Problem Set 6

Solve before the classes March 19–21.

Exercise 1

The production set $Y \subseteq \mathbb{R}^L$ is called strictly convex if it satisfies the condition

$$y, z \in Y, \alpha \in (0,1) \Rightarrow \alpha y + (1-\alpha)z \in \text{int}(Y).$$

Prove that when Y is strictly convex and $p \in \mathbb{R}^L \setminus \{0\}$, there exists at most one solution to the profit maximization problem $\max_{y \in Y} p \cdot y$.

Exercise 2

Consider a technology with only one input z and one output q. The technology is described by the production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $Y = \{(-z, q) \mid q \leq f(z) \text{ and } z \geq 0\}$. The input price is denoted by $\mathrm{w} > 0$, the output price by $p > 0$.

a) Assume that $f(z) = z/k$ for a given constant k. Argue that this technology exhibits constant returns to scale. Show that the production plan $(0,0)$ is the unique solution to the profit maximization problem when $p/\mathrm{w} < k$, that all production plans with $q = f(z)$ solve the problem for $p/\mathrm{w} = k$, and that the problem has no solution when $p/\mathrm{w} > k$.

b) Assume that $f(z) = z + 1 - 1/(1+z)$. Show that $f(0) = 0$, that f is increasing, that f' is decreasing, and that $f'(z) > 1$ for all z. Assume next that $(\mathrm{w}, p) = (1,1)$. Show that the profit maximization problem has no solution, even though $pq - \mathrm{w}z$ is bounded above.

c) The Inada conditions for the production function f state that $f(0) = 0$, that f is continuously differentiable at all $z > 0$, that f is strictly increasing, that f' is strictly decreasing, that $\lim_{z \searrow 0} f'(z) = +\infty$, and that $\lim_{z \to \infty} f'(z) = 0$. Sketch the general outline of a production function satisfying the Inada conditions. Argue that the Inada conditions imply the following (you are welcome to use your sketch): given any $(\mathrm{w}, p) \gg 0$, there exists a unique solution to the profit maximization problem, and this solution has $f'(z) = \mathrm{w}/p$.

Exercise 3

Consider technologies with a single input z and a single output q. A production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defines a production set $Y = \{(-z, q) \mid q \leq f(z) \text{ and } z \geq 0\}$. Given are two technologies Y_1 and Y_2 of this kind, with associated production functions f_1 and f_2. Assume both production functions satisfy the Inada conditions: $f_i(0) = 0$, f_i is continuously differentiable at any $z > 0$, f_i is strictly increasing, f_i' is strictly decreasing, $\lim_{z \searrow 0} f_i'(z) = +\infty$, and $\lim_{z \to \infty} f_i'(z) = 0$. We seek to describe the production set that arises when one has simultaneous access to both technologies. Thus, we aim to describe the set $Y_1 + Y_2$.

a) A given input amount z can be allocated to the two technologies as $z_1, z_2 \geq 0$, satisfying $z_1 + z_2 = z$. Given such an allocation, at most $f_1(z_1) + f_2(z_2)$ output units can be attained. Prove then, that $Y_1 + Y_2$ is described by the production function f given as

$$f(z) = \max_{z_1, z_2} [f_1(z_1) + f_2(z_2)] \text{ subject to } z_1 + z_2 = z \text{ and } z_1, z_2 \geq 0.$$

b) Argue that this maximization problem (given $z > 0$) possesses exactly one solution $(z_1, z_2) \gg 0$, satisfying the first order condition $f_1'(z_1) = f_2'(z_2)$. Hint: use the Inada conditions.

c) Apply the envelope theorem to prove $f'(z) = f_1'(z_1) = f_2'(z_2)$ where (z_1, z_2) is the solution to the maximization problem given z.

d) Explain that the above results accord well with figure 5.E.1 of the book.

Exercise 4

Exercise 5.C.10 in Mas-Colell, Whinston and Green. Rasmus will focus on the final result, not how you get there.