Problem Set 5

Solve before the classes March 12–14.

Exercise 1

Consider this constant elasticity of substitution (CES) utility function over L commodities, where $\sigma > 0$ and $a_\ell > 0$ are given constants, with $\sigma \neq 1$:

$$u(x_1, \ldots, x_L) = \left(\sum_{\ell=1}^{L} a_\ell^{\frac{1}{\sigma}} x_\ell^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}.$$

a) For any $p \gg 0$ and $u > 0$, solve the expenditure minimization problem, using Lagrange's method as follows. Show that the first order condition for x_k can be written as

$$p_k = \lambda u^{\frac{1}{\sigma}} a_k^{\frac{1}{\sigma}} x_k^{\frac{-1}{\sigma}}$$

where λ is the Lagrange multiplier. Isolate here x_k to find $x_k = a_k p_k^{-\sigma} \lambda^\sigma u$ for every $k = 1, \ldots, L$. Plug this into the constraint $u(x) = u$ and isolate λ to get

$$\lambda^\sigma = \left(\sum_{\ell=1}^{L} a_\ell p_\ell^{1-\sigma} \right)^{\frac{\sigma}{1-\sigma}}.$$

Now plug this λ back into your expression for x_k to arrive at the following:

$$h_k(p, u) = a_k p_k^{-\sigma} \left(\sum_{\ell=1}^{L} a_\ell p_\ell^{1-\sigma} \right)^{\frac{\sigma}{1-\sigma}} u.$$

b) Using the result in a), show that the expenditure function is

$$e(p, u) = \left(\sum_{\ell=1}^{L} a_\ell p_\ell^{1-\sigma} \right)^{\frac{1}{1-\sigma}} u.$$

Exercise 2

An indirect utility function $v : \mathbb{R}^L_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is said to be of the Gorman form if it can be written as $v(p, w) = a(p) + b(p)w$.

a) Assume that a preference relation $\succsim$ can be represented by a utility function that is homogeneous of degree 1 (i.e. satisfies $u(\alpha x) = \alpha u(x)$). Prove that the corresponding indirect utility function is of the Gorman form.

b) Assume that a preference relation $\succsim$ can be represented by the quasilinear utility function $u : \mathbb{R} \times \mathbb{R}^{L-1}_+ \to \mathbb{R}$ where $u(x_1, \cdots, x_L) = x_1 + \phi(x_2, \cdots, x_L)$. Prove that the corresponding indirect utility function is of the Gorman form.

Exercise 3

Exercise 3.G.11 in Mas-Colell, Whinston, and Green. Hint: Apply Roy's identity. By a wealth expansion curve we mean the same as the wealth expansion path (see pages 24–25).