Problem Set 8

Solve before the classes April 22–24.

Exercise 1 (similar to an exam question Winter 2002)

Three consumers have identical Cobb-Douglas utility functions over two goods in the consumption set $\mathbb{R}^2_+$ given by $u(x_{1i}, x_{2i}) = x_{1i}^{1/2} x_{2i}^{1/2}$. Consumer 1 has initial endowment $\omega_1 = (2, 8)$, while consumers 2 and 3 both have initial endowments $\omega_2 = \omega_3 = (8, 2)$.

a) Given a price vector $(p_1, p_2) \gg 0$, consumer i has wealth $\mathrm{w} = p \cdot \omega_i$. Find each of the three utility-maximizing consumers' demand as a function of the prices, $x_i(p)$.

b) Assume that consumers 1 and 2 live in a closed exchange economy: they can trade freely with each other, but consumer 3 is not involved in this economy. Find the Walrasian equilibrium in this two-person exchange economy. In particular, show that the equilibrium consumption bundles are $x_1^* = x_2^* = (5, 5)$.

c) In the situation of b), consumer 3 is not allowed to trade with anyone, so his consumption bundle is his initial endowment bundle $x_3^* = (8, 2)$. For all three consumers, calculate their achieved utilities $u(x_i^*)$.

d) Now assume that the world is opened to free trade: all three consumers live in a common exchange economy. Solve $x_1(p) + x_2(p) + x_3(p) = \omega_1 + \omega_2 + \omega_3$ to find the Walrasian equilibrium of this three-person exchange economy, and denote the resulting consumption bundles by x_i^{**}.

e) Calculate the equilibrium utilities $u(x_i^{**})$ from the situation of d). Show that consumers 1 and 3 are happier under the free-trade arrangement than they were before. Also show that consumer 2 is worse off, i.e. that $u(x_2^*) > u(x_2^{**})$. Comment on this: can you see an interpretation reminding you of the political free-trade disputes?

Exercise 2

Assume that an economy $(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega})$ fulfills all conditions in proposition 16.AA.1, implying that the set A of feasible allocations is non-empty and compact. Assume that every consumer i's preference relation is represented by a continuous utility function u_i.

a) Prove that the set $U' = \{(u_1(x_1), \ldots, u_I(x_I)) \mid (x, y) \in A\}$ is non-empty and compact.

b) Consider the maximization problem $\max_{u \in U'} \lambda \cdot u$, given the vector $\lambda \gg 0$. Show that if u^* is a solution to this problem, then it is a Pareto optimal allocation (x^*, y^*) which gives rise to the utility profile u^*.

c) Prove the existence of a Pareto optimal allocation of this economy.

Exercise 3

This exercise incorporates a simple externality into an Edgeworth box. It illustrates the failure of the first welfare theorem due to the externality. Either you know of externalities from your second year Micro course (in Varian's chapter 32), or the treatment of externalities in chapter 11 of Mas-Colell, Whinston and Green is required for the present course. Both books treat externalities in models with partial equilibrium features, but the welfare theorems are really connected with general equilibrium.

Consider a two person, two goods exchange economy with a consumption externality. Consumer 1 controls $x_{11} \geq 0$ and $x_{21} \geq 0$, has initial endowment $\omega_1 = (2, 0)$, and has utility function $u_1(x_{11}, x_{21}, x_{12}) = x_{11}x_{21}x_{12}$. Consumer 2 controls $x_{21} \geq 0$ and $x_{22} \geq 0$, has initial endowment $\omega_2 = (0, 2)$, and has utility function $u_2(x_{12}, x_{22}) = x_{12}x_{22}$.

a) Let consumer 1 take as given an amount $x_{12} > 0$ and goods prices $(p_1, p_2) \gg 0$, and find consumer 1's demand for (x_{11}, x_{21}).

b) Find consumer 2's demand for (x_{12}, x_{22}) as a function of the prices $(p_1, p_2) \gg 0$.

c) Determine an equilibrium price vector, such that aggregate demand equates aggregate supply, *and* such that consumer 1 takes as given the true amount x_{12} chosen by consumer 2.

d) Find an allocaton, which leaves both consumers better off than in the equilibrium above. Discuss.