Strategic Information Transmission in Networks *

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Abstract

This paper studies cheap talk in networks. We provide equilibrium and welfare characterizations that we then use to address four economic questions. We advance the study of homophily, by finding that truthful communication across communities decreases as communities become larger. We contribute to organization design, by identifying an environment where decentralized organizations maximize players' welfare. We show that decentralized communication networks, where information transmission may not be reciprocal, endogenously form in equilibrium. We finally study the problem of privately-informed policy-makers trying to coordinate their policy choices, and find that information aggregation through public conferences outperforms information exchange through closed-door bilateral meetings.

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1 Introduction

This paper studies strategic communication in networks. Each player can send a message only to the players with whom he is linked in the network.[1] The players' message may either be different for each linked player, or be common among them. Each player may exchange messages with few others, but when contemplating what to report, he must forecast how his messages will alter his counterparts' decisions, taking into account that they may also receive messages from players who are beyond his circle of contacts. Despite the intricacies of this problem, this paper provides a tractable model with sharp equilibrium and welfare characterizations. These characterizations allow us to formulate and solve a number of important theoretical economic questions.

First, we provide a new perspective to the study of homophily and segregation in communities, by studying equilibrium information transmission within and across groups with different preferences. Second, we investigate how to organize cheap talk in a minimally connected network, so as to explore the implications of our findings for the study of organization design. Third, we examine equilibrium communication networks in a model where each player can communicate with another player at a small cost paid ex-ante. This application provides a natural counterpart to the existing literature, which studies endogenous network formation under the assumption of truthful communication. Turning to political economy, we finally study the information aggregation problem of different policy-makers trying to coordinate their policy choices. This setup applies for example to a scenario where national policy makers need to implement environmental or economic policies with global consequences.

The Basic Model. Our model is a natural extension of the celebrated model of cheap talk by Crawford and Sobel (1982). There are n players, and a state of the world θ, which is unknown and uniformly distributed on the interval $[0, 1]$. Each player j simultaneously chooses an action y_j, that influences the utility of all players. If θ were known, player i would like

[1] Different network architectures represent, for example, different organization structures, existing social networks, existing diplomatic relations among countries, existing R&D collaborations among firms.

that each player's action were as close as possible to $\theta + b_i$, where b_i represents player i's bias relative to the common bliss point θ; specifically, player i's payoff is $-\sum_j (y_j - \theta - b_i)^2$. Each player i is privately informed of a signal s_i, which takes a value of one with probability θ and a value of zero with complementary probability. Before players choose their actions, they may simultaneously report their signals to others. A player can send a message only to the set of players he is linked to– his communication neighbors. In our setup, every player's neighborhood is partitioned in sets of players, that we define as *message sets*. The player can differentiate his message only across but not within message sets. Hence, our model covers both the case of *private communication*, where every player can send a message privately to each player in his neighborhood, and, the case of *public communication*, where every player's message is common to all players in his neighborhood.

As a concrete example of our model consider a cooperative designing a new product and deciding its characteristics. The profitability of the new product depends on the unknown market demand for products with different characteristics. The different divisions of the cooperative may have counteracting incentives with respect to the characteristics of the product. For example, the engineering division may have a bias to launch a new model with the most advanced technological features, while the marketing division may prefer a product with appealing design. Each division is privately informed of some features of the overall profitability of the new product, and undertakes tasks towards its design and completion. The collective problem of the cooperative is to design a network of communication between the different divisions so as to ensure that as much information as possible is aggregated. When the cooperative is facing a crisis, or when it needs to launch the new product fast to react to aggressive competition, it is reasonable to presume that communication takes place in few rounds. Our model studies the case where one round of information transmission is allowed.

As another example, consider a collection of policy makers, each accountable versus his own constituency. Each politician needs to choose and implement a policy in its polity, but the policies induce externalities across polities. Politicians differ with respect to their ideology, located on the left-right ideological, reflecting their individual views, and the views of the median voter is their constituency. Further, each politician may hold some private information on the common effects of the implemented policies. This general setup may be applied for example to the collective problem of the decision and implementation of national environmental or economic policies with global externalities. Likewise, the model is pertinent to the analysis of collective decision making by local representatives in a federal State. The collective problem of the policy-makers is to determine the mode and the network of communication so that the private information held by the different policy makers is aggregated efficiently. Again, when facing pressing global problems such as a financial crisis, or an international conflict, it is natural to assume that communication will take place in few rounds.

Basic Results. A communication strategy profile is described by a (directed) network in which each link represents a truthful message, termed *truthful network*. Our first result provides necessary and sufficient conditions for a truthful network to be an equilibrium. The characterization identifies important equilibrium effects. First, each player's incentive to misreport a low signal in order to raise the action of lower bias neighbors is tempered by the loss incurred from the increase in actions of all higher bias neighbors who belong to the same message set. Second, the composition of these gains and losses depends on the number of players truthfully communicating in equilibrium with each neighbor in the same message set. In particular, a player can gain or lose *less* in absolute terms by misreporting information to a player who receives *many* truthful messages relative to the other players in the same message set. Third, in the case of private communication, there is a strong congestion effect: the willingness of a player to communicate with one of his neighbor declines in the number of players communicating with that neighbor.[2]

We then turn to welfare analysis. In our framework, an equilibrium maximizes the *ex-ante* utility of a player if and only if it maximizes the *ex-ante* utility of each one of the other players. We find that each player i's ex-ante payoff induced by a player j's choice is an

[2]This congestion effect does not always extend beyond private communication

increasing and concave function of the number of players who truthfully communicate with j. Hence, equilibria can be ranked in the *ex-ante* Pareto sense on the basis of the distribution of in-degree that they generate in their corresponding equilibrium networks. If the in-degree distribution of an equilibrium first order stochastic dominates the in-degree distribution of another equilibrium, the former is more efficient than the latter. Moreover, if the in-degree distribution of an equilibrium network is a mean preserving spread of the in-degree distribution of another equilibrium network, then the latter is more efficient than the former.

While derived in a simple quadratic-loss Beta-binomial model, our equilibrium and welfare results are based on general features of utility functions in the Crawford and Sobel [1982] framework, and on general features of statistical Bayesian models. Specifically, the key assumptions behind our results are (i) the assumptions that utility functions are single-peaked, strictly concave and ordered through a single-crossing condition, and (ii) the fact that the effect of a signal on the posterior update decreases with the precision of prior, i.e., in a multi-player communication model, that the effect of a player's truthfully reported signal decreases with the number of truthful messages received from other players.

Private Communication. Our basic equilibrium and welfare characterizations allow us to formulate and solve a number of theoretical economic questions. In the first three settings, we suppose that players' communication is private. The first setting studies communication between two communities of players, where preferences are the same within groups, but different across groups. The analysis offers a new perspective on a phenomenon that Lazersfeld and Merton (1954) termed *homophily*: the tendency of individuals to associate and exchange information with others who are similar to themselves. Homophily has been documented across a wide range of characteristics, such as age, race, gender, religion and occupations, e.g., Moody (2001) and McPherson et al. (2001), whereas Currarini et al. (2009) provides a strategic foundation for these empirical patterns. The study of homophily has so far focused on symmetric relations such as association and friendship. In contrast, we consider the asymmetric relation of information transmission. Our results predict that there is less truthful exchange of information across individuals with different characteristics in large-population environments, such as metropolitan areas, than in small-population environments, such as rural towns. Further, we predict that large groups of individuals influence the decisions of small groups by credibly reporting information, while there is less truthful communication from small communities to large communities.

The second setting contributes to the literature on organization design. We study the optimal communication network, designed to respond to sudden crisis, in organizations where decision rights are fully decentralized. As it is common in the literature on organization design we represent the internal organization as a minimally connected communication network, see, e.g., Bolton and Dewatripont (1994), Hart and Moore (2005), Sah and Stiglitz (1986) and Radner (1992) for a survey. We show that a line where communication links are only built between players with adjacent biases maximizes the ex-ante utilities of all players among minimally connected communication networks. Therefore, we conclude that the optimal communication structure is fully decentralized. This insight complements the findings of the existing literature, which studies the optimal allocation of decision rights, as well as the optimal communication structure within organizations. Most of the literature investigates the optimal communication architecture in environments where communication is assumed to be truthful. Two exceptions are Alonso et al. (2008) and Rantakari (2008). They consider cheap talk in an organization with one central head quarter and two peripheral divisions and find that it can be optimal to decentralize decision rights to the divisions.[3] Beyond the simple structure studied in these two papers, the question of optimal allocation of decision rights within general organization architectures remains unanswered. This question may be addressed in an extension of our model, where we let the set of decision makers be a possibly proper subset of the set of players.

The analysis in the third setting contributes to the growing literature on strategic network formation, which originated with the seminal papers of Bala and Goyal (2000) and Jackson and

[3] On related topics see Dessein (2002), Harris and Raviv (2005).

Wolinsky (1996), and was extended in several other articles.[4] In these models, players choose to form costly links with others to access their information. Once a link between two players is established, communication is assumed to be truthful. A robust finding of this literature is that equilibrium networks and efficient networks are very centralized: few players have many connections, whereas the majority of players have only a few. We study a model in which each player can communicate with another player at a small cost paid *ex-ante* and where the bias difference across adjacent bias players is constant. In contrast with the findings of the existing literature, endogenous communication networks emerging from strategic communication are highly decentralized: all moderate bias players have the same in-degree, while the in-degree declines slowly as the biases become more extreme. Further, links between similar bias players are reciprocal, whereas links between players with a very different bias may be not reciprocal. In that case, moderate bias players influence the decision of extreme bias players through truthful communication, while extreme bias players do not influence the decision of moderate bias players.

Private and Public Communication. In the final part of the paper, we compare the information aggregation properties of private and public communication. We again consider the case where each player is both a sender and a receiver of information, and the bias difference across adjacent players is constant. This model advances the understanding of the information aggregation problem of a collection of privately-informed policy-makers trying to coordinate their policy choices. Hence, it is of relevance, for example, for a scenario where national policy makers need to implement environmental or economic policies with global consequences. Likewise, our setup is pertinent to the analysis of collective decision making by local representatives in a federal State. Our numerical analysis finds that, in most cases, public broadcasting of information, through public announcements or through the organization of

[4]Extensions have covered, among others, the case of players' heterogeneity (Galeotti at al. (2006), Galeotti (2006), Hojman and Szeidl (2008), Jackson and Rogers (2005)), endogenous information acquisition (Galeotti and Goyal (2008)), investment in links' reliability (Bloch and Dutta (2007) and Rogers (2008)), and investment in the quality of pairwise costly communication (Calvo-Armengol, de Marti and Prat (2009)). For a survey of the literature see Goyal (2007) and Jackson (2008).

public meetings outperforms the private disclosure of information through bilateral closed-door meetings. This result can be seen as a theoretical justification for the common practice of large intergovernmental meetings, such as the European Council, the G20, or the meetings of the general assembly of the United Nations.[5] Similarly, the above result suggests that the institution of State Chambers in federal States, such as the Senate in the US may also serve the role of aggregating the information held by policy makers accountable to individual States.

Related Literature in Communication Games. We have already discussed the relation between our article and the literature on homophily, organization design, strategic network formation. Our paper also relates to the literature on strategic information transmission, which builds on the classical model of cheap talk by Crawford and Sobel (1982). This literature is too vast for us to fully survey here, and we discuss only the papers that are more closely related to our work.[6] Morgan and Stocken (2008) study a model of communication by many senders to one receiver that adopts the same statistical structure of our paper. Indeed, our equilibrium characterization for the special case of private communication can be described as a generalization of their Proposition 2. Beyond this relation, however, the content of the two papers is completely different. Morgan and Stocken (2008) study the statistical properties of polling, and compare polling with elections. Instead, our paper considers many-to-many communication in any network structure, studies public as well as private communication, derives a completely novel welfare characterization, and, most importantly, derives a number of novel results for several important theoretical economic questions.

Another related paper is by Farrell and Gibbons (1989), who compare private and public communication by a sender to two receivers.[7] One of the equilibrium effects that we identify in our broad setup is reminiscent of their "mutual discipline" effect. In both models, the gains

[5]Indeed, while bilateral international summits may often occur, the discussions held in such meeting are seldom held with closed doors.

[6]Other influential works include Ambrus and Takahashi (2008), Austen Smith (1993), Battaglini (2002, 2004), Gilligan and Krehbiel (1987, 1989), Krishna and Morgan (2001a, 2001b), Wolinsky (2002).

[7]More recent work on cheap talk game with multiple receivers and a single sender includes Goltsmann and Pavlov (2009), Koessler and Martimort (2008), and Caillaud and Tirole (2007).

induced by misreporting a signal to bias some neighbors may be tempered by the loss induced by biasing other neighbors. But in our model, by extending the framework to many senders and many receivers, we further show that the weights of such gains and losses depend on the number of players truthfully communicating to each one of the neighbors. Furthermore, we compare welfare under private and public communication in our general model.

We conclude by discussing a nice, recent paper by Hagenbach and Koessler (2009), who study many-to-many communication.[8] The main difference between the two papers is that our basic equilibrium and welfare congestion effects are absent in their paper. This contrast is substantial because these effects drive many of our results. This difference arises because they consider the following statistical model: With probability one, the state of the world θ equals the sum of each player i's individual binary signal s_i, which are independent across players. As a result, the marginal effect of one truthful message on the action chosen by a receiver is constant in the number of truthful messages received. This implies that their condition for a player to truthfully transmit his information to an opponent does not depend on the communication strategies used by other players. In our standard Bayesian statistical model, instead, the marginal effect of one truthful message on the action chosen by a receiver decreases in the number of truthful messages received. Some further differences between the two papers are that, unlike us, they consider payoff strategic complementarities, whereas, unlike them, we introduce exogenous networks constraining communication among the players.

The rest of the paper is organized as follows. Section 2 develops the basic framework and section 3 presents the basic results on equilibrium and welfare. Section 4 studies three questions of private communication and section 5 compares welfare under private and public communication. Section 6 concludes. All proofs are in an Appendix.

[8] A few papers study many-to-many communication, but, unlike us, focus on information aggregation in committees, e.g., Ottaviani and Sorensen (2001), Austen-Smith and Feddersen (2006) and Visser and Swank (2007), Gerardi and Yariv (2007).

2 Model

The set of players is $N = \{1, 2, ..., n\}$, player i's individual bias is b_i and $b_1 \leq b_2 \leq ... \leq b_n$. The vector of biases $\mathbf{b} = \{b_1, ..., b_n\}$ is common knowledge. The state of the world θ is uniformly distributed on $[0, 1]$. Every player i receives a private signal $s_i \in \{0, 1\}$ on the realization of the state of the world, where $s_i = 1$, with probability θ.

A communication network $\mathbf{g} \in \{0, 1\}^{n \times n}$ is a (possibly directed) graph: i can send his own signal to j whenever $g_{ij} = 1$. We assume that $g_{ii} = 0$ for all $i \in N$. The communication neighborhood of i is the set of players to whom i can send his signal and it is denoted by $N_i(\mathbf{g}) = \{j \in N : g_{ij} = 1\}$. Let $\mathcal{N}_i(\mathbf{g})$ be a partition of the communication neighborhood of i, with the interpretation that player i must send the same message m_{iJ} to all agents $j \in J$, for any group of agents $J \in \mathcal{N}_i(\mathbf{g})$, where each J is denoted as a message set. We refer to $\mathcal{N}_i(\mathbf{g})$ as the communication mode available to i. A communication strategy of a player i specifies, for every $s_i \in \{0, 1\}$, a vector $\mathbf{m}_i(s_i) = \{m_{iJ}(s_i)\}_{J \in \mathcal{N}_i(\mathbf{g})}$; $\mathbf{m} = \{\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_n\}$ denotes a communication strategy profile. The mixed strategy extension of strategy $\mathbf{m}_i$ is μ_i. We let $\hat{\mathbf{m}}_i$ be the messages sent by agent i to his communication neighborhood, and $\hat{\mathbf{m}} = (\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2, ..., \hat{\mathbf{m}}_n)$.

After communication occurs, each player i chooses an action $\hat{y}_i \in \Re$. Let $N_i^{-1}(\mathbf{g}) = \{j \in N : g_{ji} = 1\}$ be the set of players communicating with agent i. Then, agent i's action strategy is $y_i : \{0, 1\}^{|N_i^{-1}(\mathbf{g})|} \times \{0, 1\} \to \Re$; $\mathbf{y} = \{y_1, ..., y_n\}$ denotes an action strategy profile. Given the state of the world θ, the payoffs of i facing a profile of actions $\hat{\mathbf{y}}$ is:

$$u_i(\hat{\mathbf{y}}|\theta) = -\sum_{j \in N} (\hat{y}_j - \theta - b_i)^2 .$$

Agent i's payoffs depend on how his own action y_i and the actions taken by other players are close to his ideal action $b_i + \theta$.[9]

[9] Depending on the particular context, a model where only a subset of players take an action and/or some players are affected only by the actions taken by a subset of the population may be more plausible. Our method of analysis and our results can easily be extended to these settings.

The equilibrium concept is Perfect Bayesian Equilibrium. To avoid dealing with payoff equivalent equilibria and off-path beliefs, we focus on equilibria where each agent i's communication strategy μ_{iJ} with a group of agents $J \in \mathcal{N}_i(\mathbf{g})$, may take only two forms: the truthful one, $m_{iJ}(s_i) = s_i$ for all s_i, and the babbling one, $\mu(\hat{m}_{iJ}|s_i) = 1/2$ for all $\hat{m}_{iJ}$ and s_i. Note that in these equilibria, all messages are on path. With some slight abuse of notation we shall therefore use $\mathbf{m}$ to indicate such communication strategies.[10]

Given the received messages $\hat{\mathbf{m}}_{N_i^{-1}(\mathbf{g}),i}$, by sequential rationality, agent i chooses y_i to maximize his expected payoff. Therefore, agent i's optimization reads

$$\begin{aligned} & \max_{y_i} \left\{ -E\left[\sum_{j\in N} (y_j - \theta - b_i)^2 \middle| s_i, \hat{\mathbf{m}}_{N_i^{-1}(\mathbf{g}),i} \right] \right\} \\ = \ & \max_{y_i} \left\{ -E\left[(y_i - \theta - b_i)^2 \middle| s_i, \hat{\mathbf{m}}_{N_i^{-1}(\mathbf{g}),i} \right] \right\}. \end{aligned}$$

Hence, agent i chooses

$$y_i\left(s_i, \hat{\mathbf{m}}_{N_i^{-1}(\mathbf{g}),i}\right) = b_i + E\left[\theta \middle| s_i, \hat{\mathbf{m}}_{N_i^{-1}(\mathbf{g}),i}\right], \tag{1}$$

where the expectation is based on equilibrium beliefs: All the messages $\hat{m}_{ji}$ received by an agent j who adopts a babbling strategy are disregarded as uninformative, and all $\hat{m}_{ji}$ received by an agent j who adopts a truthful strategy are taken as equal to s_j. Hereafter, whenever we refer to a strategy profile $(\mathbf{m}, \mathbf{y})$, each element of $\mathbf{y}$ is assumed to satisfy condition 1.

We further note that the agents' updating is based on the standard Beta-binomial model. So, suppose that an agent i holds k signals, i.e. he holds the signal s_i and $k-1$ neighbors truthfully reveal their signal to him. If l out of such k signals equal 1, then the conditional pdf is:

$$f(l|\theta, k) = \frac{k!}{l!\,(k-l)!}\theta^l (1-\theta)^{(k-l)},$$

[10]For the sake of tractability, our analysis does not characterize equilibria where players may play a mixed strategy other than the babbling strategy. However, we know that the following equilibrium property hold. In every equilibrium, a player may mix only when holding one of the two signals, and must play a pure strategy when holding the other one.

and his posterior is:

$$f(\theta|l,k) = \frac{(k+1)!}{l!\,(k-l)!}\theta^{l}(1-\theta)^{(k-l)}.$$

Consequently, $f(l|\theta,k) = f(\theta|l,k)/(k+1)$ and $E[\theta|l,k] = (l+1)/(k+2)$.

In the first stage of the game, in equilibrium, each agent i adopts either truthful communication or babbling communication with each group of agents $J \in \mathcal{N}_i(\mathbf{g})$, correctly formulating the expectation on the action chosen by agent $j \in J$ as a function of his message $\hat{m}_{iJ}$ and with the knowledge of the equilibrium strategies $\mathbf{m}_{-i}$ of the opponents.

We finally note that our framework encompasses two widely studied modes of communication: *private communication* and *public communication*. The model of private communication obtains when for each agent i the partition $\mathcal{N}_i(\mathbf{g})$ of $N_i(\mathbf{g})$ is composed of singleton sets. In this case each agent has the possibility to communicate privately with each of his neighbors. The opposite polar case is when, for each player i, the trivial partition $\mathcal{N}_i(\mathbf{g}) = \{N_i(\mathbf{g})\}$ holds, which corresponds to a model of public communication.

3 Results

We first characterize equilibria for arbitrary modes of communication. We then show that the characterization takes a simple form under private communication. We finally investigate the relationship between equilibrium communication and Pareto efficiency.

3.1 Equilibrium Networks

A communication network $\mathbf{g}$ together with a strategy profile $(\mathbf{m}, \mathbf{y})$ induces a subgraph of $\mathbf{g}$, in which each link involves truthful communication. We refer to this network as the *truthful network* and denote it by $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$. When $(\mathbf{m}, \mathbf{y})$ is equilibrium, we refer to the induced truthful network $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$ as to the *equilibrium network*. Formally, $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$ is a binary directed graph where $c_{ij}(\mathbf{m}, \mathbf{y}|\mathbf{g}) = 1$ if and only if j belongs to some element $J \in \mathcal{N}_i(\mathbf{g})$ and $m_{iJ}(s) = s$, for every $s = \{0, 1\}$. Given a truthful network $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$, let $k_j(\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g}))$ be the number of agents who send a truthful message to j. We term $k_j(\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))$ the in-degree of player j.

Theorem 1 *Consider a communication network* $\mathbf{g}$ *and a collection of communication modes* $\{\mathcal{N}_i(\mathbf{g})\}_{i\in N}$. *The strategy profile* $(\mathbf{m},\mathbf{y})$ *is an equilibrium if and only if for every truthful message from a player* i *to a group of players* $J \in \mathcal{N}_i(\mathbf{g})$, *i.e., for all* i *and* $J \in \mathcal{N}_i(\mathbf{g})$ *such that* $c_{ij}(\mathbf{m},\mathbf{y}|\mathbf{g}) = 1$ *for all* $j \in J$, *the following condition holds:*

$$2\left|b_i - \sum_{j\in J} b_j \gamma_j(\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))\right| \leq \sum_{j\in J} \frac{1}{(k_j(\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))+3)}\gamma_j(\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g})). \tag{2}$$

where for every $j \in J$, *with* $J \in \mathcal{N}_i(\mathbf{g})$,

$$\gamma_j(\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g})) = \frac{1/(k_j(\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))+3)}{\sum_{j'\in J} 1/(k_{j'}(\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))+3)}$$

Condition 2 may be interpreted as follows. The left-hand side tells us that player i is willing to truthfully communicate with players j belonging to J if and only if the weighted average $\sum_{j\in J} b_j\gamma_j$ of their biases b_j is not too different from his own bias b_i. This reflects the fact that, when contemplating whether to deviate from truthful reporting, player i can only influence the action of all players $j \in J$ in the same direction. When, for instance, $s_i = 0$ and he reports $\hat{m}_{iJ} = 1$, he will gain by biasing upwards the action of every player $j \in J$ with bias $b_j < b_i$, but, at the same time, he will lose by increasing the action of every player $j \in J$ with bias $b_j > b_i$. Overall, player i' s deviation from truthful reporting is deterred if and only if, on average, losses outweigh gains.

Turning to the specific weights γ_j in the weighted average of the biases b_j of the players $j \in J$, we observe the following. The numerator $1/(k_j + 3)$ decreases in the number of players truthfully communicating with j in equilibrium. The reason for this is that the more player j is informed in equilibrium, the less the message $\hat{m}_{iJ}$ will change his final action. Therefore, when contemplating a deviation, player i can gain or lose less in absolute terms by influencing j relative to the other players belonging to J. As a result, player i will give less weight to the bias of player j relative to the biases of the other players in J.

The right-hand side of condition 2 is a weighted average, with the same weights as the left-hand side, of the quantities $1/(k_j + 3)$. These quantities equal the difference in the expected value of θ induced by the change in one signal, when knowing k_j further signal realizations. Because player j is informed of signal s_j, receives $k_j - 1$ truthful messages from players other than i, and matches his action y_j to his expected value of θ, the quantity $1/[k_j+3]$ corresponds to the action change of player j that follows from a message deviation from equilibrium by player i.

In sum, player i will be able to truthfully communicate with the players in J if and only if twice the absolute difference between his bias and the weighted average of their bias is smaller than the weighted average change in the action of players in group J induced by a message deviation by player i. In fact, when this condition holds, player i suffers an overall loss by deviating from the truthful equilibrium strategy.

The characterization in Theorem 1 simplifies substantially under the specific case of private communication.

Corollary 1 *Consider a communication network* $\mathbf{g}$*. Under private communication a strategy profile* $(\mathbf{m}, \mathbf{y})$ *is equilibrium if and only if for every* (i, j) *with* $c_{ij}(\mathbf{m}, \mathbf{y}|\mathbf{g}) = 1$ *the following condition holds:*

$$|b_i - b_j| \leq \frac{1}{2\left[k_j(\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})) + 3\right]}. \tag{3}$$

Under private communication, the willingness of a player i to credibly communicate with a player j displays a simple dependence on their bias difference $|b_i - b_j|$ and on the number of players truthfully communicating with j. In particular, a high in-degree k_j prevents communication from i to j to be truthful. To see this, suppose that $b_i > b_j$ so that i's bliss point is smaller than j's bliss point. When many opponents truthfully communicate with j, this player is well informed. In this case, if player i deviates from the truthful communication strategy and reports $\hat{m}_{ij} = 1$ when $s_i = 0$, he will induce a small increase of j's action. Such a small increase in j's action is always beneficial in expectation to i, as it brings j's action closer to i's (expected) bliss point. Hence, player i will not be able to truthfully communicate the signal $s_i = 0$. In contrast, when j has a low in-degree, then i's report $\hat{m}_{ij} = 1$ moves j's action upwards significantly, possibly over i's bliss point. In this case, biasing upwards j's action may result in a loss for player i and so he will not be willing to deviate from the truthful communication strategy.

Returning to general communication modes, we conclude this section by showing that, unlike in the case of private communication, a player i's willingness to truthfully communicate with another player j needs not to decrease in the in-degree of j.

Example 1. Let $N = \{1, 2, 3, 4\}$ and $\mathbf{b} = \{-1, 0, \beta, \beta + c\}$, where $\beta > 1$ and c is a small positive constant. Consider the following communication network $\mathbf{g}$: $g_{21} = g_{23} = g_{43} = 1$ and no other communication links. Suppose also that player 2 must send the same message to his neighbors $\{1, 3\}$.

First, suppose that player 4 babbles to player 3. In this case, player 2 assigns the same weight to player 3 and player 4, i.e., $\gamma_1 = \gamma_3 = 1/2$. The communication strategy in which player 2 sends a truthful public message to $\{1, 3\}$ is equilibrium whenever $\beta \leq 5/4$. Second, consider that player 4 communicates truthfully with 3 (which is always possible in equilibrium for sufficiently small c). In this case, player 2 gives a higher weight to player 1 who is less informed than player 3, i.e., $\gamma_1 = 5/9 > 4/9 = \gamma_3$. The communication strategy in which player 2 sends a truthful public message to $\{1, 3\}$ is equilibrium whenever $\beta \leq 241/169$. Hence, when $\beta \in (5/4, 241/160]$ player 2 is able to report a truthful public message to his neighbors $\{1, 3\}$ only if player 4 also communicates truthfully with 3. ■

3.2 Welfare

We now consider the welfare generated in equilibrium. Because of the quadratic utility formulation, if we let $\sigma_j^2(\mathbf{m}, \mathbf{y})$ be the residual variance of θ that player j expects to have once communication has occurred, we can write player i's expected utility in equilibrium $(\mathbf{m}, \mathbf{y})$ as follows:

$$EU_i(\mathbf{m},\mathbf{y}) = -\left[\sum_{j\in N}(b_j - b_i)^2 + \sum_{j\in N}\sigma_j^2(\mathbf{m},\mathbf{y})\right].$$

This is an extension of the welfare characterization by Crawford and Sobel [1982] to multiple senders and multiple receivers. A nice feature of our model is that we can express the sum of residual variances of θ as a function of a simple property of the equilibrium network, namely its distribution of in-degrees. That is

$$\sum_{j\in N}\sigma_j^2(\mathbf{m},\mathbf{y}) = \frac{1}{6}\sum_{k=0}^{n-1}\frac{1}{k+3}P(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g})),$$

where $P(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))$ is the proportion of players with in-degree k in the equilibrium network and $P(\cdot|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g})) : \{0,...,n-1\} \to [0,1]$ is its in-degree distribution.

Inspection of the above equation shows that an equilibrium $(\mathbf{m},\mathbf{y})$ yields a higher ex-ante utility to a player i than another equilibrium $(\mathbf{m}',\mathbf{y}')$ if and only if $(\mathbf{m},\mathbf{y})$ yields higher ex-ante utility than $(\mathbf{m}',\mathbf{y}')$ to all other players j. Hence, ranking equilibria in the Pareto sense is equivalent to ranking them in the sense of utility maximization of all players. We can now state the following result.

Theorem 2 *Consider communication networks* $\mathbf{g}$ *and* $\mathbf{g}'$*. Suppose that* $(\mathbf{m},\mathbf{y})$ *and* $(\mathbf{m}',\mathbf{y}')$ *are equilibria in* $\mathbf{g}$ *and* $\mathbf{g}'$*, respectively. Equilibrium* $(\mathbf{m},\mathbf{y})$ *Pareto dominates equilibrium* $(\mathbf{m}',\mathbf{y}')$ *if and only if*

$$\sum_{k=0}^{n-1}\frac{1}{k+3}P(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g})) < \sum_{k=0}^{n-1}\frac{1}{k+3}P(k|\mathbf{c}'(\mathbf{m}',\mathbf{y}'|\mathbf{g}')). \tag{4}$$

A simple inspection of condition 4 allows us to rank equilibria in the Pareto sense based on stochastic dominance relations between the in-degree distributions of their corresponding equilibrium networks.

Corollary 2 *Consider communication networks* $\mathbf{g}$ *and* $\mathbf{g}'$*. Suppose that* $(\mathbf{m},\mathbf{y})$ *and* $(\mathbf{m}',\mathbf{y}')$ *are equilibria in* $\mathbf{g}$ *and* $\mathbf{g}'$*, respectively.*

1. *If* $P(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))$ *first order stochastic dominates* $P(k|\mathbf{c}'(\mathbf{m}',\mathbf{y}'|\mathbf{g}'))$ *then equilibrium* $(\mathbf{m},\mathbf{y})$ *Pareto dominates equilibrium* $(\mathbf{m}',\mathbf{y}')$.

2. *If* $P(k|\mathbf{c}'(\mathbf{m}',\mathbf{y}'|\mathbf{g}'))$ *is a mean preserving spread of* $P(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))$ *then equilibrium* $(\mathbf{m},\mathbf{y})$ *Pareto dominates equilibrium* $(\mathbf{m}',\mathbf{y}')$

To illustrate the first part of Corollary 2, consider an equilibrium in which i babbles with j and another equilibrium in which the only difference is that player i communicates truthfully with j. The presence of this additional truthful message only alters the equilibrium action of player j. In particular, player j's action becomes more precise and therefore the utility of each player increases. A direct consequence of this result is that if $(\mathbf{m},\mathbf{y})$ and $(\mathbf{m}',\mathbf{y}')$ are two distinct equilibria in a communication network $\mathbf{g}$ and $\mathbf{c}'(\mathbf{m}',\mathbf{y}'|\mathbf{g})$ is a subgraph of $\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g})$, then equilibrium $(\mathbf{m},\mathbf{y})$ Pareto dominates equilibrium $(\mathbf{m}',\mathbf{y}')$.

The second part of Corollary 2 allows to compare equilibria that have the same number of truthful communication links. It shows that equilibria in which truthful messages are distributed evenly across players Pareto dominate equilibria where few players receive many truthful messages, while others receive only a few. The reason is that the residual variance of θ associated to every player j is (decreasing) and *convex* in his in-degree.

Theorem 2 and Corollary 2 suggest the possibility that an equilibrium that sustains a low number of truthful messages may Pareto dominate an equilibrium with a high number of truthful messages, as long as its messages are distributed more evenly across players. We now develop an example in which this is the case.

Example 2: Even distribution of truthful messages vs. total number of truthful messages. Suppose $n = 5$ and that $b_{i+1} - b_i = \beta$, for $i = 1, 2, 3, 4$. Let $\mathbf{g}$ be a star network and player 3 be the center. When $\beta \leq 1/28$ the following two equilibrium networks are part of equilibrium. One equilibrium sustains four truthful links: each peripheral player sends a truthful message to the center, and there are no other truthful messages. The in-degree distribution of the equilibrium truthful network is then: $P(0) = 4/5$, $P(4) = 1/5$, and $P(k) = 0$, $k = 1, 2, 3$. The other equilibrium sustains three truthful links: the center sends a truthful message to players 1, 2 and 4, and there are no other truthful messages. The in-degree distribution associated to this equilibrium is: $\tilde{P}(0) = 2/5$, $\tilde{P}(1) = 3/5$ and $\tilde{P}(k) = 0$, $k = 2, 3, 4$.

Note that P and $\tilde{P}$ cannot be ranked in terms of first order or second order stochastic dominance relations. However, applying condition 4, it is easy to check that

$$\sum_{k=0}^{n-1} \tilde{P}(k) \frac{1}{k+3} = \frac{17}{60} < \frac{31}{105} = \sum_{k=0}^{n-1} P(k) \frac{1}{k+3}.$$

Hence, the second equilibrium Pareto dominates the former equilibrium, despite it sustains a lower number of truthful messages. ■

4 Private Communication

This section focuses on private communication and it explores three theoretical economic questions that can be analyzed within our framework. We first consider strategic communication between communities. We then approach a question in the study of optimal design of organizations. Finally, we analyze the properties of endogenous communication networks when information transmission is strategic. In what follows we focus on utility-maximizing equilibria. A utility-maximizing equilibrium is an equilibrium that maximizes the utility of all players, across all equilibria.[11]

4.1 Communication across groups

The purpose of the following analysis is to provide a new perspective to the study of homophily and segregation in communities, by studying equilibrium information transmission within and across groups with different preferences. We show that communication across communities decreases as communities become larger, and that communication may be asymmetric: from large communities to small ones.

[11] As we noted in section 3.2, in our setting an utility-maximizing equilibrium Pareto dominates every other equilibrium.

We study strategic communication between two communities. The set of players is partitioned in two groups, N_1 and N_2, with size n_1 and n_2, respectively. Without loss of generality, we assume that $n_1 > n_2 \geq 1$. Each member of group 1 has a bias which is normalized to 0; players in group 2 have a bias $b > 0$. Players can send a message to every other player.

Let k_i be the in-degree of an arbitrary player in group i in a utility-maximizing equilibrium network. Then $k_i = k_{ii} + k_{ij}$, where k_{ii} is the number of truthful messages that a player of group i receives from members of the same group, whereas k_{ij} is the number of truthful messages that a player in group i receives from members of the opposite community. So k_{ii} measures the level of intra-group communication and k_{ij} measures the level of cross-groups communication.

We first observe that, for every b, there always exists a utility-maximizing equilibrium in which intra-group communication is complete, i.e., $k_{ii} = n_i - 1$. To see this, consider a utility-maximizing equilibrium where $k_{11} < n_1 - 1$. This implies that there are two players i' and i'' who belong to group 1 and player i'' does not communicate truthfully with i'. If i' does not receive any truthful message from members of community 2, since i' and i'' have the same preferences, we can construct an equilibrium in which i'' communicates truthfully with i'. In light of Theorem 2 this new equilibrium Pareto dominates the original one, which contradicts our initial hypothesis. Suppose then that there is some player j' in community 2 who communicates truthfully with i'. In this case, Corollary 1 implies that we can construct another equilibrium in which j' babbles with i', whereas i'' is truthful with i'. Note that the two equilibria generate the same in-degree distribution and therefore they induce the same *ex-ante* utility for all players. Our observation then follows by iterating these two arguments for every player.

In the appendix we provide a full characterization of utility-maximizing equilibrium networks. Here, we focus on the natural subclass where there is complete intra-group communication.[12]

[12]This is a natural class of equilibria in this environment. First, these equilibria are robust to the introduction

Within this class, the only parameters that must be determined are the levels of cross-groups communication. This allows us to parsimoniously describe cross-community communication as follows. Consider a utility-maximizing equilibrium with complete intra-group communication. Since $k_{ii} = n_i - 1$ and since the in-degree within groups is the same across players, condition 3 in Corollary 1 implies that k_{ij} must satisfy

$$k_{ij} \leq \left\lfloor \frac{1}{2b} - n_i - 2 \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than x. Furthermore, in view of Theorem 2, in a utility-maximizing equilibrium network k_{ij} will be the highest possible subject to the above equilibrium condition and that $k_{ij} \leq n_j$ and $k_{ij} \geq 0$. We can now state the following result.

Proposition 1 *In every utility-maximizing equilibrium network with complete intra-group communication, the levels of cross-communities communication are:*

$$k_{ij} = \max\left\{\min\left\{\left\lfloor \frac{1}{2b} - n_i - 2 \right\rfloor, n_j\right\}, 0\right\}, i, j = 1, 2, i \neq j.$$

If $b < \frac{1}{2(n+2)}$ there is complete cross-communities communication; If $b \in \left[\frac{1}{2(n+2)}, \frac{1}{2(n_2+3)}\right]$ the level of truthful communication from group j to group i, k_{ij}, declines with the size of group i and the level of communication from large group 1 to small group 2 is higher than the level of communication from group 2 to group 1, i.e., $k_{21} > k_{12}$; If $b > \frac{1}{2(n_2+3)}$, there is not communication across communities.

The proposition shows that as communities grow larger cross-groups communication declines and that, generally, cross-communities communication is more pervasive from large to small groups, than *vice-versa*. Both effects are a simple consequence of the congestion effect that is illustrated in Corollary 1.

of infinitesimal group-sensitive preferences. For example, we can slightly modify the model so that the utility of every player l in group i is: $-(1+\epsilon)\sum_{l' \in N_i}(\hat{y}_{l'} - \theta - b_l)^2 - \sum_{l' \in N_j}(\hat{y}_{l'} - \theta - b_l)^2$, where ϵ is a small positive constant. Second, this class of utility-maximizing equilibria coincides with the set of utility-maximizing equilibria as long as the conflict of interest between the two groups is not too low. A formal result is in Appendix, see Proposition A.

4.2 Optimal Organization Network

We now explore the problem of optimal organization design in a context where decision rights are decentralized and members of the organization have different preferences. Motivated by the literature on organization design, we represent organizations as minimally connected networks. These are organizations in which there are $n-1$ undirected links and every pair of players is connected via a sequence of links. Our main insight is that the optimal organization network is the line where communication links are only built between agents with adjacent biases, i.e., *the ordered line communication network*. Hence, a "decentralized" communication architecture such as the line may outperform "centralized" communication architectures.

Formally, let $\mathbf{G}$ be the set of undirected networks, i.e., every $\mathbf{g} \in \mathbf{G}$ is such that $g_{ij} = g_{ji}$. We say that there is a path in $\mathbf{g} \in \mathbf{G}$ between i and j if either $g_{ij} = 1$ or there exist players $j_1, ..., j_m$ distinct from each other and from i and j such that $\{g_{ij_1} = g_{j_1 j_2} = ... = g_{j_m j} = 1\}$. A network $\mathbf{g} \in \mathbf{G}$ is connected if there exists a path between every pair of players; $\mathbf{g}$ is minimally connected if it is connected and there exists only one path between every pair of players. Let $\tilde{\mathbf{G}} \subset \mathbf{G}$ be the set of minimally connected networks. The ordered line network is a minimally connected network $\mathbf{g}$ where $g_{ii+1} = 1$ for all $i = 1...n-1$.

Proposition 2 *For every equilibrium* $(\mathbf{m}, \mathbf{y})$ *in organization* $\mathbf{g} \in \tilde{\mathbf{G}}$*, there exists an equilibrium* $(\mathbf{m}^*, \mathbf{y}^*)$ *in the ordered line communication network such that all players' welfare in* $(\mathbf{m}^*, \mathbf{y}^*)$ *is weakly larger than in* $(\mathbf{m}, \mathbf{y})$*.*

The proof of Proposition 2 proceeds in two steps. In the first step we show that for every equilibrium $(\mathbf{m}, \mathbf{y})$ in an arbitrary organization $\mathbf{g} \in \tilde{\mathbf{G}}$, we can construct an equilibrium $(\mathbf{m}', \mathbf{y}')$, which can be sustained in the ordered line network and where the total number of truthful communication links in equilibrium $(\mathbf{m}', \mathbf{y}')$ is the same as in the original equilibrium $(\mathbf{m}, \mathbf{y})$. This step involves substituting truthful messages in equilibrium $(\mathbf{m}, \mathbf{y})$ between non-adjacent players, i.e., (i, j) with $|i - j| > 1$, with truthful communication links between adjacent agents. The basic intuition for which this is possible comes from Corollary 1: i 's ability to credibly communicate with j is higher when the in-degree of j is low and when the absolute bias difference between the two players is small.

The second step shows that from the new equilibrium $(\mathbf{m}', \mathbf{y}')$ it is possible to construct another equilibrium $(\mathbf{m}^*, \mathbf{y}^*)$, which is sustainable in the ordered line network and has the property that the in-degree distribution associated to the original equilibrium $(\mathbf{m}, \mathbf{y})$ is a mean preserving spread of the in-degree distribution induced by the new constructed equilibrium. The result then follows from Corollary 2.

4.3 Endogenous Communication Network Formation

We study the architectural properties of endogenous communication networks in a model where players have equidistant bias: $b_{i+1} - b_i = \beta$, for all $i = 1, ..., n-1$. Instead of assuming that the communication matrix $\mathbf{g}$ is given, here we suppose that a link $g_{ij} = 1$ forms if and only if i truthfully communicates with j. This would be the case, for example, if a small cost is paid by the receiver of a link, or equivalently, ex-ante, by the sender, i.e. before knowing his signal realization.

In the previous section, we have shown that within the class of minimally connected networks the ordered line maximizes the welfare of all players. The ordered line network has two distinctive features. One feature is localization: communication links are built among players with adjacent bias. The other feature is decentralization: communication links are distributed evenly across players. We now show that there is a natural class of utility-maximizing equilibrium networks which display these two properties.[13] We first provide a complete characterization of this class of equilibria and then describe in details the properties of localization and decentralization.

[13]Further, it can be shown that for a wide range of the parameter β, this class of equilibria coincides with utility-maximizing equilibria.

Proposition 3 *Let* $V(\beta) = \max\{V \in N : \beta \leq \frac{1}{2V(2V-1+3}\}$*. For every* β*, there exists an equilibrium which maximizes the welfare of each player across all equilibria with the following properties.*

1. *Every player* $j \in \{V(\beta)+1, ..., n-V(\beta)\}$ *receives truthful information from* i *if* $|i-j| < V(\beta)$ *and from no players* i *such that* $|i-j| > V(\beta)$*;*
   - *if* $\beta > \frac{1}{2(2V(\beta)+3)V(\beta)}$*, then* j *receives truthful information from one and only one player* i *such that* $|i-j| = V(\beta)$*;*
   - *if* $\beta \leq \frac{1}{2(2V(\beta)+3)V(\beta)}$*, then* j *receives truthful information from both players* i *such that* $|i-j| = V(\beta)$*;*

2. *For all players* $j \in \{1, ..., V(\beta)\} \cup \{n-V(\beta)+1, ..., n\}$*,* j *receives truthful information from* i *if and only if* $|i-j| \leq M(j,\beta)$*, where* $M(j,\beta) = \max\{M \in N : \beta \leq \frac{1}{2(\min[j-1,n-j]+M+3)M}\}$*.*

We illustrate the equilibrium characterization in Proposition 3 in figure 1, for different values of β when $n = 6$. In the figure a solid line linking i and j signifies that i and j communicate truthfully with each other; a dash line starting from i with an arrow pointing at j means that only player i truthfully communicates with j.

By inspection of figure 1, three features of the equilibrium networks emerge. Information transmission is localized by bias differences: The individuals communicating to each one of the players have biases sufficiently close to the bias of the player. For example, when $\beta \in (1/36, 1/28]$ each player communicates with individuals of bias distance 2β. This localization property generates very decentralized network architectures. In particular, there is a set of moderate bias players who have the same in-degree, while the in-degree of the other players declines slowly as the bias becomes more extreme. For example, when $\beta \in (1/28, 1/24]$, players $2, 3, 4$ and 5 have in-degree of three, whereas players 1 and 6 have in-degree of two. Finally, information transmission may be *asymmetric*, from players with a moderate bias

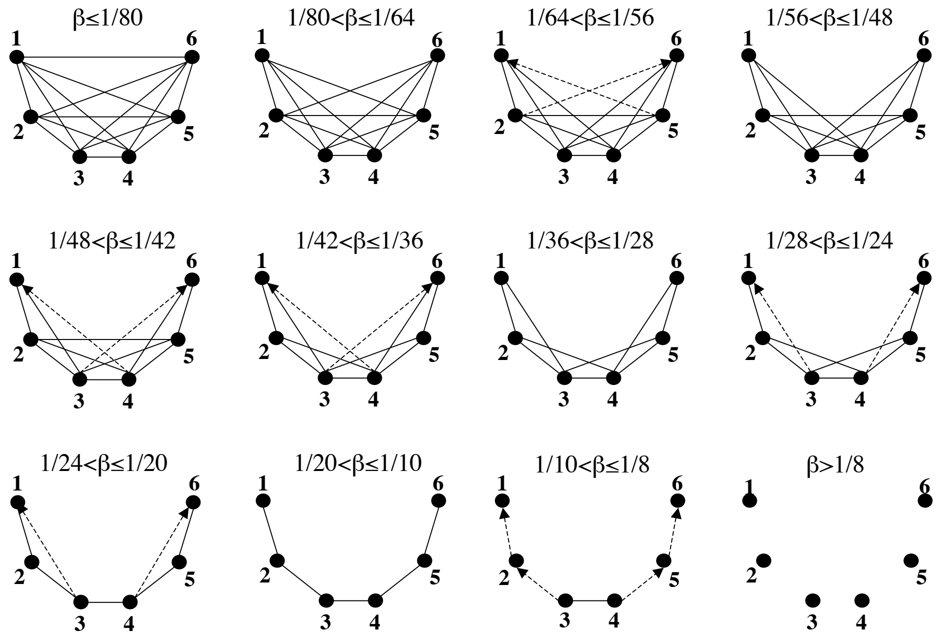

Figure 1: Equilibrium Networks in Proposition 3, $n = 6$.

to players with an extreme bias, but not *vice-versa*. For example, when $\beta \in (1/42, 1/36]$, player 3 truthfully communicates with 6 but player 6 does not truthfully communicate with 3. The basic intuition for this is that the property of localization implies that players with a moderate bias receive more truthful messages that players with an extreme bias. Because of the equilibrium congestion effect, this reduces the ability of extreme players to send truthful messages to moderate players. The following corollary formalizes these three equilibrium properties.

Corollary 3 *For every β, there exists a welfare maximizing equilibrium with the following properties:*

1. *Localization. If a player i communicates with j, then so do all players l such that $|l - j| < |i - j|$;*

2. *Decentralization. Every player $j \in \{V(\beta)+1, ..., n-V(\beta)\}$ has the same in-degree, every player $j \in \{1, ..., V(\beta)\} \cup \{n - V(\beta) + 1, ..., n\}$ has in-degree $\min[j-1, n-j] + M(j, \beta)$;*

3. *Asymmetric Communication. For every $i < j \leq \left\lfloor \frac{n+1}{2} \right\rfloor$ or $i > j \geq \left\lfloor \frac{n+1}{2} \right\rfloor$, it cannot be the case that i truthfully communicates with j and yet j does not truthfully communicate*

with i. In contrast, it can be the case that j truthfully communicates with i and yet i does not truthfully communicates with j.

5 Welfare properties: private *versus* public communication

This section considers the information aggregation problem of a set of players that are simultaneously senders and receivers of information. This model applies to the collective problem of decision and implementation of environmental or economic policies by different national policy-makers. By comparing the welfare of equilibria under private and public communication, we numerically establish that public communication usually outperforms private communication. This normative result implies that public broadcasting of information, through public announcements or through the organization of public meetings outperforms the private disclosure of information through bilateral closed-door meetings.

We use the same set up studied in section 4.3: each player can communicate to all other individuals and players have equidistant bias, i.e., $b_{i+1} - b_i = \beta$, for all $i = 1, ..., n-1$. Proposition 3 in section 4.3 characterizes a class of equilibria that maximizes all players' welfare under private communication. We now describe a class of equilibria that maximizes all players' welfare under public communication. In the definition, we shall make use of the following two functions. For any index $l = 1, ..., \lfloor n/2 \rfloor$, let

$$f(l,n) = \frac{\frac{n-2l+1}{2(n-2l+4)^2} + \frac{l-1}{(n-2l+5)^2}}{(n-2l+1)\left[\frac{n+2-2l}{2(n-2l+4)} + \frac{l-1}{n-2l+5}\right]}$$

and

$$g\left(l,n\right) = \frac{\frac{n-2l}{2(n-2l+3)^2} + \frac{2l-1}{2(n-2l+4)^2}}{(n-2l+1)\left[\frac{n-2l}{2(n-2l+3)} + \frac{l}{n-2l+4}\right]}.$$

For $l = 0$, the function $g(\cdot,\cdot)$ is defined by $g\left(0,n\right) = 0$ for all n.

Proposition 4 *Suppose that $b_{i+1} - b_i \equiv \beta$ for all players $i = 1, ..., n-1$, and that communication is public. There exists a class of welfare maximizing equilibria with the following properties:*

- *For any $l = 1, ..., \lfloor n/2 \rfloor$,*
  - *if $g(l-1, n) < \beta \leq f(l, n)$ the players who communicate truthfully are $\{l, ..., n - l + 1\}$;*
  - *if $f(l, n) < \beta \leq g(l, n)$ the players who communicate truthfully are either $\{l, ..., n - l\}$ or $\{l + 1, ..., n - l + 1\}$.*
- *If $\beta > g(\lfloor n/2 \rfloor, n)$, then no player truthfully communicates when n is even, otherwise player $(n+1)/2$ truthfully communicates.*

We now compare the welfare of utility-maximizing equilibria under private and public communication for different values of β and different number of players n. Because all welfare maximizing equilibria yield the same ex-ante utility to all players, it is sufficient to calculate the welfare of the classes of equilibria described in Proposition 3 and 4, through the condition in Theorem 2. The analysis is numerical, and conducted for all $n \leq 500$. Figure 2 summarizes some of these numerical calculations, which we now comment.

The first panel represents the welfare achieved in the two modes of communication as a function of the bias β, for $n = 16$.[14] The second panel represents the regions of the bias β for which public communication dominates (represented in white) and the regions for which the two modes give the same welfare (represented in grey), as a function of the number of agents n, for n even. It can be seen that in this case public communication always dominates private communication in terms of welfare and that the pattern is very consistent when the number of agents increases. The last two panels take the case of an odd number of players, $n = 15$ and n odd. Here, for low and high values of β, public communication is still always

[14]More precisely, the picture depicts only the part of individual welfare that depends on the extent of communication.

dominant. For intermediate values of β, private communication only occasionally outperforms public communication (represented as black areas in the forth panel). Again the pattern is consistent as the number of agents increases.

Having compared numerically the welfare of utility-maximizing equilibria under private and public communication for all values of β and for n ranging from 3 to 500, we establish the following result.

Result 1 *Suppose that $b_{i+1} - b_i \equiv \beta$ for all players $i = 1, ..., n-1$. When the number of agents n is even, public communication always yields weakly higher welfare than private communication. When n is odd, private communication may occasionally outperform public communication only for intermediate values of β. The analysis is numerical and performed for $n \leq 500$, but the results appear to be extremely stable when n increases.*

6 Conclusion

This paper provides a tractable model to study multi-person environments where players can strategically transmit their private information to individuals who are connected to them in a communication network. The players' message may either be different for each linked player (private communication), or be common among them (public communication). The first important insight that emerges from the analysis is that whether truthful communication can be sustained in equilibrium or not does not only depend on the conflict of interest between players, but also on the architecture of the communication network and the allocation of players within the network. In particular, under private communication, the willingness of a player to communicate with another individual decreases with the number of players communicating with the individual. Under public communication, the composition of biases of linked players determines whether a player is willing to communicate to them. The second general insight is that, in a multi-person environment, equilibrium welfare does not only depend on the

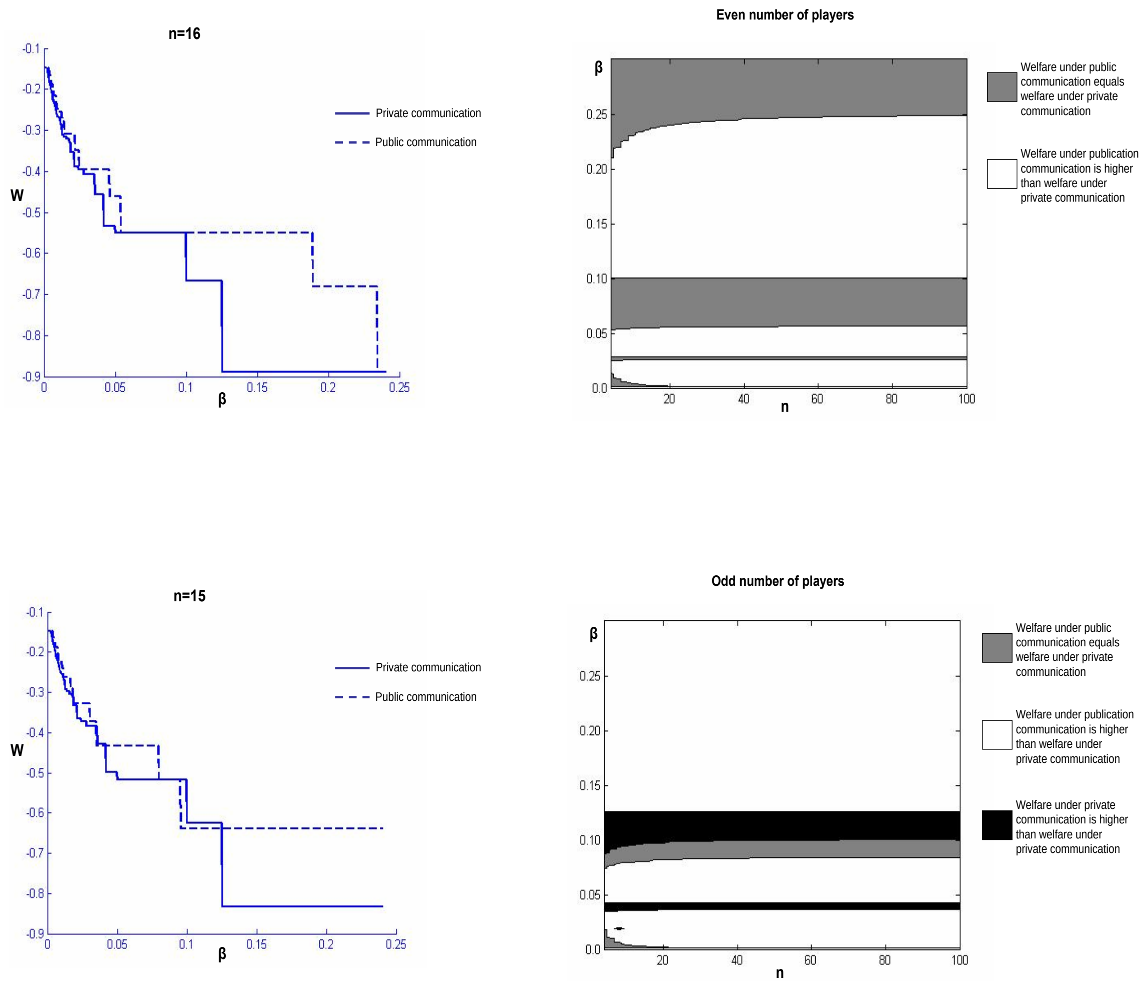

Figure 2: Welfare under Private Communication *vs.* Public Communication

amount of information aggregated in the network, but also on how evenly truthful information transmission is distributed across players.

We demonstrate the relevance of our basic results by addressing a number of theoretical economic questions. The first set of questions focuses on private communication. First, we provide a new perspective to the study of homophily and segregation in communities, by studying equilibrium information transmission within and across groups with different preferences. We show that communication across communities decreases as communities become larger, and that communication may be asymmetric: From large communities to small ones. Second, we investigate how to organize cheap talk in a minimally connected network, so as to explore the implications of our findings for the study of organization design. In our model, fully decentralized organizations maximize all players' welfare. Third, we examine equilibrium private communication in a model where each player can communicate with any other player at a small cost paid ex-ante, thereby providing a natural counterpart to the existing literature, which studies endogenous network formation where communication is assumed to be truthful. In our model, decentralized networks, where information may flow asymmetrically, endogenously form in equilibrium.

We conclude by comparing the welfare properties of private and public communication, when all players are simultaneously senders and receivers. Our analysis provides new insights on the information aggregation problem of different policy-makers trying to coordinate their policy choices. For example, it applies to a scenario where national policy makers need to implement environmental or economic policies with global consequences. We find that, in most cases, public communication dominates private communication. This implies that public broadcasting of information, through public announcements or through the organization of public meetings outperforms the private disclosure of information through bilateral closed-door meetings. Hence, this result can be seen as a theoretical justification for the common practice of large intergovernmental meetings, such as the European Council, the G20, or the meetings of the general assembly of the United Nations.

Our model can be extended in several directions. A particularly promising extension consists in letting the set of decision makers be a possibly proper subset of the set of players. In fact, our equilibrium and welfare results still hold in this extended environment, with minimal modifications. This extension would allow to study how the equilibrium communication strategies and welfare change when varying the allocation of decision making within a network of players. As this question is a main concern of the organization design literature, such an extension may uncover further exciting insights on the optimal organization of a firm.

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Appendix.

Proof of Theorem 1 Suppose that all agents in J believe that agent i reports his signal s truthfully. Let s_R be a vector containing the (truthful) signals that each j has received from his communication neighbors, i.e, from every $j' \in C_j(\mathbf{c}) \setminus \{i\}$, and his own signal. With some abuse of notation, we denote the in-degree of j in truthful network $\mathbf{c}$ by $k_j = |C_j(\mathbf{c})|$. Let also $y_{s_R,s}$ be the action that j would take if he has information s_R and player i has sent signal s; analogously, $y_{s_R,1-s}$ is the action that j would take if he has information s_R and player i has sent signal $1-s$. Agent i reports truthfully signal s to a collection of agents J if and only if

$$-\int_0^1 \sum_{j\in J} \sum_{s_R\in\{0,1\}^{k_j}} \left[(y_{s_R,s}-\theta-b_i)^2-(y_{s_R,1-s}-\theta-b_i)^2\right] f(\theta,s_R|s)d\theta \geq 0,$$

and using the identity $a^2-b^2=(a-b)(a+b)$ we get:

$$-\int_0^1 \sum_{j\in J} \sum_{s_R\in\{0,1\}^{k_j}} \left[(y_{s_R,s}-y_{s_R,1-s})\,(\frac{y_{s_R,s}+y_{s_R,1-s}}{2}-(\theta+b_i))\right] f(\theta,s_R|s)d\theta \geq 0.$$

Next, observing that

$$y_{s_R,s} = E\left[\theta + b_j | s_R, s\right],$$

we obtain

$$\begin{aligned}&-\int_0^1 \sum_{j\in J} \sum_{s_R\in\{0,1\}^{k_j}} \left[(E[\theta + b_j|s_R, s] - E[\theta + b_j|s_R, 1-s]) \right.\\ &\left. \cdot \left(\frac{E[\theta + b_j|s_R, s] + E[\theta + b_j|s_R, 1-s]}{2} - (\theta + b_i) \right) \right] f(\theta, s_R|s) d\theta \geq 0.\end{aligned}$$

Denote

$$\Omega = \left(E\left[\theta|s_R, s\right] - E\left[\theta|s_R, 1-s\right]\right).$$

Observing that:

$$f(\theta, s_R|s) = f(\theta|s_R, s)P(s_R|s),$$

and simplifying, we get:

$$-\sum_{j\in J} \sum_{s_R\in\{0,1\}^{k_j}} \int_0^1 \left[\Omega \left(\frac{E\left[\theta|s_R, s\right] + E\left[\theta|s_R, 1-s\right]}{2} + b_j - b_i - \theta \right) \right] f(\theta|s_R, s)P(s_R|s) d\theta \geq 0.$$

Furthermore,

$$\int_0^1 \theta f(\theta|s_R, s) d\theta = E[\theta|s_R, s],$$

and

$$\int_0^1 P(\theta|s_R, 1)E[\theta|s_R, s] d\theta = E[\theta|s_R, s],$$

because $E[\theta|s_R, s]$ does not depend on θ. Therefore, we obtain:

$$\begin{aligned}&-\sum_{j\in J} \sum_{s_R\in\{0,1\}^{k_j}} \left[\Omega \left(\frac{E\left[\theta|s_R, s\right] + E\left[\theta|s_R, 1-s\right]}{2} + b_j - b_i - E\left[\theta|s_R, s\right] \right) \right] P(s_R|s)\\ = &-\sum_{j\in J} \sum_{s_R\in\{0,1\}^{k_j}} \left[\Omega \left(-\frac{E\left[\theta|s_R, s\right] - E\left[\theta|s_R, 1-s\right]}{2} + b_j - b_i \right) \right] P(s_R|s) \geq 0.\end{aligned}$$

Now, note that:

$$
\begin{aligned}
\Omega &= E[\theta|s_R, s] - E[\theta|s_R, 1-s] \\
&= E\left[\theta|l+s, k_j+1\right] - E\left[\theta|l+1-s, k_j+1\right] \\
&= (l+1+s)/(k_j+3) - (l+2-s)/(k_j+3) \\
&= \begin{cases} -1/(k_j+3) & \text{if } s=0 \\ 1/(k_j+3) & \text{if } s=1. \end{cases}
\end{aligned}
$$

where l is the number of digits equal to one in s_R. Hence, we obtain that agent i is willing to communicate to agent j the signal $s = 0$ if and only if:

$$
-\sum_{j\in J}\left(\frac{-1}{k_j+3}\right)\left(-\frac{-1}{2(k_j+3)} + b_j - b_i\right) \geq 0,
$$

or

$$
\sum_{j\in J}\frac{b_j - b_i}{k_j+3} \geq -\sum_{j\in J}\frac{1}{2\left(k_j+3\right)^2}
$$

Note that this condition is redundant if $\sum_{j\in J} b_j - b_i > 0$. On the other hand, she is willing to communicate to agent j the signal $s = 1$ if and only if:

$$
-\sum_{j\in J}\left(\frac{1}{k_j+3}\right)\left(-\frac{1}{2(k_j+3)} + b_j - b_i\right) \geq 0,
$$

or

$$
\sum_{j\in J}\frac{b_j - b_i}{k_j+3} \leq \sum_{j\in J}\frac{1}{2\left(k_j+3\right)^2}.
$$

Note that this condition is redundant if $\sum_{j\in J} b_j - b_i < 0$. Collecting the two conditions:

$$
\left|\sum_{j\in J}\frac{b_j - b_i}{k_j+3}\right| \leq \sum_{j\in J}\frac{1}{2\left(k_j+3\right)^2}.
$$

This completes the proof of Theorem 1. ■

Proof of Corollary 1. Corollary 1 is a special case of theorem 1, in which for every $i \in N$ the partition $\mathcal{N}_i(\mathbf{g})$ of i's communication neighborhood, $N_i(\mathbf{g})$, is composed of singleton sets. ■

Proof of Theorem 2. Assume $(\mathbf{m}, \mathbf{y})$ is equilibrium in communication network $\mathbf{g}$. Select an arbitrary player i. The ex-ante expected utility of i is:

$$
\begin{aligned}
EU_i(\mathbf{m},\mathbf{y}) &= -E\left[\sum_{j=1}^{n}(y_j-\theta-b_i)^2|\{0,1\}^{k_j(\mathbf{c}))+1}\right] && (5)\\
&= -\sum_{j=1}^{n} E\left[(y_j-\theta-b_i)^2|\{0,1\}^{k_j(\mathbf{c})+1}\right], && (6)
\end{aligned}
$$

where, with some abuse of notation, $k_j(\mathbf{c})$ indicates j's in-degree in truthful network $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$.

Consider an arbitrary j with in-degree $k_j(\mathbf{c})$ and let l be the number of digits equal to one in a realized information vector $\{0,1\}^{k_j(\mathbf{c})+1}$. Then, we obtain:

$$
\begin{aligned}
E\left[(y_j-\theta-b_i)^2|\{0,1\}^{k_j(\mathbf{c})+1}\right] &= \int_0^1 \sum_{l=0}^{k_j(\mathbf{c})+1} \left(E\left[\theta|l,k_j(\mathbf{c})+1\right]+b_j-\theta-b_i\right)^2 f(l|k_j(\mathbf{c})+1,\theta)d\theta\\
&= \int_0^1 \sum_{l=0}^{k_j(\mathbf{c})+1} \left(E\left[\theta|l,k_j(\mathbf{c})+1\right]+b_j-\theta-b_i\right)^2 \frac{f\left(\theta|l,k_j(\mathbf{c})+1\right)}{k_j(\mathbf{c})+1+1}d\theta,
\end{aligned}
$$

where the second equality follows from $f(l|k_j(\mathbf{c})+1,\theta) = f(\theta|l,k_j(\mathbf{c})+1)/(k_j(\mathbf{c})+2)$. Let $\Pi = \left(E\left[\theta|l,k_j(\mathbf{c})+1\right]-\theta\right)^2$. Then we have:

$$
\begin{aligned}
&E\left[(y_j-\theta-b_i)^2|\{0,1\}^{k_j(\mathbf{c})+1}\right]\\
=\ & \frac{1}{k_j(\mathbf{c})+2}\int_0^1 \sum_{l=0}^{k_j(\mathbf{c})+1} \left(\Pi+(b_j-b_i)^2+2(b_j-b_i)\left(E\left[\theta|l,k_j(\mathbf{c})+1\right]-\theta\right)\right) f\left(\theta|l,k_j(\mathbf{c})+1\right)d\theta\\
=\ & (b_j-b_i)^2+\frac{1}{k_j(\mathbf{c})+2}\left[\int_0^1 \sum_{l=0}^{k_j(\mathbf{c})+1} \left(\Pi+2(b_j-b_i)\left(E\left[\theta|l,k_j(\mathbf{c})+1\right]-\theta\right)\right) f\left(\theta|l,k_j(\mathbf{c})+1\right)d\theta\right]\\
=\ & (b_j-b_i)^2+\frac{1}{k_j(\mathbf{c})+2}\left[\sum_{l=0}^{k_j(\mathbf{c})+1}\left(\int_0^1 \left(E\left[\theta|l,k_j(\mathbf{c})+1\right]-\theta\right)^2 f\left(\theta|l,k_j(\mathbf{c})+1\right)d\theta\right)\right].
\end{aligned}
$$

Next, let $V(\theta|l,k)$ be the variance of a beta distribution of parameters l and k, i.e.,

$$V(\theta|l,k) = \int_0^1 (E[\theta|l,k]-\theta)^2 f(\theta|l,k)d\theta.$$

It is well known that:

$$V(\theta|l,k) = \frac{(l+1)(k-l+1)}{(k+2)^2(k+3)}.$$

Hence,

$$\begin{aligned}
E\left[(y_j-\theta-b_i)^2\,|\{0,1\}^{k_j(\mathbf{c})+1}\right] &= (b_j-b_i)^2+\frac{1}{k_j(\mathbf{c})+2}\left[\sum_{l=0}^{k_j(\mathbf{c})+1} V\left(\theta|l,k_j(\mathbf{c})+1\right)\right] \\
&= (b_j-b_i)^2+\sum_{l=0}^{k_j(\mathbf{c})+1}\frac{(l+1)\left(k_j(\mathbf{c})-l+2\right)}{\left(k_j(\mathbf{c})+2\right)\left(k_j(\mathbf{c})+3\right)^2\left(k_j(\mathbf{c})+4\right)} \\
&= (b_j-b_i)^2+\frac{1}{6(k_j(\mathbf{c})+3)}.
\end{aligned}$$

We can then write the ex-ante expected utility of player i in equilibrium $(\mathbf{m},\mathbf{y})$ as follows:

$$\begin{aligned}
EU_i(\mathbf{m},\mathbf{y}) &= -\sum_{j=1}^{n}\left[(b_j-b_i)^2+\frac{1}{6(k_j(\mathbf{c})+3)}\right] \\
&= -\sum_{j=1}^{n}(b_j-b_i)^2-\frac{1}{6}\sum_{j=1}^{n}\frac{1}{k_j(\mathbf{c})+3} \\
&= -\sum_{j=1}^{n}(b_j-b_i)^2-\frac{1}{6}\sum_{k=0}^{n-1}\frac{|I(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))|}{k+3},
\end{aligned}$$

where $|I(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))|$ is the set of players with in-degree k, i.e., $I(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))=\{i\in N: k_i(\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))=k\}$. Therefore,

$$EU_i(\mathbf{m},\mathbf{y})\geq EU_i(\mathbf{m}',\mathbf{y}')$$

if and only if:

$$\sum_{k=0}^{n-1}\frac{|I(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))|}{k+3}\leq\sum_{k=0}^{n-1}\frac{|I(k|\mathbf{c}'(\mathbf{m}',\ \mathbf{y}'|\mathbf{g}'))|}{k+3},$$

which is equivalent to

$$\sum_{k=0}^{n-1}P(k|\mathbf{c}(\mathbf{m},\mathbf{y}|\mathbf{g}))\frac{1}{k+3}\leq\sum_{k=0}^{n-1}P(k|\mathbf{c}'(\mathbf{m}',\mathbf{y}'|\mathbf{g}'))\frac{1}{k+3}.$$

This concludes the proof of Theorem 2. ■

Proof of Corollary 2. The proof of Corollary 2 follows from standard arguments of stochastic dominance and therefore the details are omitted. ■

Appendix A. Not For Publication.

Proof of Proposition 1. We fist need to characterize all utility-maximizing equilibria in the two communities case developed in Section 4.1. This is done in Proposition A below.

However, before stating the result we need to introduce some definitions. A $k^1 \times k^2$-network is a network where k^x is the in-degree of players in group x, $x = 1, 2$. A segregated network is a $(n_1 - 1) \times (n_2 - 1)$-network with no links across communities. A partially segregated network is a $(n_1 - 1) \times k^2$-network where there are no links going from players in community 2 to players in community 1 and there are some links going from community 1 to community 2, i.e., $k^2 \in \{n_2, ..., n_1 - 1\}$. A complete network is a $(n - 1) \times (n - 1)$-network.

Proposition A. Consider the two-communities model.

I. The complete network is a utility-maximizing equilibrium network if and only if $b \leq \frac{1}{2(n+2)}$;

II. A $k \times k$–network with $k \in \{n_1, ..., n - 2\}$ is a utility-maximizing equilibrium if and only if $b \in \left(\frac{1}{2(k+4)}, \frac{1}{2(k+3)}\right]$;

III. A partially segregated network with $k^2 \in \{n_2, ..., n_1 - 1\}$ is a utility-maximizing equilibrium network if and only if $b \in \left(\frac{1}{2(k+4)}, \frac{1}{2(k+3)}\right]$;

IV. A segregated network is a utility-maximizing equilibrium network if and only if $b > \frac{1}{2(n_2+3)}$.

Proof of Proposition A. We first need to show the following Lemma.

Lemma 1 *Suppose* $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$ *is a utility-maximizing equilibrium network (TPEN). Then,* $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$ *is a* $k^1 \times k^2$*-communication network, i.e., all players in a group have same degree and this is larger or equal than the size of the group minus one.*

Proof of Lemma 1. With some abuse of notation we indicate a TPEN $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$ as to $\mathbf{c}$, and the in-degree of a player i in $\mathbf{c}$ as to k_i. Let $M(\mathbf{c})$ be the number of (directed) links in $\mathbf{c}$, i.e., the total number of truthful communications. First, note that the segregate communication network, $\mathbf{c}^s$, is always equilibrium and that $M(\mathbf{c}^s) = n_1(n_1 - 1) + n_2(n_2 - 1)$. Since $\mathbf{c}$ is a TPEN equilibrium, and each player in $\mathbf{c}^{\mathbf{s}}$ has the same in-degree, Theorem 2 implies that $M(\mathbf{c}) \geq M(\mathbf{c}^s)$. We now divide the analysis in two parts.

Part A. If $b > \frac{1}{2(n_2+3)}$ then $\mathbf{c} = \mathbf{c}^s$. To see this, suppose, for a contradiction, that $\mathbf{c} \neq \mathbf{c}^s$. Let $I_{12} = \{i \in N_1 : c_{ji} = 1 \text{ for some } j \in N_2\}$ and $I_{21} = \{j \in N_2 : c_{ij} = 1 \text{ for some } i \in N_1\}$. If $|I_{12}| = |I_{21}| = 0$, then, since $\mathbf{c} \neq \mathbf{c}^s$, $M(\mathbf{c}) < M(\mathbf{c}^s)$, a contradiction.

Next, assume that $|I_{12}| \neq 0$ and that $|I_{21}| \neq 0$. Since $\mathbf{c}$ is a TPEN equilibrium, it cannot be the case that $k_i < n_1 - 1$, for all $i \in I_{12}$ and $k_j < n_2 - 1$ for all $j \in I_{21}$; for otherwise $M(\mathbf{c}) < M(\mathbf{c}^s)$. Note also that for all $i \in I_{12}$ it must be the case that $k_i < n_1 - 1$. Indeed, if it exists some $i \in I_{12}$ with $k_i \geq n_1 - 1$, then, since $\mathbf{c}$ is equilibrium, it must hold that $b(k_i + 3) \leq 1/2$, which contradicts our initial hypothesis that $b(n_2 + 3) > 1/2$, because $k_i + 3 \geq n_1 - 1 + 3 \geq n_2 + 3$. These two observations imply that there must exist $j \in I_{21}$ such that $k_j \geq n_2 - 1$. Furthermore, if all these players like j have $k_j = n_2 - 1$, then $M(\mathbf{c}) < M(\mathbf{c}^s)$. So, there exists $j \in I_{21}$ such that $k_j > n_2 - 1$. In such a case, equilibrium implies that $b[k_j + 3] \leq 1/2$. But, since $k_j + 3 \geq n_2 + 3$, this contradicts our initial hypothesis that $b[n_2 + 3] > 1/2$.

Hence, it must be the case that either $|I_{12}| \neq 0$ and $|I_{21}| = 0$ or $|I_{12}| = 0$ and $|I_{21}| \neq 0$. Each of these two cases can be ruled out using the same arguments adopted for the case in which $|I_{12}| \neq 0$ and $|I_{21}| \neq 0$; details are omitted. This completes the proof of part A.

Part B. Suppose that $b(n_2 + 3) < 1/2$. We first prove that each player in group 2 must have the same in-degree, i.e., $k_i = k^2$ for all $i \in N_2$. Given $\mathbf{c}$, without loss of generality, all players in group 2 are ordered according to their in-degrees, i.e., $k_1 \leq k_2 \leq .... \leq k_{n_2}$. Assume, for a contradiction, that $k_1 < k_{n_2}$. We consider three sub-cases.

Part B, Case 1. Suppose $k_{n_2} > n_2 - 1$. This implies that $c_{jn_2} = 1$ for some $j \in N_1$, and since $\mathbf{c}$ is equilibrium, it must hold that $b[k_{n_2} + 3] \leq 1/2$. Next, since $k_1 < k_{n_2}$, it must exist a $j \in N$ such that $c_{j1} = 0$. But then the network $\mathbf{c}' = \mathbf{c} + c_{j1}$ is also equilibrium. In fact, every agent communicating in $\mathbf{c}$ with a player different from player 1 can still communicate in $\mathbf{c}'$, because the in-degrees of these players have not changed, and every agent l that was communicating with 1 in $\mathbf{c}$ still communicates in $\mathbf{c}'$ because $k_1(\mathbf{c}') = k_1(\mathbf{c} + 1) \leq k_{n_2}$ and $b[k_{n_2} + 3] \leq 1/2$. But then $\mathbf{c}$ is a subgraph of $\mathbf{c}'$, which, in view of Theorem 2, contradicts our initial hypothesis that $\mathbf{c}$ is a TPEN.

Part B, Case 2. Suppose $k_{n_2} = n_2 - 1$. We first note that $c_{jn_2} = 0$ for all $j \in N_1$; otherwise, we can replicate the argument developed in Part A, Case 1 to show a contradiction. Next, let player $l \in N_2$ such that $k_l < k_{n_2}$ and $k_{l+1} = k_{n_2}$. Note that for all $l' \in N_2$ with $l' \leq l$, there must exist some $j \in N_1$ such that $c_{jl'} = 1$. Indeed, if there exists a $l' \in N_2$ with $l' \leq l$, such that $c_{jl'} = 0$ for all $j \in N_1$, then, since $k_{l'} < n_2 - 1$, there exists a $i \in N_2$ such that $c_{il'} = 0$. But then, the network $\mathbf{c}' = \mathbf{c} + c_{il'}$ is also equilibrium, and in view of Theorem 2, this contradicts that $\mathbf{c}$ is a TPEN.

Now, for an arbitrary $l' \in N_2$ with $l' \leq l$, define $A(l')$ as the number of links that l' receives from players in group N_1. Define also $W(l')$ as the number of links that l' receives from players in group N_2. Then, the number of players in group N_2 who do not communicate with l' is

$$\bar{W}(l') = n_2 - 1 - k_{l'} + A(l') > A(l').$$

From network $\mathbf{c}$, construct $\mathbf{c}'$ in the following way: one, delete all links from group 1 to players $l' \in N_2$, with $l' \leq l$, and, two, for each $j \in N_2$ such that $c_{jl'} = 0$, $l' \in N_2$, $l' \leq l$, set $c'_{jl'} = 1$. Note that since $\mathbf{c}$ is equilibrium, then $\mathbf{c}'$ is also equilibrium, because each of the new links in $\mathbf{c}'$ are between members of the same community. Note also that

$$M(\mathbf{c}') - M(\mathbf{c}) = \sum_{l' < l, l' \in N_2} \left(\bar{W}(l') - A(l') \right) > 0,$$

and, by construction, the in-degree distribution in $\mathbf{c}'$ first order stochastic dominates the in-degree distribution of $\mathbf{c}$. Corollary 2 then implies that $\mathbf{c}'$ Pareto dominates $\mathbf{c}$, which contradicts that $\mathbf{c}$ is a TPEN equilibrium.

The final case in which $k_{n_2} < n_2 - 1$ is easy to rule out and details are omitted. We have shown that players in group 2 must have the same in-degree. The arguments developed here, can then be used to show that all players in group 1 must have the same in-degree. This concludes the proof of Lemma 1. ■

Finally, note that part IV of Proposition A follows from the proof of Part A of Lemma 1. Parts I-III of Proposition A simply follows by comparing the total number of links that can be sustained in $k^1 \times k^2$-communication network equilibrium. This concludes the proof of Proposition A. ■

The proof of Proposition 1 simply follows from Proposition A and the details are omitted. ■

Proof of Proposition 2. Let $\mathbf{g} \in \tilde{\mathbf{G}}$ be a minimally connected network and let $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$ be part of an equilibrium. Hereafter, when there is no confusion we write $\mathbf{c}$ to indicate $\mathbf{c}(\mathbf{m}, \mathbf{y}|\mathbf{g})$. We say that the link $c_{ij} = 1$ is a jump link if $|i - j| > 1$. The set of jump links in network $\mathbf{c}$ is $P(\mathbf{c}) = \{(i, j) : c_{ij} = 1 \text{ and } |i - j| > 1\}$ and we partition it in two sets: $P_1(\mathbf{c}) = \{(i, j) \in P(\mathbf{c}) : k_j(\mathbf{c}) = 1\}$ and $\tilde{P}_2(\mathbf{c}) = \{(i, j) \in P(\mathbf{c}) : k_j(\mathbf{c}) > 1\}$. We also single out two subsets of $\tilde{P}_2(\mathbf{c})$: $P_{2A}(\mathbf{c}) = \{(i, j) \in \tilde{P}_2(\mathbf{c}) : c_{ji} = 0\}$ and $P_{2B}(\mathbf{c}) = \{(i, j) \in \tilde{P}_2(\mathbf{c}) : c_{ji} = 1 \text{ and } i < j\}$. Define $P_2 = P_{2A} \cup P_{2B}$. Let $A(\mathbf{c}) = \{l : c_{ll-1} = c_{l-1l} = 0\}$ and, with some abuse of terminology, we term this as the set of unused adjacent links in network $\mathbf{c}$.

We now provide a procedure which substitutes jump links in $\mathbf{c}$ with unused adjacent links. This procedure leads to a network $\mathbf{c}'$ such that there exists a strategy profile $(\mathbf{m}', \mathbf{y}')$ which is equilibrium in the ordered line communication network $\mathbf{g}'$ and $\mathbf{c}'(\mathbf{m}', \mathbf{y}'|\mathbf{g}') = \mathbf{c}'$.

We start with two claims, which are key for the proof.

Claim 1. For every jump link $(i,j) \in P_2$ there exists a $l \in A(\mathbf{c})$ where $\min\{i,j\} < l \leq \max\{i,j\}$. This defines a non-empty correspondence $\Sigma : P_2 \rightarrow A$.

Proof of Claim 1: Suppose, by contradiction, that such l does not exist. Then the closure of $\mathbf{g}$, $\bar{\mathbf{g}}$, cannot be minimal, because there would be a cycle $\{(\min\{i,j\}, \min\{i,j\}+1), (\min\{i,j\}+1, \min\{i,j\}+2, ..., (\max\{i,j\}-1, \max\{i,j\}), (i,j)\}$. ■

Claim 2. There exists a selection σ of Σ with a well defined inverse σ^{-1}.

Proof of Claim 2: We proceed by contradiction. Suppose that there are two pairs $(i,j) \in P_2$ and $(i',j') \in P_2$ such that $\sigma(i,j) = \sigma(i',j')$ for all selections σ of Σ where $\sigma(i,j)$ is a singleton $l \in A(\mathbf{c})$. Suppose without loss of generality that $\min\{i,j\} \leq \min\{i',j'\}$. Further, because $\min\{i,j\} < l \leq \max\{i,j\}$ and $\min\{i',j'\} < l \leq \max\{i',j'\}$, it must be that $\min\{i',j'\} < \max\{i,j\}$. We distinguish two cases.

First, suppose $\max\{i,j\} \leq \max\{i',j'\}$. Then, because $\min\{i,j\} < l \leq \max\{i,j\}$ and $\min\{i',j'\} < l \leq \max\{i',j'\}$, it must be that $\min\{i',j'\} < l \leq \max\{i,j\}$. But this means that $\bar{\mathbf{g}}$ has the cycle $\bar{g}_{i,j} = 1$, $\bar{g}_{l',l'-1} = 1$ for all $\min\{i,j\} < l' \leq \min\{i',j'\}$, $\bar{g}_{i',j'} = 1$, $\bar{g}_{l',l'-1} = 1$ for all $\max\{i,j\} < l' \leq \max\{i',j'\}$.

In the second case, $\max\{i',j'\} < \max\{i,j\}$. Then, because $\min\{i,j\} < l \leq \max\{i,j\}$ and $\min\{i',j'\} < l \leq \max\{i',j'\}$, it must be that $\min\{i',j'\} < l \leq \max\{i',j'\}$. Because $\sigma(i,j) = \sigma(i',j')$ is a singleton, it must be that $\bar{g}_{l',l'-1} = 1$ for all $\min\{i,j\} < l' \leq \min\{i'.j'\}$ and $\bar{g}_{l',l'-1} = 1$ for all $\max\{i',j'\} < l' \leq \max\{i,j\}$. But this means that $\bar{\mathbf{g}}$ has the cycle $\bar{g}_{i,j} = 1$, $\bar{g}_{l',l'-1} = 1$ for all $\min\{i,j\} < l' \leq \min\{i',j'\}$, $\bar{g}_{i',j'} = 1$, $\bar{g}_{l',l'-1} = 1$ for all $\max\{i,j\} < l' \leq \max\{i',j'\}$. ■

We are now ready to prove Proposition 2.

Part A. Jump links in $P_1(\mathbf{c})$. Substitute any jump link $(i,j) \in P_1(\mathbf{c})$ such that $i < j$, with the unused adjacent link $(j-1,j)$. This is possible because, since $(i,j) \in P_1(\mathbf{c})$, then $k_j(\mathbf{c}) = 1$ and, since $c_{ij} = 1$, it follows that $c_{j-1j} = 0$. Analogously, substitute any jump link $(i,j) \in P_1(\mathbf{c})$ such that $i > j$, with the unused adjacent link $(j+1,j)$.

Part B: jump links in $\mathbf{P}_2(\mathbf{c})$. We now take up jump links in the set $\mathbf{P}_2(\mathbf{c})$. Here, we use extensively Claim 1 and Claim 2. Note that the two claims imply that there is an invertible function, σ, which maps for every jump link in P_2, say jump link (i, j), to an unused adjacent link $l \in A(\mathbf{c})$ with $c_{ll-1} = c_{l-1l} = 0$ and $\min\{i, j\} < l \leq \max\{i, j\}$. We first consider jump links in P_{2A} and then jump links in P_{2B}.

Jump links in P_{2A}. *First,* substitute any $(i, j) \in P_{2A}$ such that there is no jump link $(i', j') \in P_1$ where $j' = \sigma(i, j) - 1$ and $i' > j'$, with the unused adjacent link $(\sigma(i, j), \sigma(i, j) - 1)$ if $i > j$, while, if $i < j$, with the unused adjacent link $(\sigma(i, j) - 1, \sigma(i, j))$. Make the same substitution for $(i, j) \in P_{2A}$ such that there is no jump link $(i', j') \in P_1$, with $j' = \sigma(i, j)$ and $i' < j'$. *Second,* Substitute any $(i, j) \in P_{2A}$ such that there is a jump link $(i', j') \in P_1$ where $j' = \sigma(i, j) - 1$ and $i' > j'$, with the unused adjacent link $(\sigma(i, j) - 1, \sigma(i, j))$. *Third,* substitute any $(i, j) \in P_{2A}$ such that there is a $(i', j') \in P_1$ with $j' = \sigma(i, j)$ and $i' < j'$, with the unused adjacent link $(\sigma(i, j) + 1, \sigma(i, j))$. *Fourth,* substitute any $(i, j) \in P_{2A}$ such that there is a $(i', j') \in P_1$ with $j' = \sigma(i, j) - 1$ and $i' > j'$ as well as a jump link $(i', j') \in P_1$ with $j' = \sigma(i, j)$ and $i' < j'$, with unused adjacent link $(\sigma(i, j) + 1, \sigma(i, j))$.

Jump links in P_{2B}. *First,* for any $(i, j) \in P_{2B}$ such that there is no jump link $(i', j') \in P_1$ where $j' = \sigma(i, j) - 1$ and $i' > j'$, substitute the link $c_{ij} = 1$ with the unused adjacent link $(\sigma(i, j) - 1, \sigma(i, j))$ and the link $c_{ji} = 1$ with the unused adjacent link $(\sigma(i, j), \sigma(i, j) - 1)$. Make the same substitutions for $(i, j) \in P_{2B}$ such that there is no jump link $(i', j') \in P_1$, with $j' = \sigma(i, j)$ and $i' < j'$. *Second,* for any $(i, j) \in P_{2B}$ such that there is a jump link $(i', j') \in P_1$ where $j' = \sigma(i, j) - 1$ and $i' > j'$, substitute the jump link $c_{ij} = 1$ with $(\sigma(i, j) - 1, \sigma(i, j))$ and the jump link $c_{ji} = 1$ with $(\sigma(i, j) - 2, \sigma(i, j) - 1)$. Here note that $c_{\sigma(i,j)-2,\sigma(i,j)-1} = 0$ because, since $(i', j') \in P_1$, $k_{\sigma(i,j)-1} = 1$. *Third,* for any $(i, j) \in P_{2B}$ such that there is a jump link $(i', j') \in P_1$ where $j' = \sigma(i, j)$ and $i' < j'$, substitute the jump link $c_{ij} = 1$ with $(\sigma(i, j), \sigma(i, j) - 1)$ and the jump link $c_{ji} = 1$ with $(\sigma(i, j) + 1, \sigma(i, j))$. *Fourth,* for any $(i, j) \in P_{2B}$ such that there is a jump link $(i', j') \in P_1$ with $j' = \sigma(i, j) - 1$ and $i' > j'$ as well as a jump link $(i', j') \in P_1$ with $j' = \sigma(i, j)$ and $i' < j'$, substitute the jump link $c_{ij} = 1$ with

$(\sigma(i,j)-2, \sigma(i,j)-1)$ and the jump link $c_{ji}=1$ with $(\sigma(i,j)+1, \sigma(i,j))$.

By construction, when applying simultaneously to $\mathbf{c}$ all these substitutions we obtain a new network $\mathbf{c}'$, which can be supported in equilibrium in a ordered line communication network. Note also that, by construction, the total number of (directed) links in $\mathbf{c}$ is the same as the total number of (directed) links in $\mathbf{c}'$.

We now show that from $\mathbf{c}'$ we can construct a new equilibrium $\mathbf{c}''$, which can be supported in the ordered line communication network and in which the expected utility of each player is higher than in the original equilibrium $\mathbf{c}$.

Let $N^+(\mathbf{c}') = \{i \in N : k_i(\mathbf{c}') > k_i(\mathbf{c})\}$ and $N^-(\mathbf{c}') = \{j \in N : k_j(\mathbf{c}') < k_j(\mathbf{c})\}$, and we recall that $k_j(\mathbf{c})$ denotes the in-degree of j in network $\mathbf{c}$. Define $S(\mathbf{c}') = N \setminus \{N^+(\mathbf{c}) \cup N^-(\mathbf{c})\}$. Clearly, if $S(\mathbf{c}') = N$, then $EU_i(\mathbf{c}) = EU_i(\mathbf{c}')$, for all $i \in N$, and the claim follows.

Suppose instead that $S(\mathbf{c}') \subset N$. By construction of $\mathbf{c}'$, it follows that $\sum_{i \in N^+(\mathbf{c}')}[k_i(\mathbf{c}') - k_i(\mathbf{c})] = \sum_{j \in N^i(\mathbf{c}')}[k_j(\mathbf{c}) - k_j(\mathbf{c})]$, and therefore $S(\mathbf{c}') \subset N$ if and only if $N^+(\mathbf{c}')$ and $N^-(\mathbf{c}')$ are both non-empty sets. Furthermore, each player in $i \in N^+(\mathbf{c}')$ is such that: (a) $k_i(\mathbf{c}') = 1$ and $k_i(\mathbf{c}) = 0$, or, (b) $k_i(\mathbf{c}') = 2$ and $k_i(\mathbf{c}) = 1$, or, (c) $k_i(\mathbf{c}') = 2$ and $k_i(\mathbf{c}') = 0$.

Take a player $i \in N^+(\mathbf{c}')$ with $k_i(\mathbf{c}') = 1$ and $k_i(\mathbf{c}) = 0$. Select, if there exists, a $j \in N^-(\mathbf{c}')$ with $k_j(\mathbf{c}') < 2$. Delete the link that i receives, and add an adjacent link to j which can be sustained in equilibrium. Clearly, such link exists because $k_j(\mathbf{c}) > k_j(\mathbf{c}') \in \{0, 1\}$. Call this new profile $\mathbf{c}''$. This is equilibrium and note that $S(\mathbf{c}') \subset S(\mathbf{c}'')$. By repeating this procedure, we end up with an equilibrium, say $\hat{\mathbf{c}}$ such that if there exists $i \in N^+(\hat{\mathbf{c}})$ with $k_i(\hat{\mathbf{c}}) = 1$ and $k_i(\mathbf{c}) = 0$, then every $j \in N^-(\hat{\mathbf{c}})$ has $k_j(\hat{\mathbf{c}}) = 2$.

Take a player $i \in N^+(\hat{\mathbf{c}})$ with $k_i(\hat{\mathbf{c}}) = 2$ and $k_i(\mathbf{c}) = 1$. Select, if there exists, a $j \in N^-(\hat{\mathbf{c}})$ with $k_j(\hat{\mathbf{c}}) < 2$. Delete a link that i receives, and add an adjacent link to j which can be sustained in equilibrium. Clearly, such link exists because $k_j(\mathbf{c}) > k_j(\hat{\mathbf{c}}) \in \{0, 1\}$. Call this new profile $\hat{\mathbf{c}}'$. This is equilibrium and note that $S(\hat{\mathbf{c}}) \subset S(\hat{\mathbf{c}}')$. By repeating this procedure, we end up with an equilibrium, say $\tilde{\mathbf{c}}$ such that if there exists $i \in N^+(\tilde{\mathbf{c}})$ with $k_i(\tilde{\mathbf{c}}) = 1$ and $k_i(\mathbf{c}) = 0$, then every $j \in N^-(\tilde{\mathbf{c}})$ has $k_j(\tilde{\mathbf{c}}) = 2$.

Take a player $i \in N^+(\tilde{\mathbf{c}})$ with $k_i(\tilde{\mathbf{c}}) = 2$ and $k_i(\mathbf{c}) = 0$. Select, if there exists, a $j \in N^-(\tilde{\mathbf{c}})$ with one of the following property: (1) $k_j(\tilde{\mathbf{c}}) = 0$ and $k_j(\mathbf{c}) = 1$, (2) $k_j(\tilde{\mathbf{c}}) = 0$ and $k_j(\mathbf{c}) \geq 2$, (3) $k_j(\tilde{\mathbf{c}}) = 1$ and $k_j(\mathbf{c}) = 2$. In case (1) delete a link that i receives and add an adjacent link to j; in case (2), delete the two links that i receives and add two adjacent links to j; in case (3) delete a link that i receives and add an adjacent link to j. Call this new profile $\tilde{\mathbf{c}}'$. This is equilibrium and note that $S(\tilde{\mathbf{c}}) \subset S(\tilde{\mathbf{c}}')$. By repeating this procedure, we end up with an equilibrium, say $\tilde{\tilde{\mathbf{c}}}$, such that if there exists $i \in N^+(\tilde{\tilde{\mathbf{c}}})$ with $k_i(\tilde{\tilde{\mathbf{c}}}) = 2$ and $k_i(\mathbf{c}) = 0$, then every $j \in N^-(\tilde{\tilde{\mathbf{c}}})$ has $k_j(\tilde{\tilde{\mathbf{c}}}) \in \{1, 2\}$ and $k_j(\mathbf{c}) \geq 3$.

For a given $\mathbf{c}$, the three procedures given above transform a profile on the line with the same total number of (directed) links as in $\mathbf{c}$ into another profile on the line with the same number of (directed) links as in $\mathbf{c}$ and $\mathbf{c}'$. We note this transformation Φ, so that $\tilde{\tilde{\mathbf{c}}} = \mathbf{\Phi}(\mathbf{c}^{'})$. The procedure can be iterated until a profile $\mathbf{c}^*$ such that $S(\mathbf{c}^*) = S(\mathbf{\Phi}(\mathbf{c}^*))$ is reached. If $S(\mathbf{c}^*) = N$, then clearly the in-degree distribution in $\mathbf{c}^*$ equals the in-degree distribution in $\mathbf{c}$. Hence, the expected utility of each player is the same in the two equilibria and the proof follows. Suppose that $S(\mathbf{c}^*) \subset N$. Let $N_a^+(\mathbf{c}^*) = \{i \in N^+(\mathbf{c}^*) : k_i(\mathbf{c}^*) = 2 \text{ and } k_i(\mathbf{c}) = 1\}$, $N_b^+(\mathbf{c}^*) = \{i \in N^+(\mathbf{c}^*) : k_i(\mathbf{c}^*) = 2 \text{ and } k_i(\mathbf{c}) = 0\}$ and $N_b^+(\mathbf{c}^*) = \{i \in N^+(\mathbf{c}^*) : k_i(\mathbf{c}^*) = 1 \text{ and } k_i(\mathbf{c}) = 0\}$. Let also $n_x = |N_x^+(\mathbf{c}^*)|$, $x = a, b, c$. By construction of $\mathbf{c}^*$ we have that $N^+(\mathbf{c}^*) = \cup_{x \in \{a,b,c\}} N_x^+(\mathbf{c}^*)$. Furthermore, note that

$$\sum_{i \in N^+(\mathbf{c}^*)} [k_i(\mathbf{c}^*) - k_i(\mathbf{c})] = \sum_{j \in N^-(\mathbf{c}^*)} [k_j(\mathbf{c}) - k_j(\mathbf{c}')]$$

and since

$$\sum_{i \in N^+(\mathbf{c}^*)} [k_i(\mathbf{c}^*) - k_i(\mathbf{c})] = n_a + 2n_b + n_c$$

it follows that

$$n_a + 2n_b + n_c = \sum_{j \in N^-(\mathbf{c}^*)} [k_j(\mathbf{c}) - k_j(\mathbf{c}^*)]. \tag{7}$$

Using the expression 4 in Theorem 2, we can see that the expected utility of an arbitrary i in $\mathbf{c}^*$ is at least as high as her expected utility in equilibrium $\mathbf{c}$ if and only if

$$\frac{n_a+n_b}{5}+\frac{n_c}{4}+\sum_{j\in N^-(\mathbf{c}^*)}\frac{1}{k_j(\mathbf{c}^*)+3}\leq\frac{n_a}{4}+\frac{n_b+n_c}{3}+\sum_{j\in N^-(\mathbf{c}^*)}\frac{1}{k_j(\mathbf{c})+3}.$$

This is satisfied if only if

$$\sum_{j\in N^-(\mathbf{c}^*)}\frac{k_j(\mathbf{c})-k_j(\mathbf{c}^*)}{(k_j(\mathbf{c})+3)(k_j(\mathbf{c}^*)+3)}\leq\frac{3n_a+8n_b+5n_c}{60}.$$

Note that by construction $j\in N^-(\mathbf{c}^*)$ if and only if $k_j(\mathbf{c}^*)\in\{1,2\}$ and $k_j(\mathbf{c})\geq 3$. Hence,

$$\begin{aligned}\sum_{j\in N^-(\mathbf{c}^*)}\frac{k_j(\mathbf{c})-k_j(\mathbf{c}^*)}{(k_j(\mathbf{c})+3)(k_j(\mathbf{c}^*)+3)} &\leq \sum_{j\in N^-(\mathbf{c}^*)}\frac{k_j(\mathbf{c})-k_j(\mathbf{c}^*)}{(3+3)(1+3)}\\ &= \sum_{j\in N^-(\mathbf{c}^*)}\frac{k_j(\mathbf{c})-k_j(\mathbf{c}^*)}{24}\\ &= \frac{n_a+2n_b+n_c}{24}\\ &< \frac{3n_a+8n_b+5n_c}{60},\end{aligned}$$

where the second equality follows by using (7), while the last inequality is easily verified. This completes the proof of the proposition. ■

Proof of Proposition 3. The proof proceeds in three steps. The first step of the proof shows that a strategy profile $(\mathbf{m},\mathbf{y})$ that satisfies the conditions in Proposition 3 is equilibrium. The second step shows the set of these equilibrium strategies is a subset of the set of utility-maximizing equilibria. The last step shows the second part of the Proposition 3.

Step I. Let $(\mathbf{m},\mathbf{y})$ be a strategy profile that satisfies the conditions in Proposition 3. We show that $(\mathbf{m},\mathbf{y})$ is an equilibrium. Select $l=1,...,2j-n-1$. When $\beta\leq[2(j-l)(n-l+3)]^{-1}$, the strategy profile such that the $n-l$ players $\{l,...,j-1,j+1,...,n\}$ truthfully communicate with j is part of an equilibrium. Indeed, as $j-l>n-j$, i.e., $l<2j-n$, it follows that

$$\{l\}=\arg\max_{i\in\{l,...,j-1,j+1,...,n\}}|b_i-b_j| \text{ and that } j-l=\max_{i\in\{l,...,j-1,j+1,...,n\}}|b_i-b_j|,$$

and Theorem 1 implies that the requirement for the strategy profile to be an equilibrium is exactly $\beta \leq [2(j-l)(n-l+3)]^{-1}$.

Next, select $l = 2j-n, ...j-1$. For $\beta \leq [2(j-l)(2(j-l)+3)]^{-1}$ the profile such that the $2(j-l)$ players who truthfully communicate with j are $\{l, ..., j-1, j+1, 2j-l\}$ or $\{l+1, ..., j-1, j+1, 2j-l+1\}$ is part of an equilibrium. Indeed, suppose the players who truthfully communicate with j are $\{l, ..., j-1, j+1, 2j-l\}$ (the other case being symmetric). As $j-l = (2j-l)-j$ it follows that:

$$\{l, 2j-l\} = \arg \max_{i \in \{l, ..., j-1, j+1, ..., 2j-l\}} |b_i - b_j| \text{ and that } j-l = \max_{i \in \{l, ..., j-1, j+1, ..., 2j-l\}} |b_i - b_j|,$$

and Theorem 1 implies that the requirement for the profile to be an equilibrium is exactly $\beta \leq [2(j-l)(2(j-l)+3)]^{-1}$.

To conclude the first step, note that for $\beta \leq [2(j-l)(2(j-l)-1+3)]^{-1}$, the profile such that the $2(j-l)-1$ players who truthfully communicate with player j are $\{l+1, ..., j-1, j+1, 2j-l\}$ is part of an equilibrium. Indeed, as $j-l = (2j-l)-j$ it follows that:

$$\{2j-l\} = \arg \max_{i \in \{l+1, ..., j-1, j+1, ..., 2j-l\}} |b_i - b_j| \text{ and that } j-l = \max_{i \in \{l+1, ..., j-1, j+1, ..., 2j-l\}} |b_i - b_j|,$$

and theorem 1implies that the requirement for the strategy profile to be an equilibrium is exactly $\beta \leq [2(j-l)(2(j-l)-1+3)]^{-1}$.

Step II. Suppose that $(\mathbf{m}, \mathbf{y})$ belongs to the set of equilibrium strategy profiles considered in Step I above. We now show that this strategy profile is such that $\mathbf{m}$ is a utility-maximizing equilibrium. We start by noting that for every $l = 1, ..., 2j-n-1$ if a set of players C_j communicates with j and $|C_j| = n-l$, then $\beta \leq [2(j-l)(n-l+3)]^{-1}$. Indeed, since $n-l$ players communicate with j, there must be a player $i \in C_j$ such that $i \leq l$, and the equilibrium condition for player i to communicate with j is:

$$\beta \leq [2(j-i)(n-l+3)]^{-1} \leq [2(j-l)(n-l+3)]^{-1},$$

where the inequality follows because $i \leq l$. Because $[2(j-l)(n-l+3)]^{-1}$ increases in l, it follows that if a set of players C_j truthfully communicates with j and $|C_j| = n-v \geq n-l$, then

$\beta \leq [2(j-v)(n-v+3)]^{-1} \leq [2(j-l)(n-l+3)]^{-1}$. Hence, for every $l = 1, ..., 2j-n-1$, if $\beta > [2(j-l)(n-l+3)]^{-1}$, then there is no equilibrium where $n-l$ players truthfully communicate to j. So, the proposed profile where $n-l-1$ players truthfully communicate to j, achieves the maximal number of communication links to player j and it is part of a utility-maximizing equilibrium.

We now turn to the case of $l = 2j-n, ..., j-1$, and show equilibrium communication by $2(j-l)$ players to j requires that $\beta \leq [2(j-l)(2(j-l)+3)]^{-1}$. To see this, suppose that a set C_j of $2(j-l)$ players communicate with j. Then, there must be a player $i \in C_j$ such that $|j-i| \geq j-l$. Consequently, the equilibrium condition for player i to communicate with j is:

$$\beta \leq [2(j-i)(2(j-l)+3)]^{-1} \leq [2(j-l)(2(j-l)+3)]^{-1}.$$

Because $[2(j-l)(2(j-l)+3)]^{-1} < [2(j-l)(2(j-l)-1+3)]^{-1}$ holds for all l and the fact that $[2(j-l)(2(j-l)+3)]^{-1}$ increases in l, we can conclude that communication by at least $2(j-l)$ players to j requires $\beta \leq [2(j-l)(2(j-l)+3)]^{-1}$. Hence, for $\beta > [2(j-l)(2(j-l)+3)]^{-1}$, the specified strategy profile where $2(j-l)-1$ players communicate with j, is part of a utility maximizing equilibrium.

To conclude this second step we need to show that equilibrium communication by $2(j-l)-1$ players to j requires that $\beta \leq [2(j-l)(2(j-l)-1+3)]^{-1}$. Indeed, if a set C_j of $2(j-l)-1$ players communicates with j, then, there must be a player $i \in C_j$ such that $|j-i| \geq j-l$. Then, the equilibrium condition for player i to communicate with j is:

$$\beta \leq [2(j-i)(2(j-l)-1+3)]^{-1} \leq [2(j-l)(2(j-l)-1+3)]^{-1}.$$

Because $[2(j-l)(2(j-l)-1+3)]^{-1} < [2(j-(l+1))(2(j-(l+1))+3)]^{-1}$ holds and the fact that $[2(j-l)(2(j-l)-1+3)]^{-1}$ increases in l, we can conclude that communication by $2(j-l)-1$ players with j requires $\beta \leq [2(j-l)(2(j-l)-1+3)]^{-1}$. Hence, for $\beta > [2(j-l)(2(j-l)-1+3)]^{-1}$, the specified strategy profile where $2(j-(l-1))$ players communicate with j, is part of a utility-maximizing equilibrium.

Step III. We now show the last part of the Proposition. We have already shown that for any $l = 1, ..., 2j - n - 1$ if a set of players C_j communicates to j and $|C_j| = n - l$, then $\beta \leq [2(j-l)(n-l+3)]^{-1}$. If the $n-l$ players who communicate are not $\{l, ..., j-1, j+1, ..., n\}$, then there must be a player $i < l$, and the equilibrium condition for player i to communicate with j is:

$$\beta \leq [2(j-i)(n-l+3)]^{-1} < [2(j-l)(n-l+3)]^{-1},$$

where the inequality follows because $i < l$. Therefore for every configuration where the $n-l$ players who communicate are not $\{l, ..., j-1, j+1, ..., n\}$ there exists some $\beta \in B(\mathbf{m}, \mathbf{y})$ (recall $\mathbf{m}, \mathbf{y} \in S^*(\beta)$) such that such configuration is not an equilibrium.

Consider now the case of $l = 2j - n, ...j - 1$. Suppose that a set C_j of $2(j-l)$ players communicates with j, other than the specified configurations. Then, there must be a player $i \in C_j$ such that $|j - i| > j - l$, so that the equilibrium condition for player i to communicate with j is:

$$\beta \leq [2(j-i)(2(j-l)+3)]^{-1} < [2(j-l)(2(j-l)+3)]^{-1}.$$

Consequently, there exists some $\beta \in B(\mathbf{m}, \mathbf{y})$ such that such configuration is not an equilibrium.

Finally, if a set C_j of $2(j-l)-1$ players communicate with j, other than the specified configurations, then, there must be a player $i \in C_j$ such that $|j-i| > j-l$. Hence, the equilibrium condition for player i to communicate with j is:

$$\beta \leq [2(j-i)(2(j-l)-1+3)]^{-1} < [2(j-l)(2(j-l)-1+3)]^{-1}.$$

Again there exists some $\beta \in B(\mathbf{m}, \mathbf{y})$ such that such configuration is not an equilibrium. This concludes the proof of Proposition 3. ■

Proof of Corollary 3. The first part of the corollary is obvious. We prove the second part. For all β, we show that $M(j, \beta)$ is weakly decreasing in j for $j \in \{1, ..., V(\beta)\}$. In fact, solving $2\beta(j - 1 + M + 3)M - 1 = 0$, we obtain that $M(j, \beta) = \left\lfloor \frac{1}{2}\left(-(j+2) + \sqrt{2/\beta + (j+2)^2}\right)\right\rfloor$ and that

$$\frac{dM(j,\beta)}{dj} = \frac{1}{2}\left(\frac{j+2}{\sqrt{2/\beta + (j+2)^2}} - 1\right) < 0$$

Consequently, $d(j)$ decreases in j. Furthermore, j's in-degree is equal to $j-1+M(j,\beta)$, and it is easy to check that it increases in j. Also, we note that, by construction, $M(V(\beta),\beta)=V(\beta)$. Hence, when $\beta > \frac{1}{2(2V(\beta)+3)V(\beta)}$, j's in-degree increases in j until reaching $2V(\beta)-1$ for $j=V(\beta)$. On the other hand, when $\beta \leq \frac{1}{2(2V(\beta)+3)V(\beta)}$, j's in-degree increases in j until reaching $2V(\beta)$ for $j=V(\beta)+1$ and then stays constant. ■

Proof of Proposition 4 The proof proceeds in two steps. In the first step we show that the described profile of strategies is equilibrium. The second step shows that the constructed equilibria are utility maximizing equilibria. In what follows $(\mathbf{m},\mathbf{y})$ denotes the equilibrium, $\mathbf{c}(\mathbf{m},\mathbf{y})$ the truthful communication network and k_j is the in-degree of j in truthful communication network $\mathbf{c}(\mathbf{m},\mathbf{y})$. Note that, with some abuse of notation, we have suppressed the qualification that the communication network $\mathbf{g}$ is complete.

First Step. We show that the described strategy profiles are equilibria. First, note that Theorem 1 implies that, when $\beta \leq f(k,n)$, the profile $(\mathbf{m},\mathbf{y})$ such that $c_{i,j}(\mathbf{m},\mathbf{y})=1$ if and only if $i \in \{l,...,n-l+1\}$ is equilibrium if and only if

$$\left|\sum_{j\in N\setminus\{i\}}\frac{b_j-b_i}{k_j+3}\right| \leq \sum_{j\in N\setminus\{i\}}\frac{1}{2\left(k_j+3\right)^2}$$

for all $i \in \{l,...,n-l+1\}$. To see this note that in $\mathbf{c}(\mathbf{m},\mathbf{y})$ there are $n-2l+2$ players communicating truthfully, $k_j=n-2l+1$ for all $j \in \{l,...,n-l+1\}$, whereas $k_j=n-2l+2$ for all $j \notin \{l,...,n-l+1\}$. Because $b_j-b_i=\beta(j-i)$, the above equilibrium condition simplifies to:

$$\left|\sum_{j\in\{l,...,n-l+1\}\setminus\{i\}}\frac{\beta(j-i)}{n-2l+1+3}+\sum_{j=1}^{l-1}\frac{\beta(j-i)}{n-2l+2+3}+\sum_{j=n-l+2}^{n}\frac{\beta(j-i)}{n-2l+2+3}\right|$$
$$\leq \sum_{j\in\{l,...,n-l+1\}\setminus\{i\}}\frac{1}{2\left(n-2l+1+3\right)^2}+\sum_{j=1}^{l-1}\frac{1}{2\left(n-2l+2+3\right)^2}+\sum_{j=n-l+2}^{n}\frac{1}{2\left(n-2l+2+3\right)^2},$$

for all $i \in \{l,...,n-l+1\}$. This condition can be further simplified as follows: $\beta \leq \min_{i\in\{l,...,n-l+1\}}\phi(i,l,n)$, where

$$\phi(i,l,n) = \frac{\frac{n-2l+1}{2(n-2l+4)^2}+\frac{2(l-1)}{2(n-2l+5)^2}}{\left|\frac{\sum_{j\in\{l,\ldots,n-l+1\}\setminus\{i\}}(j-i)+(n-2l+4)\left[\sum_{j\neq i}(j-i)\right]}{(n-2l+4)(n-2l+5)}\right|}$$
$$= \frac{\frac{n-2l+1}{2(n-2l+4)^2}+\frac{2(l-1)}{2(n-2l+5)^2}}{\frac{1}{2}\left|n+1-2i\right|\frac{(n+(n-2l+4)(n-2l+2))}{(n-2l+4)(n-2l+5)}}.$$

The numerator of this expression does not depend on i, whereas the denominator is decreasing for $i<(n+1)/2$, it is increasing for $i>(n+1)/2$ and symmetric around $(n+1)/2$. Thus, the denominator is maximized for $i=l$ and $i=n-l+1$. This implies that $\min_{i\in\{l,\ldots,n-l+1\}}\phi(i,l,n)=\phi(l,l,n)$ and, by definition, $f(l,n)=\phi(l,l,n)$. Hence we have recovered the condition that $\beta\leq f(l,n)$. For future reference, we stress that $\min_{i\in\{l,\ldots,n-l+1\}}\phi(i,l,n)=\phi(n-l+1,l,n)=\phi(l,l,n)$.

Next, using a similar approach, we note that when $\beta\leq g(l,n)$, the strategy profile $(\mathbf{m},\mathbf{y})$ such that $c_{i,j}(\mathbf{m},\mathbf{y})=1$ if and only if $i\in\{l,...,n-l\}$ is equilibrium if and only if

$$\left|\sum_{j\in\{l,\ldots,n-l\}\setminus\{i\}}\frac{\beta(j-i)}{n-2l+3}+\sum_{j=1}^{l-1}\frac{\beta(j-i)}{n-2l+1+3}+\sum_{j=n-l+1}^{n}\frac{\beta(j-i)}{n-2l+1+3}\right|$$
$$\leq \sum_{j\in\{l,\ldots,n-l\}\setminus\{i\}}\frac{1}{2(n-2l+3)^2}+\sum_{j=1}^{l-1}\frac{1}{2(n-2l+1+3)^2}+\sum_{j=n-l+1}^{n}\frac{1}{2(n-2l+1+3)^2},$$

for all $i\in\{l,...,n-l\}$. The above condition simplifies as: $\beta\leq\min_{i\in\{l,\ldots,n-l\}}\gamma(i,l,n)$, where

$$\gamma(i,l,n) = \frac{\frac{n-2l}{2(n-2l+3)^2}+\frac{2l-1}{2(n-2l+4)^2}}{\left|\frac{\sum_{j\in\{l,\ldots,n-l\}\setminus\{i\}}(j-i)+(n-2l+3)\sum_{j\neq i}(j-i)}{(n-2l+3)(n-2l+4)}\right|}$$
$$= \frac{\frac{n-2l}{2(n-2l+3)^2}+\frac{2l-1}{2(n-2l+4)^2}}{\frac{1}{2}\frac{\left|(n-2i)(n-2l+1)+(n-2l+3)n(n+1-2i)\right|}{(n-2l+3)(n-2l+4)}}.$$

In $\gamma(i,l,n)$, the numerator does not depend on i; the denominator is maximal for $i=l$, because $\{l,n-l\}=\arg\max_{i\in\{l,\ldots,n-l\}}|n-2i|$ and $\{l\}=\arg\max_{i\in\{l,\ldots,n-l\}}|n+1-2i|$. Hence, $\min_{i\in\{l,\ldots,n-l\}}\gamma(i,l,n)=\gamma(l,l,n)$ and, by definition, $g(l,n)=\gamma(l,l,n)$. Hence we have recovered the condition that $\beta\leq g(l,n)$.

Second Step. We now show that the equilibria described are utility-maximizing equilibria. This amounts to show that: 1) when $g(l,n) < \beta$, there is no equilibrium where strictly more than $n-2l$ players truthfully communicates, and 2) when $f(l,n) < \beta$, there is no equilibrium where strictly more than $n-2l+1$ players truthfully communicate. To see that this is sufficient, note that the welfare of each player i when L players communicate truthfully is: $W_i(L) = -\sum_{j\in N}(b_i-b_j)^2 - (n-L)\frac{1}{6(L+2)} - L\frac{1}{6(L-1+2)}$. Indeed, each of the L players who communicate truthfully receives $L-1$ truthful messages, whereas each of the remaining players who do not communicate truthfully receives L messages. It is easy to see that $W_i(L)$ is increasing in L, i.e., $W_i'(L) = \frac{1}{6}\frac{n(1+L)^2+L^2-2}{(L+1)^2(L+2)^2} > 0$ for $n > 2$.

We start by noting that because $g(v-1,n) < f(v,n) < g(v,n)$ for all $v = 1,...,\lfloor\frac{n}{2}\rfloor$, it follows that for $\beta > f(l,n)$ there are no equilibria where strictly more than $n-2l+2$ players communicate, and that for $\beta > g(l,n)$ there are no equilibria where strictly more than $n-2l$ players communicate.

Next, suppose that $n-2l+2$ players communicate in an equilibrium $(\mathbf{m}',\mathbf{y}')$. Let the set of players who truthfully communicate in $(\mathbf{m}',\mathbf{y}')$ be C', so that $|C'| = n-2l+2$. Then, since $(\mathbf{m}',\mathbf{y}')$ is equilibrium, Theorem 3 implies that for all $i \in C'$ it must be that

$$\beta \leq \min_{i\in C'}\phi(i,|C'|,n) \text{ where } \phi(i,|C'|,n) = \frac{\frac{n-2l+1}{2(n-2l+4)^2}+\frac{2l-2}{2(n-2l+5)^2}}{\left|\frac{\left[\sum_{j\in N\setminus\{i\}}(j-i)\right][n-2l+4]+\left[\sum_{j\in C'\setminus\{i\}}(j-i)\right]}{[n-2l+4][n-2l+5]}\right|}.$$

We now claim that the set $C^* = \{l,...,n-l+1\}$ has the property that

$$\{C^*\} = \arg\max_{C:|C|=n-2l+2}\min_{i\in C}\phi(i,C,n).$$

Note that this claim would imply that an equilibrium where $n-2l+2$ players communicate truthfully exists if and only if $\beta \leq \min_{i\in C^*}\phi(i,C^*,n)$. But, since we have earlier proved that $l \in \arg\min_{i\in C^*}\phi(i,|C^*|,n)$ and that $f(l,n) = \min_{i\in C^*}\phi(i,|C^*|,n)$, this implies that if $\beta > f(l,n)$ then there are no equilibria where $n-2l+2$ players communicate.

To prove the claim, first note that the numerator of $\phi(i,|C|,n)$ depends neither on i nor on $|C|$. Consider the denominator of $\phi(i,|C|,n)$, and suppose that $C \neq \{l,...,n-l+1\}$. Let v be one of the most extreme players in C, i.e., $v \in \arg\max_{i\in C}|i-(n+1)/2|$. We must consider two sub-cases.

The first sub-case is when $v<(n+1)/2$. Here, note that

$$\sum_{j\in N\backslash\{v\}}(j-v) > \sum_{j\in N\backslash\{l\}}(j-l) > 0 \textit{ and } \sum_{j\in C\backslash\{v\}}(j-v) \geq \sum_{j\in C^*\backslash\{l\}}(j-l) > 0.$$

These inequalities follow from noticing that: 1) since $C \neq \{l,...,n-l+1\}$ and $v \in \arg\max_{i\in C}|i-(n+1)/2|$, it must be the case that $v<l$, and, 2) since because $l=\min\{i : i\in C^*\}$ and $v=\min\{i : i\in C\}$, we have then that $j-v>0$ for all $j\in C\backslash\{v\}$ and $j-l>0$ for all $j\in C^*\backslash\{l\}$. Hence, we can now conclude that:

$$f(l,n)=\phi(l,|C^*|,n)=\min_{i\in C^*}\phi(i,|C^*|,n)>\phi(v,|C|,n)\geq\min_{i\in C}(i,|C|,n).$$

The sub-case when $v>(n+1)/2$, can be ruled out using similar arguments, and therefore details are omitted. Hence, we can conclude that an equilibrium where $n-2l+2$ players communicate truthfully exists if and only if $\beta\leq f(l,n)$.

Suppose now that $n-2l+1$ players communicate in equilibrium $(\mathbf{m}',\mathbf{y}')$; again, C' is the set of players communicating truthfully and $|C'|=n-2l+1$. Since $(\mathbf{m}',\mathbf{y}')$ is equilibrium, Theorem 3 implies that for all $i\in C(m,y)$ it must be that:

$$\beta\leq\min_{i\in C'}\gamma(i,|C'|,n) \text{ where } \gamma(i,|C'|,n)=\frac{\frac{n-2l}{2(n-2l+3)^2}+\frac{2l-1}{2(n-2l+4)^2}}{\left|\frac{\left[\sum_{j\in N\backslash\{i\}}(j-i)\right][n-2l+3]+\left[\sum_{j\in C'\backslash\{i\}}(j-i)\right]}{[n-2l+3][n-2l+4]}\right|}.$$

Consider the sets $C^*=\{l,...,n-l\}$ and its symmetric counterpart around $(n+1)/2$, denoted $C^{**}=\{l+1,...,n-l+1\}$. Let $h=n-l+1$. By symmetry, it is easy to see that

$$\min_{i\in C^*}\phi(i,|C^*|,n)=\phi(l,|C^*|,n)=\phi(h,|C^{**}|,n)=\min_{i\in C^{**}}\phi(i,|C^{**}|,n).$$

We now claim that

$$\{C^*,C^{**}\}=\arg\max_{C:|C|=n-2l+1}\min_{i\in C}\phi(i,|C|,n).$$

As in the case covered above for $f(l,n)$, this result concludes that if $\beta > g(l,n)$ then there are no equilibria where $n-2l+1$ players communicate.

To prove the claim, note that the numerator of $\gamma(i,|C|,n)$ does not depend on i nor on $|C|$. Consider the denominator of $\gamma(i,|C|,n)$. Suppose that $C \notin \{\{l,...,n-l\},\{l+1,...,n-l+1\}\}$. Let v be one of the most extreme players in C, i.e., $v \in \arg\max_{i\in C}|i-(n+1)/2|$. Proceeding in exactly the same way as for the case of $f(k,n)$, we show that for $v<(n+1)/2$, $g(l,n) = \gamma(l,|C^*|,n) = \min_{i\in C^*}\gamma(i,|C^*|,n) > \gamma(v,|C|,n) \geq \min_{i\in C}\gamma(i,|C|,n)$; and that for $v > (n+1)/2$, $g(l,n) = \gamma(h,|C^*|,n) = \min_{i\in C^*}\gamma(i,|C^*|,n) > \gamma(v,|C|,n) \geq \min_{i\in C}\gamma(i,|C|,n)$. Because $g(l,n) \geq \min_{i\in C}(i,|C|,n)$ for all C such that $|C| = n-2l+1$, we conclude that an equilibrium where $n-2l+1$ players communicate truthfully exists if and only if $\beta \leq g(l,n)$.

■