Herding with Costly Observation*

PRELIMINARY AND INCOMPLETE DRAFT
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Abstract

We characterize optimal strategies in a simple herding model where observations have a small cost. We assume that there are two states and two possible signals that each agent may get. The prior distribution is biased towards adopting behavior. That is ex-ante adopting gives a higher expected utility than not adopting. Contrary to Kultti & Miettinen (2005) herding does not arise deterministically in this model when the cost of observation is small.

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1 Introduction

The simplest model of herding is probably the following. There are two possible states and the agents' decision consists of either investing or refraining from investing. In the good state the investment yields a positive return and in the bad state it yields zero. Each agent gets an informative signal about the true state, and the signals are i.i.d. The cost of investment is one half of the return in the good state, and the prior probability of the good state is one half. This means that the agents are indifferent between investing and refraining when they have but the prior information. If they get a good signal they strictly prefer to invest and if they get a bad signal they strictly prefer to refrain. More generally, if an agent, by observing the actions of his predecessors, can infer that the number of good signals is greater than the number of bad signals he prefers to invest. In the opposite case he prefers to refrain.

Kultti & Miettinen (2005) study this model with a twist that observing the act of any previous agent involves a fixed costs $c > 0$. We find that no matter how small the cost is the agents from the third agent on observe only the act of the previous agent, and that a herd necessarily starts from the third agent on.

Even though this is a neat result it is not particularly general as the parameters of the model above are very special. There are several ways to address the problem in a more general setting. First, the signal technology could allow for a larger set of signals. This would probably be just a complication; any agent would have a threshold value for the signal so that signals below this value would lead him to refrain and signals above this value would lead him to invest. The threshold would depend on the position of the agent in the queue and on the actions of his predecessors.

Second, one could study situations where the agents are not indifferent between the two actions given the prior information. There are two interesting cases that are mirror images. The prior is such that without any signals the agents prefer to invest, and if they get a bad signal they update the probability of the good state so low that they prefer to refrain. This is the problem we aim to solve in this article. Notice that one could also study different costs of investment but there are two 'free' parameters, namely the cost of investment and the prior probability and it is sufficient to let one of them vary.

2 Related Literature

Seminal contributions by Banerjee (1992) and Bikhchandani, Hirshleifer & Welch (1992). A more involved herding model by Smith & Sorensen (2000). With imperfect observability include Banerjee & Fudenberg (2004), Celen & Kariv (2004) and Smith & Sorensen (1997). (To be completed...)

3 Model

There is a set of agents $\{1, 2, \ldots, N-1, N\}, N \in \mathbb{Z}_+$, that have to make a decision to adopt the new technology or product. The agents make their decisions sequentially where the order is exogenous and common knowledge among the agents. The new technology/product can be either good or bad. In what follows we customarily talk about adopting. There are two possible states of the world $\Omega = \{\omega, \bar{\omega}\}$ where ω denotes the good and $\bar{\omega}$ the bad state of the world. If the state of the world is good then the adopting agent receives a benefit of b. If the state of the world is bad then the adopting agent receives a benefit of b^L that is normalized to zero. The cost of adoption $c = \frac{b}{2}$ is same across the states and agents. The prior probability that the state of the world is good (or bad) is $\Pr(\omega) = q, \quad (\Pr(\bar{\omega}) = 1 - q)$. Each agent receives a signal $\hat{S}$ about the state of the world from the set $\mathcal{S} = \{S, \bar{S}\}$, where S denotes a signal that the state is good and $\bar{S}$ is a signal that the state is bad. Agent i's signal $\hat{S}_i = S$ is denoted by S_i and $\hat{S}_i = \bar{S}$ by $\bar{S}_i$. The conditional probabilities of the signals are $\Pr(S_i \mid \omega) = p = \Pr(\bar{S}_i \mid \bar{\omega}) > \frac{1}{2}$ and $\Pr(S_i \mid \bar{\omega}) = (1-p) = \Pr(\bar{S}_i \mid \omega) < \frac{1}{2}$. Thus the signals are informative about the state of the world. The agents do not automatically observe the actions taken by their predecessors, but they can get the information about what some preceding agent did by paying a fee of $\rho > 0$ per observed predecessor. When an agent observes his predecessor he learns only whether the predecessor adopted or not. No information is learned about the observations that the predecessor has done. We assume that the agents make their observation decisions sequentially. That is, an agent first observes a predecessor, then conditional on his signal and the observed behavior he will decide whether to make further observations or to adopt on the basis of the current information. Observation cost is incurred at the time of observation. This amounts to assuming that the observation cost is sunk once a predecessor is observed. Once the agent decides that he does not want to make any more observations he makes the decision about adopting. In the following A_i denotes agent i's decision to adopt and $\bar{A}_i$ denotes the decision to refrain. With W we denote the agent's decision to wait and observe more preceding agents.

3.1 Assumptions

Throughout the paper we make the following two assumptions

Assumption 1 $P(\omega) = q > 1/2$

Assumption 2 $q < p$.

Assumption 1 states that adopting is a profitable option ex-ante. In the case that this inequality is reversed, our results hold "as mirror images" when "adopting" is replaced with "not adopting". Assumption 2 guarantees that the agent's problem is non-trivial. We need that the agent's posterior belief about the good state is less than 1/2 if he observes a bad signal. If this is not satisfied the first agent buys irrespective of his signal. Our requirement amounts to

$$P(\omega|\bar{S}_i) = \frac{(1-p)q}{(1-p)q + p(1-q)} < \frac{1}{2} \iff q < p.$$

Assumption 3 *The observation cost ρ is small enough to accommodate any additional observations that may benefit the decision maker.*

We elaborate more on what happens when this assumption is relaxed at the end of the paper.

3.2 Strategies

A strategy for player i defines the predecessors to be observed and the action taken by the agent as a function of the observed behavior.

Formally let $a_i^1 : \{S, \bar{S}\} \longrightarrow \{A, \bar{A}, W\}$ denote the action that agent i takes based on his own signal alone. Let $O_i^1 : \{S, \bar{S}\} \times \{a_i^1(\hat{S})\} \longrightarrow \{1, 2, \ldots, i-1\} \cup \{\emptyset\}$ determine the first person that agent i with signal $\hat{S} \in \{S, \bar{S}\}$ observes[1]. We denote the observed[2] behavior of agent j by $b_j \in \{A, \bar{A}\}$.

Then let $a_i^2 : \{S, \bar{S}\} \times \mathcal{B}^1 \longrightarrow \{A, \bar{A}, W\}$ denote the action that agent i takes based his own signal and the first observation, where we denote the set of observed behavior as $\mathcal{B}^1 := \{\{b_j\} : b_j \in \{A, \bar{A}\}, j = O_i^1(\hat{S})\}$. Similarly let $O_i^2 : \{S, \bar{S}\} \times \mathcal{B}^1 \times \{a_i^2(\hat{S}, \cdot)\} \longrightarrow \{1, 2, \ldots, i-1\} \cup \{\emptyset\}$, denote the second person that is observed.

Now let $a_i^3 : \{S, \bar{S}\} \times \mathcal{B}^2 \longrightarrow \{A, \bar{A}, W\}$ denote the action that agent i takes based on his own signal and the two observations where the set of observed behavior is now $\mathcal{B}^2 := \{\{b_i, b_j\} : b_i, b_j \in \{A, \bar{A}\}, i, j \in O_i^1 \cup O_i^2\}$. Also let $O_i^3 : \{S, \bar{S}\} \times \mathcal{B}^2 \times \{a_i^3(\hat{S}, \cdot, \cdot)\} \longrightarrow \{1, 2, \ldots, i-1\} \cup \{\emptyset\}$, denote the third person that is observed.

Then define recursively $a_i^t : \{S, \bar{S}\} \times \mathcal{B}^{t-1} \longrightarrow \{A, \bar{A}, W\}$ as the action that agent i takes based on his own signal and the observation history $\{b_j\}_{j \in \mathcal{O}^{t-1}}$, where $\mathcal{B}^{t-1} = \{\{b_j\}_{j \in \mathcal{O}^{t-1}} : b_j \in \{A, \bar{A}\}, \forall j \in O^{t-1}\}$ denotes the set of possible observations and $\mathcal{O}^{t-1} := \cup_{j=1}^{t-i} O_i^j(\hat{S}) \in \mathcal{P}(\{1, 2, \ldots, i-1\})$ the set of observed agents.

Also define recursively $O_i^t : \{S, \bar{S}\} \times \mathcal{B}^{t-1} \times \{a_i^{t-1}(\cdot)\} \longrightarrow \{1, 2, \ldots, i-1\}$, to be the map that determines the t^{th} person that agent i observes with signal $\hat{S} \in \{S, \bar{S}\}$ and observation history $\{b_j\}_{j \in \mathcal{O}^{t-1}} \in \mathcal{B}^{t-1}$.

Then agent i's strategy can be defined as

$$\sigma_i(\hat{S}) := (a_i^t(\hat{S}, \{b_j\}_{j \in \mathcal{O}^{t-1}}), O_i^t(\hat{S}, a_i^{t-1}(\hat{S}), \{b_j\}_{j \in \mathcal{O}^{t-1}}))_{t=1}^{i-1}.$$

Since the agents make a once-and-for-all decision it is reasonable to require that any strategy satisfies $O_i^t(\cdot, a_i^{t-1}(\hat{S}), \cdot) = \emptyset$ whenever $a_i^{t-1}(\hat{S}) \in \{A, \bar{A}\}$ and that $a_i^t(\hat{S}, \{b_j\}_{j \in \mathcal{O}^{t-1}}) = a_i^{t-1}(\cdot)$, if $a_i^{t-1}(\cdot) \in \{A, \bar{A}\}$ and there exists $k \leq t-1$ such

[1] We take $O_i^k(\hat{S}) = \emptyset$ to denote that agent i observes no predecessors at round $k \geq 1$.

[2] The agent making the observations should be clear from the context.

that $O_i^k = \emptyset$. This requirement amounts to saying that once the agent makes his decision, he makes no more observations, and that he "sticks" to his decision, as it is irreversible.

Hence the agent's decision problem is divided to stages. At each stage agent decides either to observe one more predecessor or to decide about adopting the good.

4 Optimal Strategies

All agents $i \geq 3$ have the same optimal strategy. We will first elaborate on agents 1,2 and 3 and then finally prove optimality for agent $k \geq 3$.

The choice of the first agent is dictated by his signal.

$$\sigma_1(\hat{S}) = \begin{cases} (A_1, \emptyset) & \text{if } \hat{S} = S_1 \\ (\bar{A}_1, \emptyset) & \text{if } \hat{S} = \bar{S}_1 \end{cases} \tag{1}$$

The second agent will observe the first agent only if the second agent has a bad signal. If the signal is good observing that the first agent has a bad signal cannot affect the second agent's decision to adopt. Hence the second agent will make no observations with a good signal. When the second agent has a bad signal, observing the first agent is profitable. This is true since

$$P(\omega|A_1, \bar{S}_2) = q.$$

The second agent will optimally adopt, when he observes that the first agent adopted. Naturally, if the first agent did not adopt then this only reinforces the second agent's belief that the state is bad and he refrains. Below we denote the fact that agent j is observed by O_j. No confusion should arise about the agent that makes the observation as it is clear whose strategy we consider.

$$\sigma_2(\hat{S}) = \begin{cases} (A_2, \emptyset) & \text{if } \hat{S} = S_2 \\ (W, O_1) & \text{if } \hat{S} = \bar{S}_2 \text{ and } a_2(\bar{S}_2, b_1) = \begin{cases} A_2 \text{ if } b_1 = A_1 \\ \bar{A}_2 \text{ if } b_1 = \bar{A}_1 \end{cases} \end{cases} \tag{2}$$

Third agent's problem is more subtle. If the third agent has a good signal he wants to make sure that there is at least one good signal, because in this case adopting is profitable. The "cheapest" way to verify this is to observe the second agent's action. If the second agent did not adopt, it implies that two bad signals are observed. Observing that the second agent adopted will again imply that at least one good signal is observed by one of the third agent's predecessors. Observing that the first agent did not adopt, is not sufficient to rule out adopting when the third agent has a good signal. If this observation is made the third agent wants to further observe the action of the second agent, which by itself is sufficient to provide him the information that he needs.

When the third agent has a bad signal observing the first agent is the best policy. The third agent wants to be sure that he refrains only in the case that the number of "inferable" bad signals is larger than the number of "inferable" good signals. By observing that the first agent did not adopt the first agent can assure himself that there are two bad signals and that adopting is not profitable irrespective of the signal of the second agent. Similarly if the first agent is observed to adopt then there is no way to infer the signal of the second agent by observing his action. Hence the third agent can infer that there is one good and one bad signal. He then updates his beliefs about the state to be equal to the prior belief and adopts.

Consider what happens when the third agent observes the second agent instead. In the case that the second agent did not adopt then all predecessors signals are inferable and the same decision is made as if the third agent observed the first agent's action. However, if the second agent adopts then the third agent wants to see the first agent's choice. In the case that the first agent adopted then, as above, the number of inferable good signals is equal to the number of inferable bad ones. In the case that the first agent refrains, all signals are inferable, but the first agent's action is sufficient to determine the third agent's optimal action. In the appendix we show that the third agent always prefers to observe the first agent to observing the second agent when he has a bad signal.

Proposition 1 *When players 1 and 2 act according to equations (1) and (2) the optimal strategy for player i even ($i \geq 4$) is*

$$\sigma_i(\hat{S}) = \begin{cases} (W, O_{i-2}) & \text{if } \hat{S} = S_i \text{ and } a_i(S_i, b_{i-2}) = \begin{cases} A_i & \text{if } b_{i-2} = A \\ \bar{A}_i & \text{if } b_{i-2} = \bar{A} \end{cases} \\ (W, O_{i-1}) & \text{if } \hat{S} = \bar{S}_i \text{ and } a_i(\bar{S}_i, b_{i-1}) = \begin{cases} A_i & \text{if } b_{i-1} = A \\ \bar{A}_i & \text{if } b_{i-1} = \bar{A} \end{cases} \end{cases} \tag{3}$$

and the optimal strategy for player i odd ($i \geq 3$) is

$$\sigma_i(\hat{S}) = \begin{cases} (W, O_{i-1}) & \text{if } \hat{S} = S_i \text{ and } a_i(S_i, b_{i-1}) = \begin{cases} A_i & \text{if } b_{i-1} = A \\ \bar{A}_i & \text{if } b_{i-1} = \bar{A} \end{cases} \\ (W, O_{i-2}) & \text{if } \hat{S} = \bar{S}_i \text{ and } a_i(\bar{S}_i, b_{i-2}) = \begin{cases} A_i & \text{if } b_{i-2} = A \\ \bar{A}_i & \text{if } b_{i-2} = \bar{A} \end{cases} \end{cases} \tag{4}$$

We prove the proposition through the following five lemmata.

Lemma 1 *If all agents play according to the strategies (3) and (4) then a herd gets started if two consecutive agents take the same action.*

Proof. Suppose that players i and $i+1$ ($i \geq 1$) take action $\bar{A}$. Now the strategy for agent $i+2$ prescribes that he observes either agent i or agent $i+1$ and mimics his action. ■

Lemma 2 *If all agents play according to the strategies (3) and (4) then the only signal realization with which herd does not get started is the one where all odd agents have bad signals and all even agents have good signals.*

Proof. Suppose to the contrary, that there exist some other realization of signals with which the herd does not start. This string of signals must have an instance where there are either two consecutive good or bad signals. Suppose that the players i and $i+1$. (W.l.o.g. assume that i is even.) get the same signals. If the two signals are both good. Then agent i observes agent $i-2$ and agent $i+1$ observes agent i. Now to avoid herd from starting we need that agents i and $i+1$ take different actions. But now whatever agent i chooses will be copied by agent $i+1$ and by the lemma 2 above this leads to a herd getting started. Similarly if the signals are bad then agent i observes agent $i-1$ but then agent $i+1$ also observes agent $i-1$ and hence the agents take the same action, a contradiction. ■

Denote the set of predecessors of agent i by Φ_i. Fix the signal of agent i, $\hat{S}_i \in \{S, \bar{S}\}$ and let $Z \subset \Phi_i$. Partition the set of observations into sets $\mathcal{O}_i(Z)$ and $\bar{\mathcal{O}}_i(Z)$ where

$$\mathcal{O}_i(Z) = \Big\{\{b_j\}_{j\in Z} : P(\omega|\{b_j\}_{j\in Z}, \hat{S}_i) \geq 1/2\Big\}$$
$$\bar{\mathcal{O}}_i(Z) = \Big\{\{b_j\}_{j\in Z} : P(\omega|\{b_j\}_{j\in Z}, \hat{S}_i) < 1/2\Big\}.$$

Definition 1 *Agent i is adequately informed by observing the set of predecessors $Z \subset \Phi_i$ at $\hat{S}_i$, if for all $B \subset \Phi_i \setminus Z$ we have*

$$\{b_j\}_{j\in Z} \in \mathcal{O}_i(Z) \Longrightarrow P(\omega|\{b_j\}_{j\in B\cup Z}, \hat{S}) \geq 1/2$$
$$\textit{and}$$
$$\{b_j\}_{j\in Z} \in \bar{\mathcal{O}}_i(Z) \Longrightarrow P(\omega|\{b_j\}_{j\in B\cup Z}, \hat{S}) < 1/2$$

for all (equilibrium) observations $\{b_j\}_{j\in B} \in \{A, \bar{A}\}^{|B|}$.

I.e. agent i is "adequately informed" by observing the predecessors in Z if there does not exist any set $B \subset \Phi_i \setminus Z$ such that by observing the predecessors in B in addition to Z causes agent i to change his decision about adopting.

Lemma 3 *If σ_i is an optimal strategy for agent i, then agent i is adequately informed with σ_i.*

Proof. Suppose not. Let $\hat{S}_i \in \{S, \bar{S}\}$ and Z be the set of observed predecessors under σ_i. Then there exists a set $B \subset \Phi_i$ and a *realization of observed actions* $\{b_j\}_{j\in B}$ such that either

$$P(\omega|\{b_j\}_{j\in Z}, \hat{S}) \geq 1/2 \text{ and } P(\omega|\{b_j\}_{j\in Z}, \{b_j\}_{j\in B}, \hat{S}) < 1/2$$
$$\text{or}$$
$$P(\omega|\{b_j\}_{j\in Z}, \hat{S}) < 1/2 \text{ and } P(\omega|\{b_j\}_{j\in Z}, \{b_j\}_{j\in B}, \hat{S}) \geq 1/2,$$

for some *realization of actions* $\{b_j\}_{j\in Z}$. Assume that the first case holds[3]. Then

$$\begin{aligned} E[u_i(\sigma_i)|\{b_j\}_{j\in Z},\hat{S}] &= \sum_{\{b_j\}_{j\in B}\in\mathcal{B}} P(\{b_j\}_{j\in B}|\{b_j\}_{j\in Z},\hat{S})\Big(P(\omega|\{b_j\}_{j\in Z},\{b_j\}_{j\in B},\hat{S})b-c\Big) \\ &< \sum_{\{b_j\}_{j\in B}\in\mathcal{B}\setminus\bar{B}} P(\{b_j\}_{j\in B}|\{b_j\}_{j\in Z},\hat{S})\Big(P(\omega|\{b_j\}_{j\in Z},\{b_j\}_{j\in B},\hat{S})b-c\Big) \\ &= E[u_i(\sigma_i')|\{b_j\}_{j\in Z},\hat{S}], \end{aligned}$$

where $\mathcal{B}$ denotes the possible (equilibrium) observations from B,

$$\bar{B} := \Big\{\{b_j\}_{j\in B}\in\mathcal{B} : P(\omega|\{b_j\}_{j\in Z},\{b_j\}_{j\in B},\hat{S}) < 1/2\Big\}$$

and under σ_i' the decision to adopt is made after observing additionally the predecessors in B. The behavior with regard to adopting is the same in σ_i' as in σ_i whenever $P(\omega|\{b_j\}_{j\in Z},\{b_j\}_{j\in B},\hat{S}) > 1/2$ and differs otherwise. Since we assumed that the observation costs $\rho(B)$ are small enough to allow any beneficial observations, the inequality holds even when the costs are accounted for. But this contradicts the optimality of σ_i. ■

Lemma 4 *If all agents $k < i$ behave according to the strategies in the proposition 1 then there exist agents $a_i(S)$ and $a_i(\bar{S})$ in P_i such that agent i is adequately informed by observing $a_i(S)$ or $a_i(\bar{S})$.*

Proof. First note that if an agent can detect that a herd has started, then it is optimal for the agent to join the herd. By lemma 3 above, observing that an even agent refrains implies that $\bar{A}$ herd has started. Similarly observing that an odd agent accepts implies that A herd has started. We show that using the strategy in the proposition 1 agent i becomes adequately informed.

1.) i is even:

a.) If i has a signal S_i then he observes $i-2$. Observing

- $\bar{A}_{i-2} \Rightarrow \bar{A}$ herd $\Rightarrow \bar{A}_i$
- $A_{i-2} \Rightarrow \begin{cases} A \text{ herd } \Rightarrow A_i \text{ or} \\ \text{no herd before agent } i-2 \end{cases}$

If no herd has started before agent $i-2$ then there are $\frac{i-2}{2}$ good and bad signals among the predecessors of agent $i-1$. Now in case agent $i-1$ has a bad signal the agents up until i have received exactly $\frac{i}{2}$ good and bad signals. The posterior probability of the good state in this case is

$$P\Big(\omega\Big|\frac{i}{2}\text{ good and bad signals}\Big) = \frac{[p(1-p)]^{\frac{i}{2}}q}{[p(1-p)]^{\frac{i}{2}}[q+(1-q)]} = q.$$

[3] The other case is similar and omitted for brevity.

Hence it is worth adopting for i irrespective of the signal of $i-1$.

b.) If i has a signal $\bar{S}_i$ then he observes $i-1$. Observing

- $\bar{A}_{i-1} \Rightarrow \begin{cases} \bar{A} \text{ herd } \Rightarrow \bar{A}_i \text{ or} \\ \text{no herd before agent } i-1 \end{cases}$

If no herd has started before agent $i-1$ then there are $\frac{i-2}{2}$ good and $\frac{i-2}{2}+1$ bad signals among the predecessors of agent i. Since i has a bad signal the agents up until i have received exactly $\frac{i}{2}+1$ good and $\frac{i}{2}-1$ bad signals. The posterior probability of the good state in this case is

$$P\Big(\omega\Big|\frac{i}{2}-1 \text{ good and } \frac{i}{2}+1 \text{ bad signals}\Big) = \frac{[p(1-p)]^{\frac{i}{2}-1}(1-p)^2 q}{[p(1-p)]^{\frac{i}{2}-1}[(1-p)^2 q + p^2(1-q)]} \leq 1/2$$

$$(1-p)^2 q - p^2(1-q) \leq 0$$

$$q - p(2q(1-p)+p) \leq q - p((1-p)+p) = q - p < 0.$$

Hence it not is worth adopting for i.

- $A_{i-1} \Rightarrow A \text{ herd} \Rightarrow A_i$

2.) i is odd:

a.) If i has a signal S_i then he observes $i-1$. Observing

- $\bar{A}_{i-1} \Rightarrow \bar{A} \text{ herd} \Rightarrow \bar{A}_i$

- $A_{i-1} \Rightarrow \begin{cases} A \text{ herd } \Rightarrow A_i \text{ or} \\ \text{no herd before agent } i-1 \end{cases}$

If no herd has started before agent $i-1$ then there are $\frac{i-1}{2}$ good and bad signals among the predecessors of agent i. Since i has a good signal the agents up until i have received exactly $\frac{i-1}{2}+1$ good and $\frac{i-1}{2}$ bad signals. The posterior probability of the good state in this case is

$$P\Big(\omega\Big|\frac{i-1}{2}+1 \text{ good and} \frac{i-1}{2} \text{ bad signals}\Big) = \frac{[p(1-p)]^{\frac{i-1}{2}} pq}{[p(1-p)]^{\frac{i-1}{2}}[qp+(1-p)(1-q)]} \geq 1/2.$$

Hence it is worth adopting for i.

b.) If i has a signal $\bar{S}_i$ then he observes $i-2$. Observing

- $\bar{A}_{i-2} \Rightarrow \begin{cases} \bar{A} \text{ herd } \Rightarrow \bar{A}_i \text{ or} \\ \text{no herd before agent } i-2 \end{cases}$

If no herd has started before agent $i-2$ then there are $\frac{i-3}{2}$ good and $\frac{i-3}{2}+1$ bad signals among the predecessors of agent $i-1$. Now if agent $i-1$ has a good signal the agents up until i have received exactly $\frac{i-1}{2}$ good and $\frac{i-1}{2}+1$ bad signals. The posterior probability of the good state in this case is

$$P\Big(\omega\Big|\frac{i-1}{2} \text{ good and } \frac{i-1}{2}+1 \text{ bad signals}\Big) = \frac{[p(1-p)]^{\frac{i-1}{2}}(1-p)q}{[p(1-p)]^{\frac{i-1}{2}}[(1-p)q+p(1-q)]} \leq 1/2.$$

Hence it not is worth adopting for i irrespective of the signal of $i-1$.

- $A_{i-2} \Rightarrow A$ herd $\Rightarrow A_i$

■

Note that in the proof there are two cases where agent i cannot infer what the signal of his immediate predecessor is[4]. However observing agent i's predecessor's action and (possibly) inferring his signal does not change the decision whether to adopt or refrain by agent i. It is immediate that an agent, who decides about adopting without any observations, is not adequately informed. Since the proposed strategy involves only one observed predecessor it is at least as good as any other strategy that makes the agent adequately informed. Hence the strategies are optimal as further observation is costly.

Lemma 5 *The strategies in proposition 1 are optimal.*

Proof. Suppose not. Denote the optimal strategy and the strategy of proposition 1 for agent i by σ_i^* and σ_i respectively. Then by lemmas 3 and 4 agent i is adequately informed under both strategies. The only strategy where there are strictly less observation costs than under σ_i is when agent i makes no observations at all, but this cannot be optimal for any agent $i \geq 3$. Hence the observation costs in σ_i^* are at least as high as in σ_i. Disregarding the possibly different observations costs in σ_i^* and σ_i we have that

$$E[u_i(\sigma_i^*)] > E[u_i(\sigma_i)],$$

which implies that there must exist a realization of behavior $\{b_j\}_{j<i}$ such that the decision to adopt under σ_i^* differs from the decision to adopt under σ_i. By lemma 4 behavior in σ_i is optimal with respect to the information observed. But this implies that under σ_i^* there is some additional information that causes a decision different from σ_i and contradicts that agent i is adequately informed under σ_i. ■

[4]Cases 1.a. and 2.b.

Corollary 1 *Optimal strategy is unique.*

Proof. Since the strategy of proposition 1 is optimal each player can make at most one observation in any optimal strategy.

For any agent i observing a predecessor k such that $|i-k|>2$ is not part of an optimal strategy, since there are realizations of signals for which neither A nor $\bar{A}$ herd starting after agent k can be ruled out. Hence agent i is not adequately informed by such an observation.

If i has a signal S_i then observing his immediate odd predecessor j is not a part of an optimal strategy. If $\bar{A}_j$ is observed then either a herd where nobody adopts or no herd has started. Agent i can find out which case prevails with probability 1 by observing one more agent, namely his immediate even predecessor. (Observing any other even predecessor k leaves the possibility that $\bar{A}$ herd starts after k.) Hence agent i is not adequately informed when he observes his immediate odd predecessor.

If i has a signal $\bar{S}_i$ then observing his immediate even predecessor j is not a part of an optimal strategy. If A_j is observed then either a herd where everyone adopts or no herd has started. Agent i can find out with probability 1 which case prevails by observing one more agent, namely his immediate odd predecessor. (Observing any other odd predecessor k leaves the possibility that A herd starts after k.) Hence agent i is not adequately informed when he observes his immediate even predecessor. ■

5 Comparison to the Basic Model

Consider the herding model above in the "regular" scenario where all agents observe all their predecessors' actions. Then it is easy to verify that the herd gets started if either the first agent adopts or if two consecutive agents take the same action. Hence whenever the observation costs are low enough to allow for observations, the probability with which a herd gets started after any agent $k \geq 1$ is same as if there were no costs of observation. However this relies on the assumption that costs are low enough to allow for every agent to observe according to the strategy of proposition 1. In the appendix we show that if the second and third agents behave according to the prescribed strategy then all following agents will do the same. If the cost $\rho > 0$ is such that all agents until i act according to the prescribed strategy then the cost ρ' that makes agent $i+1$ prefer acting on his own signal alone is larger than the cost ρ. As the observation costs are same for all agents the they will behave according the prescribed equilibrium whenever the second and third agents do. Denote this bound for the cost by $\bar{\rho}$. There also exists an upperbound $\hat{\rho} > 0$ such that if $\rho > \hat{\rho}$ then no observations are made. We know what happens whenever $\rho \in [0, \bar{\rho}) \cup [\hat{\rho}, \infty)$. In the case that $\rho \in [\bar{\rho}, \hat{\rho})$ there are many things that may take place in equilibrium and we do not understand this region completely as different cases emerge that depend on the values of p and q.

6 Conclusion

In this paper we find the optimal strategy for the agents in a simple herding model where observations are costly. We find that contrary to the case where the prior probability of good and bad state is equal, herding arises probabilistically when observation costs are small. In the symmetric setup considered by Kultti & Miettinen (2005) herding arises necessarily in equilibrium. We conjecture that this equilibrium is a limit result of the current model in a specific region of the parameter space as the prior probability of the good state goes to 1/2. However we do not study this case here. We don't know yet if there is an equilibrium where herding is neccessary in the case of asymmetric priors.

7 Appendix

7.1 Third agent's observations with a bad signal

Expected utility from observing the first agent (3rd agent buys after having observed that the 1st agent bought)

$$
\begin{aligned}
P(A_1|\bar{S}_3)\Big(P(\sigma|A_1,\bar{S}_3)2b-b\Big) &= \frac{P(A_1,\bar{S}_3)}{P(\bar{S}_3)}\Big(\frac{P(A_1,\bar{S}_3|\sigma)P(\sigma)}{P(A_1,\bar{S}_3|\sigma)P(\sigma)+P(A_1,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}2b-b\Big)\\
&= \frac{P(A_1,\bar{S}_3|\sigma)P(\sigma)+P(A_1,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}{P(\bar{S}_3|\sigma)P(\sigma)+P(\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}\times\\
&\quad\Big(\frac{2P(A_1,\bar{S}_3|\sigma)P(\sigma)-P(A_1,\bar{S}_3|\sigma)P(\sigma)-P(A_1,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}{P(A_1,\bar{S}_3|\sigma)P(\sigma)+P(A_1,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}b\Big)\\
&= \Big(\frac{P(A_1,\bar{S}_3|\sigma)P(\sigma)-P(A_1,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}{P(\bar{S}_3|\sigma)P(\sigma)+P(\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}b\Big)\\
&= \frac{p(1-p)(2q-1)}{(1-p)q+p(1-q)} > 0, \text{ since } q > \frac{1}{2}. \qquad (5)
\end{aligned}
$$

Expected utility from observing the second agent (2nd agent has a good signal)

$$
\begin{aligned}
P(\sigma|A_2,\bar{S}_3)2b-b &= \frac{P(A_2,\bar{S}_3|\sigma)P(\sigma)}{P(A_2,\bar{S}_3|\sigma)P(\sigma)+P(A_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}2b-b\\
&= \frac{P(A_2,\bar{S}_3|\sigma)P(\sigma)-P(A_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}{P(A_2,\bar{S}_3|\sigma)P(\sigma)+P(A_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}b\\
&= \frac{P(S_1,\bar{S}_3\vee\bar{S}_1,S_2,\bar{S}_3|\sigma)P(\sigma)-P(S_1,\bar{S}_3\vee\bar{S}_1,S_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}{P(A_2,\bar{S}_3|\sigma)P(\sigma)+P(A_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}b\\
&= \frac{p(1-p)(1+(1-p))q-p(1-p)(1+p)(1-q)}{P(A_2,\bar{S}_3|\sigma)P(\sigma)+P(A_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}b\\
&= \frac{p(1-p)[(2-p)q-(1+p)(1-q)]}{P(A_2,\bar{S}_3|\sigma)P(\sigma)+P(A_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}b
\end{aligned}
$$

This is non-negative if

$$
\begin{aligned}
(2-p)q-(1+p)(1-q) &\geq 0\\
&\text{i.e. if,}\\
q &\geq \frac{1+p}{3}.
\end{aligned}
$$

Expected utility from observing the second agent (when the assumption above holds)

$$
\begin{aligned}
P(A_2|\bar{S}_3)\Big(P(\sigma|A_2,\bar{S}_3)2b-b\Big) &= \frac{P(A_2,\bar{S}_3)}{P(\bar{S}_3)} \times \frac{P(A_2,\bar{S}_3|\sigma)P(\sigma)-P(A_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}{P(A_2,\bar{S}_3|\sigma)P(\sigma)+P(A_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}b \\
&= \frac{P(A_2,\bar{S}_3|\sigma)P(\sigma)+P(A_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}{P(\bar{S}_3|\sigma)P(\sigma)+P(\bar{S}_3|\bar{\sigma})P(\bar{\sigma})} \times \\
&\quad \frac{P(A_2,\bar{S}_3|\sigma)P(\sigma)-P(A_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}{P(A_2,\bar{S}_3|\sigma)P(\sigma)+P(A_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}b \\
&= \frac{P(S_1,\bar{S}_3 \vee \bar{S}_1,S_2,\bar{S}_3|\sigma)P(\sigma)-P(S_1,\bar{S}_3 \vee \bar{S}_1,S_2,\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}{P(\bar{S}_3|\sigma)P(\sigma)+P(\bar{S}_3|\bar{\sigma})P(\bar{\sigma})}b \\
&= \frac{p(1-p)[(2-p)q-(1+p)(1-q)]}{(1-p)q+p(1-q)} > 0, \text{ by assumption.}
\end{aligned} \tag{6}
$$

Then the difference between equations (5) and (6) is

$$
p(1-p)\frac{(2q-1)-(3q-1-p)}{(1-p)q+p(1-q)} = p(1-p)\frac{p-q}{(1-p)q+p(1-q)} > 0.
$$

7.2 Cost boundary

In this section it is assumed that all preceding agents are behaving according to the strategy of proposition 1.

Utility for an agent with signal $\hat{S} \in \{S, \bar{S}\}$ from observing the k^{th} person is given by

$$
\begin{aligned}
P(A_k|\hat{S})(P(\omega|A_k,\hat{S})2b-b) &= \frac{P(A_k,\hat{S})}{P(\hat{S})}\Big(\frac{P(A_k,\hat{S}|\omega)P(\omega)}{P(A_k,\hat{S})}2b-b\Big) \\
&= \frac{2bP(A_k,\hat{S}|\omega)P(\omega)-b\big(P(A_k,\hat{S}|\omega)P(\omega)+P(A_k,\hat{S}|\bar{\omega})P(\bar{\omega})\big)}{P(\hat{S})} \\
&= \frac{P(A_k,\hat{S}|\omega)P(\omega)-P(A_k,\hat{S}|\bar{\omega})P(\bar{\omega})}{P(\hat{S})}b
\end{aligned} \tag{7}
$$

We first define a limit for the cost of observation $\bar{\rho}_2$ such that if $\rho > \bar{\rho}_2$ then the second agent will rather act upon his own signal. It is clear that we only need to consider the case where the second agent has a bad signal. In this case we compare the expected utility from observing to the utility from not adopting.

Expected utility form observing agent 1 when 2 has a bad signal is

$$
\frac{P(A_1,\bar{S}|\omega)P(\omega)-P(A_1,\bar{S}|\bar{\omega})P(\bar{\omega})}{P(\bar{S})}b-\rho = \frac{p(1-p)(2q-1)}{q(1-p)+(1-q)p}b-\rho.
$$

The expected utility from not adopting is zero hence we have that agent 2 will observe 1 with a bad signal only if

$$\rho \leq \frac{p(1-p)(2q-1)}{q(1-p)+(1-q)p}b \equiv \bar{\rho}_2. \tag{8}$$

The expected utility for the third agent from observing agent 2 with a good signal is

$$\frac{P(A_2, S|\omega)P(\omega) - P(A_2, S|\bar{\omega})P(\bar{\omega})}{P(S)}b - \rho = \frac{p^2q(2-p) - (1-p)^2(1-q)(1+p)}{pq + (1-p(1-q)}b - \rho. \tag{9}$$

The utility from adopting without any observations is

$$P(\sigma|S_3)2b - b = \frac{P(S_3|\sigma)P(\sigma)}{P(S_3)}2b - b = \frac{pq - (1-p)(1-q)}{pq + (1-p)(1-q)}b. \tag{10}$$

Now agent 3 behaves according to the strategy of proposition 1 if (9) is larger than the utility from relying on her own signal alone (10). This is true only if

$$\rho \leq \frac{p(1-p)[p(1-q) - q(1-p)]}{pq + (1-p)(1-q)}b \equiv \rho_3.$$

The limit that determines whether agent 3 wants to observe agent 1 with a bad signal is equal to (8). We claim that if $\rho \leq \bar{\rho} \equiv min\{\bar{\rho}_2, \rho_3\}$ then the agents will behave according to the equilibrium strategies of proposition 1. We show that when all predecessors of an agent act according to the strategies of proposition 1 the expected utility from using the strategy of proposition 1 increases with the agent's number in the sequence. This implies that the cost has to be even higher than $\bar{\rho}$ for an agent $k > 3$ to prefer acting on his own signal alone.

Suppose that k is even. Then the probability that A_k is observed when the state is good or bad respectively is

$$P(A_k|\sigma) = p\sum_{i=0}^{k/2-1}(1-p)^i p^i + (1-p)^{k/2}p^{k/2-1}$$

and

$$P(A_k|\bar{\sigma}) = (1-p)\sum_{i=0}^{k/2-1}(1-p)^i p^i + p^{k/2}(1-p)^{k/2-1}$$

Hence the utility for agent i from observing an even predecessor k with a good signal is

$$\begin{aligned}\frac{P(A_k,S|\omega)P(\omega)-P(A_k,S|\bar\omega)P(\bar\omega)}{P(S)}b &= \frac{q}{P(S)}\Big[p^2\sum_{i=0}^{k/2-1}(1-p)^ip^i+(1-p)^{k/2}p^{k/2}\Big]- \\ &\quad \frac{(1-q)}{P(S)}\Big[(1-p)^2\sum_{i=0}^{k/2-1}(1-p)^ip^i+p^{k/2}(1-p)^{k/2}\Big] \\ &= \frac{[qp^2-(1-q)(1-p)^2]\sum_{i=0}^{k/2-1}(1-p)^ip^i+(1-p)^{k/2}p^{k/2}(2q-1)}{P(S)} \\ &= \frac{[qp^2-(1-q)(1-p)^2]}{P(S)}\frac{1-(1-p)^{k/2}p^{k/2}}{1-p+p^2}+ \\ &\quad \frac{(1-p)^{k/2}p^{k/2}(2q-1)}{P(S)}.\end{aligned}$$

Now as $k\longrightarrow\infty$ we get that

$$\frac{P(A_k,S|\omega)P(\omega)-P(A_k,S|\bar\omega)P(\bar\omega)}{P(S)}b \underset{k\longrightarrow\infty}{\longrightarrow} \frac{qp^2-(1-q)(1-p)^2}{P(S)(1-p+p^2)}.$$

To see that agent i wants to rather observe agent k than agent $k-2$ note that

$$\begin{aligned}&[P(A_k,S|\omega)P(\omega)-P(A_k,S|\bar\omega)P(\bar\omega)]b-[P(A_{k-2},S|\omega)P(\omega)-P(A_{k-2},S|\bar\omega)P(\bar\omega)]b= \\ &b[qp^2-(1-q)(1-p)^2][(1-p)^{k/2-1}p^{k/2-1}]-(2q-1)(1-p)^{k/2-1}p^{k/2-1}(1-(1-p^2)p^2)= \\ &b(1-p)^{k/2-1}p^{k/2-1}\Big[qp^2-(1-q)(1-p)^2-(2q-1)(1-p(1-p))\Big]>0\end{aligned}$$

Since

$$\begin{aligned}qp^2-(1-q)(1-p)^2-(2q-1)(1-p(1-p)) = \; & qp^2-1+q+2p-2pq-p^2+p^2q \\ & -2q+2pq-2p^2q+1-p+p^2 \\ & =p-q.\end{aligned}$$

The utility for agent i from observing an even predecessor k with a bad signal is

$$\frac{P(A_k,\bar{S}|\omega)P(\omega)-P(A_k,\bar{S}|\bar{\omega})P(\bar{\omega})}{P(\bar{S})}b = \frac{1}{P(\bar{S})}\left[\begin{array}{c} q(1-p)[p\sum_{i=0}^{k/2-1}(1-p)^i p^i + (1-p)^{k/2}p^{k/2-1} - \\ (1-q)p[(1-p)\sum_{i=0}^{k/2-1}(1-p)^i p^i + p^{k/2}(1-p)^{k/2-1}\end{array}\right]$$

$$= \begin{array}{l} p(1-p)\sum_{i=0}^{k/2-1}(1-p)^i p^i + \\ {[q(1-p)^2-(1-q)p^2]p^{k/2-1}(1-p)^{k/2-1}}. \end{array}$$

Again as $k \longrightarrow \infty$ we get that

$$\frac{P(A_k,\bar{S}|\omega)P(\omega)-P(A_k,\bar{S}|\bar{\omega})P(\bar{\omega})}{P(\bar{S})}b \longrightarrow \frac{p(1-p)(2q-1)}{(1-p+p^2)P(\bar{S})}.$$

Now the term $g(\cdot,\cdot) = q(1-p)^2-(1-q)p^2 < 0$. When $p = q$ we get that $q(1-p)^2-(1-q)q^2 = q(1-q)(1-2q) < 0$. Taking the derivative of g with respect to p we get that $-2q(1-p)-2(1-q)p < 0$. Hence whenever p increases from $p = q$ this term becomes more negative. It should be clear that agent i is better off observing agent $k = n$ than agent $k = n-2$ as the positive term, $p(1-p)\sum_{i=0}^{k/2-1}(1-p)^i p^i$, is larger and the negative term, $q(1-p)^2-(1-q)p^2$, is multiplied with a smaller number.

Consider next the case where agent i has a bad signal. Now suppose that k is odd. Then the probability that A_k is observed when the state is good or bad respectively is

$$P(A_k|\sigma) = p\sum_{i=0}^{\frac{k-1}{2}}(1-p)^i p^i$$

and

$$P(A_k|\bar{\sigma}) = (1-p)\sum_{i=0}^{\frac{k-1}{2}}(1-p)^i p^i.$$

Now the expected utility from observing an odd predecessor k with a good signal is

$$\frac{P(A_k,S|\omega)P(\omega)-P(A_k,S|\bar{\omega})P(\bar{\omega})}{P(S)}b = \frac{[p^2q-(1-p)^2(1-q)]\sum_{i=0}^{\frac{k-1}{2}}(1-p)^i p^i}{P(S)}b.$$

Now as $k \longrightarrow \infty$ this expected utility goes to

$$\frac{P(A_k, \bar{S}|\omega)P(\omega) - P(A_k, \bar{S}|\bar{\omega})P(\bar{\omega})}{P(\bar{S})} b \longrightarrow \frac{p^2 q - (1-p)^2(1-q)}{(1-p+p^2)P(\bar{S})} b.$$

Similarly the expected utility when the signal is bad is

$$\frac{P(A_k, \bar{S}|\omega)P(\omega) - P(A_k, \bar{S}|\bar{\omega})P(\bar{\omega})}{P(\bar{S})} b = \frac{p(1-p)(2q-1) \sum_{i=0}^{\frac{k-1}{2}} (1-p)^i p^i}{P(\bar{S})} b.$$

As $k \longrightarrow \infty$ we get that

$$\frac{P(A_k, \bar{S}|\omega)P(\omega) - P(A_k, \bar{S}|\bar{\omega})P(\bar{\omega})}{P(\bar{S})} b \longrightarrow \frac{p(1-p)(2q-1)}{(1-p+p^2)P(\bar{S})} b.$$

Now it is immediate that the expected utility from observing an odd predecessor k is larger than the expected utility from observing $k-2$ predecessor as terms is the series increase with k. Hence whenever it is worth for the second and the third agent to observe one predecessor (with both signals) all other agents will also observe according to the equilibrium strategy. Hence there exists a lower bound $\bar{\rho} > 0$ such that if $\rho < \bar{\rho}$ the proposed equilibrium emerges as a result.

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