d x_t-\theta \alpha x_t p^{\prime}\left(x_{t+1}\right) d x_{t+1}-\theta \alpha d x_{t+1}\left(1-x_t\right) p^{\prime}\left(x_{t+1}\right) \\
\frac{d x_{t+1}}{d x_t} &= \frac{1-\theta(1-\alpha)\left(1+p^{\prime}\left(x_t\right)\right)-\theta \alpha}{1+\theta \alpha p^{\prime}\left(x_{t+1}\right)} \\
\left.\frac{d x_{t+1}}{d x_t}\right|_{x_t=x^*} &= \frac{1-\theta-\theta p^{\prime}\left(x_t\right)+\alpha \theta p^{\prime}\left(x^*\right)}{1+\theta \alpha p^{\prime}\left(x^*\right)}
\end{aligned}
$$

It is straightforward that $\left.\frac{d x_{t+1}}{d x_t}\right|_{x_t=x^*}<1$. For local stability at x^*, we also need

$$
\begin{aligned}
\frac{1-\theta-\theta p^{\prime}\left(x_t\right)+\alpha \theta p^{\prime}\left(x^*\right)}{1+\theta \alpha p^{\prime}\left(x^*\right)} &> -1 \\
\Leftrightarrow 2+(2 \alpha-1) \theta p^{\prime}\left(x^*\right) &> \theta .
\end{aligned}
$$

That $\alpha>\frac{1}{2}$ implies the last inequality proves that it is also a sufficient condition to guarantee the rest point x^* to be locally stable. ■

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