Sequential Decisions with Tests

David Gill
Trinity College
University of Oxford

Daniel Sgroi [1]
Faculty of Economics and Churchill College
University of Cambridge

Address for correspondence:
Dr. Daniel Sgroi
Faculty of Economics
Austin Robinson Building
Sidgwick Avenue
Cambridge
CB3 9DE
United Kingdom
daniel.sgroi@econ.cam.ac.uk
Tel: 00 44 1223 335244
Fax: 00 44 1223 335299

[1] For financial support, Daniel Sgroi would like to thank AEA Technology plc and David Gill would like to thank the Economic and Social Research Council. Both authors would like to thank Mark Armstrong, Simon Board, Douglas Gale, Michael Grubb, Godfrey Keller, Paul Klemperer, Clare Leaver, Meg Meyer, David Myatt, Hamid Sabourian, Rebecca Stone and participants at presentations in Birkbeck, Cambridge, Essex, Oxford and the Second World Congress of the Game Theory Society in Marseille for helpful comments and suggestions.

Abstract

We consider a principal-agent problem where the principal wishes to be endorsed by a sequence of agents, but cannot truthfully reveal type. In the standard "herding" model, the agents learn from each other's decisions, which can lead to cascades on a given decision when later agents' private information is swamped. We augment the standard model to allow the principal to subject herself to a test designed to provide public information about her type. She must decide how tough a test to attempt from a continuum of test types, which involves trading off the higher probability of passing an easier test against the greater impact from passing a tougher test. We find that the principal will always choose to be tested, and will prefer a tough test to a neutral or easy one.

Keywords: Bayesian updating, endorsements, herding, sequential decision-making, tests

1. Introduction

When a firm develops a new product, it will often consider where to send it for pre-launch review or accreditation. A job applicant facing a sequence of interviews must select suitable referees. Film studios decide who to invite to pre-launch viewings and premieres. A politician may wish to leak a new policy to the media, and has a choice of which media outlet to target. A firm launching an initial public offering can choose from auditors of different reputations to review its accounts. The list goes on. In each case a principal ultimately wants to sell products, win votes, or generally be endorsed by a group of agents, and has the option of being publicly tested before seeking endorsement. Despite the powerful effects that success or failure of such a test may have on the performance of the principal, the literature has paid little attention to the role of such public testing, especially the decision that a principal must make when a variety of tests are available which differ in their degree of toughness. The present paper attempts to correct for this omission.

The principal is assumed to be either good or bad for agents, who need to estimate the relative likelihood of the two types before making their endorsement decision. The agents decide in sequence and are granted three sources of information. As in the standard "herding" model, they receive private information, which perhaps relates to prior experience of the principal or her product or policies, and can observe each other's endorsement decisions in an attempt to learn something about other agents' private information. The sequential nature of decision-making can allow cascades on a given decision to develop when later agents' private information is swamped by the information revealed by the decisions of earlier agents. We introduce a third source of information by allowing the principal to subject herself to a test designed to provide public information about her type before any agent has made his endorsement decision. She must decide how tough a test to attempt from a continuum of test types, which involves trading off the higher probability of passing an easier test against the greater impact from passing a tougher test.

As seems reasonable in this context, a bad type of principal can costlessly duplicate the choice of test chosen by a good principal. As a result, all of our outcomes will be pooling, and there will be no issues of incentive compatibility or scope for a separating equilibrium. Therefore the choice of a tough test does not have the advantage of signaling the type of principal. Without a role for signaling by choice of test, we might assume that since an easy test is by definition the most likely to be passed, it must naturally be the first choice for any principal. However tough tests have two innate advantages. They generate a stronger impact on agents in the event of a pass, and they are less damaging in the event of a fail. We find that the principal will always choose to be tested, and perhaps surprisingly will prefer a tough test to a neutral or easy one despite the lack of an immediate signaling advantage through the choice of test.

1.1. **Related Literature.** The paper most closely related to ours is Lerner and Tirole's (2004) recent working paper concerning the role of technology standard setting authorities as certifiers. These certifiers, who act in a similar way to our tests, are assumed to have an arbitrary bias towards the technology sponsor. The model has significant differences to ours: Lerner and Tirole's certifiers discover with certainty the quality of the technology they are asked to review, while consumers in their model do not receive any private information and cannot learn from other agents' actions. Therefore, as certifiers cannot overwhelm bad private information, Lerner and Tirole do not find any role for certifiers biased against the technology. Instead they find that the sponsor prefers the certifier most biased in favor of the new technology on offer, subject to users adopting following a positive decision by the certifier.

In a model with sequential sales, Sgroi (2002) examines the use of small groups of consumers who are encouraged to decide early and hence act in a similar way to tests in this paper, providing additional information for later consumers. Sgroi finds that irrespective of product quality firms would like to use these "guinea pigs". However, there is no notion of bias or toughness and so the choice of "guinea pigs" is not analogous to the choice of a test from a continuum of toughness levels.

In more specific industrial organization contexts, Taylor (1999) and Ottaviani (1999) find that high prices can be optimal in a similar way to tough tests in this paper. In Ottaviani, the firm wishes to set a high initial price (relative to perceived quality) to encourage the transmission of information. If price is too low, everybody buys, so consumers do not learn from each other's decisions, while if an expensive good becomes successful, this conveys strong positive information to later buyers. Taylor, concentrating on the housing market, finds a high price to be optimal as a failure to sell a house early can then be attributed to overpricing rather than low quality.

Our work should be contrasted with the literature on experts, in which self-interested experts filter information about the true state of the world (see chapter 10 of Chamley (2004) for a survey). These experts' self-interest gives rise to incentives to manipulate the messages they send. Our tests, on the other hand, are purely mechanical: the level of toughness is fixed and commonly known. Our work is also different from the literature on payment structures to certification intermediaries, who may play a similar role to tests - see for example Albano and Lizzeri (2001). In these papers, the intermediary has all the bargaining power as it sets the terms of trade via a price and disclosure rule. Our focus is different as we do not consider explicitly how tests come into existence or take on a particular toughness. However, the preference of different principals for different types of test may justify the existence of different toughness levels, which would make an interesting topic for future research.

Finally, our analysis is different from Ottaviani and Pratt (2001), who find that a monopolist may wish to use a public signal of quality such as an outside certifier. Our principal is informed about her type, while in Ottaviani and Pratt both the buyer and seller are uninformed, so a public signal affiliated with the buyer's private information reduces the buyer's informational rents in a second-degree price discrimination setting.

2. The Model

2.1. **A Simple Herd Model.** A principal, whose objective is to maximize the number of endorsements from a group of $C \in \mathbb{N}_{++}$ agents, may be good or bad for the agents. The principal knows her type, but the agents do not, and there is no easy means of truthful revelation. The agents act in an exogenously ordered sequence deciding whether to endorse the principal or not, so the action of agent i $A_i \in \{Y, N\}$ where Y denotes an endorsement and N a rejection. This "endorsement" is a general concept which could, for example, encompass adopting some new technology, voting for a candidate in an election, purchasing a product, watching a movie, making a job offer etc.[2] The payoff to an agent is simply V which has prior probability $q = \frac{1}{2}$ of returning 1 or -1, depending on whether the principal is a good or bad type, leaving agents indifferent before additional information is obtained. The agents each receive a conditionally independent signal about V defined as $X_i \in \{H, L\}$ for agent i. The signals are informative in the following sense.

Definition 1. *Signals are informative, but not fully-revealing, in the sense that:*

$$\begin{aligned} \Pr\left[X_i = H \mid V = 1\right] &= \Pr\left[X_i = L \mid V = -1\right] = p \in \left(\tfrac{1}{2}, 1\right) \\ \Pr\left[X_i = H \mid V = -1\right] &= \Pr\left[X_i = L \mid V = 1\right] = 1 - p \in \left(0, \tfrac{1}{2}\right) \end{aligned}$$

Agents Bayesian update their beliefs using their private information and inferences from the observed actions of their predecessors in the sequence, endorsing the principal if $E[V] > 0$. The belief-updating model we use is a variant of the seminal herd paper by Bikhchandani, Hirshleifer and Welch (1992). Define the history to agent c as the set of actions of agents 1 to $c-1$ so $H_{c-1} \equiv \{A_1, A_2, ..., A_{c-1}\}$. Now define the information set of agent i as $I_i \equiv \{H_{i-1}, X_i\}$. In certain circumstances X_i will be inferable from A_i but this will not always be true. Now $X_1 = H \Leftrightarrow A_1 = Y$ and $X_1 = L \Leftrightarrow A_1 = N$. Agent 2 can infer agent 1's signal, X_1, from his action, A_1, and so has an information set $I_2 = \{X_1, X_2\}$. If $X_2 = H$ and $A_1 = Y \Rightarrow X_1 = H$ then agent 2 endorses so $A_2 = Y$. If $X_2 = H$ and $A_1 = N \Rightarrow X_1 = L$ or if $X_2 = L$ and $A_1 = Y \Rightarrow X_1 = H$ agent 2 will have two conflicting signals and will be indifferent,

[2] In many specific cases we might need to add more content to the model such as prices in an industrial organization context, a voting rule in a political economy context, etc. We wish to leave this open, but the addition of such features is straightforward. See Gill and Sgroi (2004) for a specific application to purchasing decisions with flexible prices, though in a simultaneous sales context.

so we require a tie-breaking rule. We use a simple coin-flipping rule which is common knowledge to all agents:[3]

Condition 1. *(Tie-breaking) If* $\Pr[V = 1 \mid I_i] = \frac{1}{2}$*, so* $E[V] = 0$*, then* $\Pr[A_i = Y] = \Pr[A_i = N] = \frac{1}{2}$.

Consider a possible chain of events. The *first agent* will endorse if $X_1 = H$ and reject if $X_1 = L$. The *second agent* can infer the signal of the first agent from his action. He will then endorse if $X_2 = H$ having observed adoption by the first agent. If he observed rejection but receives the signal $X_2 = H$ then he will flip a coin following the tie-breaking rule. If he receives $X_2 = L$ and $A_1 = N$ then he too will choose $A_2 = N$. If the first agent endorsed then he would be indifferent and so flip a coin. The *third agent* is the first to face the possibility of a cascade. If he observed two endorsements, so $H_2 = \{Y, Y\}$ then $A_3 = Y$ for all X_3 since he knows that $X_1 = H$ and the second agent's signal is also more likely to be H than L, so the weight of evidence is in favor of endorse *regardless of* X_3. This initiates a Y *cascade*: the third agent will endorse, revealing no information, so the *fourth agent* will also endorse, as will the fifth, etc. Similarly if the third agent observes that both previous choices were rejections then he too will reject, so a N *cascade* is initiated. An informational cascade occurs if an agent's action does not depend upon his private information. The individual, having observed the actions of those ahead of him in a sequence, who follows the behavior of the preceding individual without regard to his own information, is said to be in a cascade.

Definition 2. ***Informational Cascades.*** *A* Y *cascade is said to have started by agent* c *if* $A_c = Y$ *and* $A_{i-1} = Y \Rightarrow A_i = Y$ *for all* X_i *with* $i > c$*. A* N *cascade is said to have started by agent* c *if* $A_c = N$ *and* $A_{i-1} = N \Rightarrow A_i = N$ *for all* X_i *with* $i > c$*.*

We should note that a cascade, once started, will last forever as no further information is revealed by agents' actions.[4] This is so even if it is based on an action which would not be chosen if all agents' signals were common knowledge. The possibility of convergence to the incorrect outcome through the loss of information contained in later agents' private signals might be phrased in terms of a discernible negative *herd externality* as suggested in Banerjee (1992).

From the model specifications the (conditional) *ex ante* probabilities of a Y cascade, N cascade, or no cascade after c agents can be derived. Define $Y(c)$ to be a Y cascade which has started by agent

[3] Coin flipping is the standard tie-break rule used in herding models. See for example Bikhchandani, Hirshleifer and Welch (1992). Equivalently each agent may be following a fixed selection rule, so long as in expectation half of indifferent agents select the principal and half do not. Banerjee (1992) instead uses a "follow your own signal" rule, but does this specifically to minimize the chance of a herd.

[4] Given the simplicity of our herding structure there will be no differentiation between herding and cascading by agents. If p were different for each agent, for some agents p_i might be sufficiently high that they gain nothing from observation. This would therefore confound the learning process and might allow herds to be broken. By restricting $p_i = p \; \forall i$, we do not consider such confounded learning. See Smith and Sorensen (2000) for further details.

c. Similarly define $N(c)$ for a N cascade by agent c. For example, $\Pr[Y(2)]$ is simply the probability that the first two agents both choose Y. After an even number c of agents, the relevant functions can be derived using geometric progressions:

$$\Pr[Y(c) \mid V=1] = \tfrac{p(p+1)}{2}\tfrac{1-\left(p-p^2\right)^{\frac{c}{2}}}{1-(p-p^2)} \tag{2.1}$$

$$\Pr[N(c) \mid V=1] = \tfrac{(p-2)(p-1)}{2}\tfrac{1-\left(p-p^2\right)^{\frac{c}{2}}}{1-(p-p^2)} \tag{2.2}$$

These expressions allow us to make a number of clarifying remarks. It can be shown that (2.1) is increasing in p and c, but (2.2) registers a high probability even for p much greater than $\frac{1}{2}$. For example, for $p=\frac{3}{4}$, there is a 20% chance of an incorrect cascade (a cascade on endorse with a bad principal, or a cascade on reject with a good principal) having started by the 10th agent. Therefore, even when the principal is a good type (which is likely to be reflected by the great majority of the signals), she still faces the prospect of a possible cascade of rejections, because there is a reasonable chance that a few early incorrect signals start an incorrect cascade. This is worrying both for agents and for principals of the good type. The symmetric case where $V=-1$ would apply when the principal is a bad type, and the results provide some hope for such principals, since there is always the chance of a Y cascade. Of course there is no reason for a principal to stay passive in the face of such potential cascades, and by selecting a suitable test the principal can hope to raise the chance of a cascade in her favor and diminish the chance of a cascade going against her.

2.2. **Adding Tests.** Before facing the stream of agents, the principal can opt to be publicly tested. We want to think of the test as by its very nature revealing more information about the principal's type than a single typical private signal. The simplest way of modeling this is to allow the test to involve the draw of two signals about the principal's type, instead of just the one received by agents.

The test generates a decision $d \in \{P, F\}$ whether to pass (P) or fail (F) the principal. Of course, tests may make a finer distinction than simply passing the principal or not. However, we want to think of the result of the test as being quickly and easily disseminated throughout the population of agents, e.g., through word of mouth, written reports concerning the test and so on. Thus we are thinking of a process through which even sophisticated tests quickly get shortened to a binary decision through this process of dissemination.[5]

We consider a continuum of test types which receive two i.i.d. draws from the same signal distribution as agents. Tests are passed with a signal draw of HH, pass with probability $\phi \in [0,1]$ on a draw of HL

[5]Modeling an evaluation as condensing more complex information into a simple binary decision follows for example Calvert (1985) who notes that: "This feature represents the basic nature of advice, a distillation of complex reality into a simple recommendation."

or LH and fail on a draw of LL. The value of ϕ encapsulates the type of the test. The lower the value of ϕ, the tougher the test that is chosen by the principal.

Definition 3. *Tests with* $\phi \in [0, \frac{1}{2})$ *are termed "tough", those with* $\phi \in (\frac{1}{2}, 1]$ *"easy", and those with* $\phi = \frac{1}{2}$, *which pass on a coin flip on observing a set of mixed signals, "neutral".*

The toughness of a particular test is common knowledge, perhaps generated through a known history of pass or fail decisions, and the principal is able to choose the test type. We have left the notion of test fairly abstract, but depending on the application, the choice of test might consist of a choice between different reviewers, interviewers, referees, accreditation bodies and so on.

The simple herd model assumes that the sequence of agents is of a known determinate length, allowing the use of geometric progressions to solve for herd probabilities. Once we introduce a choice of test, using this method to calculate and compare the number of endorsements for the principal under different symmetric and asymmetric scenarios quickly becomes excessively complex. Instead we introduce a stream of agents of uncertain length, which allows us to use a recursive solution method to readily solve for and compare expected numbers of endorsements, even where an asymmetry is introduced by the choice of a tough or easy test. We assume that the length C of the sequence of agents is not known with certainty to the principal. Instead, after each agent decides whether or not to endorse, there is a probability $(1 - \theta)$ that the sequence of agents comes to an end, where $\theta \in (0, 1)$. Thus, the expected number of agents $E[C] = 1 + \theta + \theta^2 + ... = \frac{1}{1-\theta}$, and $\theta = \frac{E[C]-1}{E[C]}$. Note $\frac{dE[C]}{d\theta} > 0$.

There are several interpretations of $E[C]$. Firstly, the principal may simply not know how many agents she faces. Secondly, the method used is formally equivalent to introducing a standard discount factor where the principal faces an infinite stream of agents, and so can equally well be used to assess the principal's choice of test when she discounts later endorsements. If the principal is attempting to sell a product, push a new technology or advance a particular policy then, $E[C]$ may also represent a measure of how quickly the principal expects a rival product, technology or policy to be developed which will make her own obsolete. Although θ can range from 0 to 1, for reasonable $E[C]$ values it will be in the upper part of this range. For example, for $E[C] \geq 10$, $\theta \geq 0.9$.

2.3. **Restriction to High Ability Types of Principal.** Throughout, for conciseness, we consider just the good type of principal. By standard signaling considerations, a bad type of principal will be forced to copy the choice of the good type to avoid immediately revealing type and so receive no endorsements. A separating equilibrium is not possible, as the bad type would copy the choice of the good principal, and so be believed to be good and obtain the same outcome as the good type (receiving endorsements from all agents). Thus, we restrict attention to pooling equilibria in which the bad type of principal is forced to follow the good type's preference. Such equilibria can always be supported by the belief that any principal who deviates from the good type's preferences must be a bad type. Note that in such pooling equilibria, agents will be unable to infer anything about the type of the principal from the principal's *choice of test*, and so will have to rely on observing the *outcome of the test* to provide additional information about the principal's type.

3. Agent Learning

Once the principal has selected a test of a publicly known type, and the test result has been publicly announced, agents decide in a fixed sequence. The sequential ordering allows agents to learn from each other's decisions as well as from the result of the test, combining this information with their own private signals. Potentially, more information may be transmitted to agents later in the sequence. However, agents may fall into an informational cascade in the sense of Bikhchandani, Hirshleifer and Welch (1992), where public information swamps private information. Once a given agent in the sequence rationally disregards his own private information, nothing further is revealed to later agents who will then also all disregard their private information and copy the choice of their predecessor agent. In this section we focus on the information possessed, and choices made, by the agents and the potential herding which their ability to observe other agents may induce.

3.1. **Agents' Beliefs and Endorsement Decisions.**

Agent i will observe the test result and the actions of his predecessors, which will allow him to update his prior belief that the principal is good from $q = \frac{1}{2}$ to $q = q_i^*$. We start by deriving two remarks, which are used implicitly throughout. The first says that agents when applying Bayes' Rule to calculate the ratio of the probability of principal being good to the probability of her being bad simply need to calculate the ratio of the probability of the private signal they have observed if the principal were good to the probability if the principal were bad, suitably weighted by the updated prior. The second is self-explanatory.

Remark 1.

$$\frac{\Pr\left[V=1|X_i\right]}{\Pr\left[V=-1|X_i\right]} = \frac{\frac{\Pr[X_i|V=1]\Pr[V=1]}{\Pr[X_i]}}{\frac{\Pr[X_i|V=-1]\Pr[V=-1]}{\Pr[X_i]}} = \frac{\Pr\left[X_i|V=1\right]q_i^*}{\Pr\left[X_i|V=-1\right](1-q_i^*)}$$

Remark 2. *When calculating beliefs, agents can cancel and ignore opposing H and L signals.*

Proof. Suppose the agent infers an information set I_i. Now suppose that instead of I_i, the agent infers I_i^+, which we define as the set I_i plus a further two opposing H and L signals. Then, using Remark 1 (replacing q_i^* with the initial prior q):

$$\frac{\Pr\left[V=1|I_i^+\right]}{\Pr\left[V=-1|I_i^+\right]} = \frac{\Pr\left[I_i^+|V=1\right]q}{\Pr\left[I_i^+|V=-1\right](1-q)} = \frac{\Pr[I_i|V=1]p(1-p)q}{\Pr[I_i|V=-1](1-p)p(1-q)} = \frac{\Pr[V=1|I_i]}{\Pr[V=-1|I_i]} \qquad \square$$

We can now determine agent $i's$ endorsement decision. Where $q_i^* > p$, the i^{th} agent will endorse. Agent i is least likely to endorse if $X_i = L$. Taking this case, we have an odds ratio $\frac{\Pr[V=1|L]}{\Pr[V=-1|L]} =$

$\frac{(1-p)q_i^*}{p(1-q_i^*)} = \frac{q_i^*-pq_i^*}{p-pq_i^*} > 1$ since $q_i^* > p$. A symmetrical argument shows that where $q_i^* < 1-p$, agent i will reject. Where $q_i^* \in (1-p, p)$, following a H signal $\frac{\Pr[V=1|H]}{\Pr[V=-1|H]} = \frac{pq_i^*}{(1-p)(1-q_i^*)} > 1$ as $q_i^* > (1-p)$, so i endorses. Following a L signal $\frac{\Pr[V=1|L]}{\Pr[V=-1|L]} = \frac{(1-p)q_i^*}{p(1-q_i^*)} < 1$ as $q_i^* < p$, so i rejects. Where $q_i^* = p$, following a L signal, $\frac{\Pr[V=1|L]}{\Pr[V=-1|L]} = \frac{(1-p)q_i^*}{p(1-q_i^*)} = 1$ as $q_i^* = p$, so the agent is indifferent and flips a coin. A H signal is more positive, so the agent endorses. By symmetry, where $q_i^* = 1-p$, following a H signal the agent flips a coin, while following a L signal the agent rejects. The following lemma summarizes this information.

Lemma 1. *The i^{th} agent will respond to an updated prior as follows: (a) if $q_i^* > p$ then i will endorse; (b) if $q_i^* = p$, following a H signal i endorses, while following a L signal, he flips a coin; (c) if $q_i^* \in (1-p, p)$ then i will endorse if and only if $X_i = H$; (d) if $q_i^* = 1-p$, following a H signal i flips a coin, while following a L signal he rejects; (e) if $q_i^* < 1-p$ then i rejects.*

This lemma is crucial to our understanding of the impact of test results for different test types. As we will see, by selecting a particular type of test, the principal can effectively partition the updated prior for the first agent.

Where q_i^* is strongly positive or negative, it outweighs any possible private signal agent i might receive, leading to a cascade. Where $q_i^* > p$, from Lemma 1 agent i endorses whatever the signal received. The decision is thus uninformative, so agent $i+1$ also endorses, and so on. A symmetrical argument applies where $q_i^* < 1-p$.

Lemma 2. *Where $q_i^* > p$, we have a cascade on endorsement, i.e., agent i and all subsequent agents endorse. Where $q_i^* < 1-p$, we have a cascade on rejection, i.e., agent i and all subsequent agents fail to endorse.*

3.2. **Impact of Tests on Beliefs.** Here, we determine how different test results impact on agents' beliefs. We begin by finding the updated prior faced by the first agent in the event of a pass (denoted by $q_1^* = q_P^*$) and a fail (denoted by $q_1^* = q_F^*$).

$$q_P^* \equiv \Pr[V = 1 \mid \text{Pass}] = \frac{[p^2+2p(1-p)\phi]}{[p^2+2p(1-p)\phi]+[(1-p)^2+2(1-p)p\phi]} = \frac{p^2+2p(1-p)\phi}{p^2+(1-p)^2+4p(1-p)\phi}$$

$$q_F^* \equiv \Pr[V = 1 \mid \text{Fail}] = \frac{[(1-p)^2+2p(1-p)(1-\phi)]}{[(1-p)^2+2p(1-p)(1-\phi)]+[p^2+2(1-p)p(1-\phi)]} = \frac{(1-p)^2+2p(1-p)(1-\phi)}{(1-p)^2+p^2+4p(1-p)(1-\phi)}$$

Next we note some properties of these expressions. Firstly, note that q_P^* is decreasing in ϕ:

$$\frac{dq_P^*}{d\phi} = \frac{2p(1-p)[p^2+(1-p)^2+4p(1-p)\phi]-4p(1-p)[p^2+2p(1-p)\phi]}{[p^2+(1-p)^2+4p(1-p)\phi]^2} = \frac{2p(1-p)(1-2p)}{[p^2+(1-p)^2+4p(1-p)\phi]^2} < 0$$

since for $p > \frac{1}{2}$, the denominator is always strictly positive, $2p(1-p) > 0$, but $(1-2p) < 0$. Secondly, note that q_F^* is also decreasing in ϕ:

$$\frac{dq_F^*}{d\phi} = \frac{-2p(1-p)\left[(1-p)^2+p^2+4p(1-p)(1-\phi)\right]+4p(1-p)\left[(1-p)^2+2p(1-p)(1-\phi)\right]}{\left[(1-p)^2+p^2+4p(1-p)(1-\phi)\right]^2} = \frac{2p(1-p)(1-2p)}{\left[(1-p)^2+p^2+4p(1-p)(1-\phi)\right]^2} < 0$$

similarly to the q_P^* case. That these updated priors should be decreasing in ϕ is perfectly natural: a pass is better news the tougher the test, while a fail is not such bad news.

With a neutral test, so setting $\phi = \frac{1}{2}$, we have:

$$q_P^*\left(\phi = \tfrac{1}{2}\right) = \frac{p^2+p(1-p)}{p^2+(1-p)^2+2p(1-p)} = p \tag{3.1}$$

$$q_F^*\left(\phi = \tfrac{1}{2}\right) = \frac{(1-p)^2+p(1-p)}{(1-p)^2+p^2+2p(1-p)} = 1-p \tag{3.2}$$

As q_P^* and q_F^* are both strictly decreasing in ϕ, this immediately implies that $q_P^*\left(\phi < \frac{1}{2}\right) > p$, $q_P^*\left(\phi > \frac{1}{2}\right) < p$, $q_F^*\left(\phi < \frac{1}{2}\right) > 1-p$ and $q_F^*\left(\phi > \frac{1}{2}\right) < 1-p$. Furthermore, $q_P^*(\phi = 1) > \frac{1}{2}$ and $q_F^*(\phi = 0) < \frac{1}{2}$:

$$\begin{aligned} q_P^*(\phi = 1) &= \frac{p^2+2p(1-p)}{p^2+(1-p)^2+4p(1-p)} > \tfrac{1}{2} \Leftrightarrow p^2 > (1-p)^2 \\ q_F^*(\phi = 0) &= \frac{(1-p)^2+2p(1-p)}{(1-p)^2+p^2+4p(1-p)} < \tfrac{1}{2} \Leftrightarrow (1-p)^2 < p^2 \end{aligned}$$

Using the fact that q_P^* and q_F^* are both strictly decreasing in ϕ, it follows that $q_P^* > \frac{1}{2}$ and $q_F^* < \frac{1}{2}$ $\forall\phi$. All this information is summarized in the following lemma:

Lemma 3. *(a)* $q_P^*\left(\phi < \frac{1}{2}\right) > p$; *(b)* $q_P^*\left(\phi = \frac{1}{2}\right) = p$; *(c)* $q_P^*\left(\phi > \frac{1}{2}\right) \in \left(\frac{1}{2}, p\right)$; *(d)* $q_F^*\left(\phi < \frac{1}{2}\right) \in \left(1-p, \frac{1}{2}\right)$; *(e)* $q_F^*\left(\phi = \frac{1}{2}\right) = 1-p$; *(f)* $q_F^*\left(\phi > \frac{1}{2}\right) < 1-p$.

3.3. **Expected Number of Endorsements.** In the previous section, we calculated the effect of different test results on q_1^*; now we calculate the expected number of endorsements, π, for the good type of principal $\forall q_1^*$.[6] From Lemma 2, $q_1^* > p$ leads to an immediate cascade on endorsement for the principal, so $\pi_{q_1^*>p} = \frac{1}{1-\theta}$, while $q_1^* < (1-p)$ leads to an immediate cascade on rejection, so $\pi_{q_1^*<1-p} = 0$. Next we find the expected number of endorsements where $q_1^* = \frac{1}{2}$.

Lemma 4. *Where* $q_1^* = \frac{1}{2}$, *the expected number of endorsements for a good principal is*

$$\pi_{q_1^*=\frac{1}{2}} = \frac{p\left[2-(1-p)\theta^2\right]}{2\left[1-p(1-p)\theta^2\right](1-\theta)} \tag{3.3}$$

Proof. See Appendix. The proof is based on a recursive solution to the appropriate decision tree. □

[6]If the principal was a firm attempting to sell products, then π would measure revenue. Note that since there is no cost to the principal, this would also equal profits. In a voting model π would be the number of votes obtained.

Now suppose $q_1^* = p$. From Lemma 1 if the first agent gets a positive signal he endorses, while if he gets a negative signal this exactly cancels the positive prior so he is indifferent and flips a coin. We can think of the second and subsequent agents as starting a new sequence with updated prior q_2^*. If the first agent rejects, later agents infer $X_1 = L$, so $q_2^* = \frac{1}{2}$. If the first endorses, then he is more likely to have observed H than L, sending a positive signal, thus increasing q_2^* above q_1^*, so $q_2^* > p$.[7] Thus, using Lemma 2:

$$\pi_{q_1^*=p} = \left[p + \tfrac{1}{2}(1-p)\right]\left(\tfrac{1}{1-\theta}\right) + \tfrac{1}{2}(1-p)\theta\pi_{q_1^*=\frac{1}{2}} \tag{3.4}$$

Suppose instead $q_1^* = 1-p$. This case is the symmetric opposite. If the first agent gets a L signal, he rejects, while if he gets a H signal, this exactly cancels the negative prior so he flips a coin. Thus, if the first agent endorses, later agents infer $X_1 = H$, so $q_2^* = \frac{1}{2}$. If the first rejects, then he is more likely to have observed L than H, sending a negative signal, so $q_2^* < 1-p$. Thus, using Lemma 2:

$$\pi_{q_1^*=1-p} = \tfrac{1}{2}p\left(1 + \theta\pi_{q_1^*=\frac{1}{2}}\right) \tag{3.5}$$

Next, we calculate the expected number of endorsements where $q_1^* \in (\frac{1}{2}, p)$ or $q_1^* \in (1-p, \frac{1}{2})$. Note that the expected number of endorsements is independent of the specific q_1^* value in the two ranges.

Lemma 5. *Where $q_1^* \in (\frac{1}{2}, p)$, the expected number of endorsements for a good principal is*

$$\pi_{q_1^*\in(\frac{1}{2},p)} = \frac{p\left[1 + (1-p)\theta\,(1-\theta)\right]}{\left[1 - p(1-p)\theta^2\right](1-\theta)} \tag{3.6}$$

Proof. See Appendix. Again, a recursive decision tree is used. □

Lemma 6. *Where $q_1^* \in (1-p, \frac{1}{2})$, the expected number of endorsements for the good principal is*

$$\pi_{q_1^*\in(1-p,\frac{1}{2})} = \frac{p\left[p + (1-p)\,(1-\theta)\right]}{\left[1 - p(1-p)\theta^2\right](1-\theta)} \tag{3.7}$$

Proof. See Appendix. □

Of course, $\pi_{q_1^*>p} > \pi_{q_1^*=p} > \pi_{q_1^*\in(\frac{1}{2},p)} > \pi_{q_1^*=\frac{1}{2}} > \pi_{q_1^*\in(1-p,\frac{1}{2})} > \pi_{q_1^*=1-p} > \pi_{q_1^*<1-p}$.

4. Choice of Test

We now have the building blocks which allow us to easily calculate and compare the expected number of endorsements for the principal from choosing different test types. Suppose first that the test is neutral. Then, using Lemma 3 and the fact that $\Pr[P] = p^2 + 2p(1-p)\phi$ while $\Pr[F] = 2p(1-p)(1-\phi) + (1-p)^2$:

$$\Pi\left[\phi = \tfrac{1}{2}\right] = \Pr[P].\pi_{q_1^*=p} + \Pr[F].\pi_{q_1^*=1-p} = p\pi_{q_1^*=p} + (1-p)\pi_{q_1^*=1-p} \tag{4.1}$$

[7] Formally, $\frac{q_2^*}{1-q_2^*} = \frac{\Pr[V=1|A_1=Y]}{\Pr[V=-1|A_1=Y]} = \frac{p+\frac{1}{2}(1-p)}{(1-p)+\frac{1}{2}p}\frac{q_1^*}{1-q_1^*} > \frac{q_1^*}{1-q_1^*}$.

With a tough test, using Lemmas 3 and 2, a pass starts an immediate cascade on endorsement with $q_1^* > p$, while a fail leaves $q_1^* \in (1-p, \frac{1}{2})$. Thus:

$$\Pi\left[\phi \in [0, \tfrac{1}{2})\right] = \left\{p^2 + 2p(1-p)\phi\right\} \pi_{q_1^*>p} + \left\{2p(1-p)(1-\phi) + (1-p)^2\right\} \pi_{q_1^* \in (1-p, \frac{1}{2})} \tag{4.2}$$

Note that $\frac{d\left(\Pi\left[\phi \in [0,\frac{1}{2})\right]\right)}{d\phi} > 0$ as $\pi_{q_1^*>p}$ and $\pi_{q_1^* \in (1-p,\frac{1}{2})}$ are constant in ϕ (see (3.7) and Lemma 2), $\pi_{q_1^*>p} > \pi_{q_1^* \in (1-p,\frac{1}{2})}$, $\Pr[P] = p^2 + 2p(1-p)\phi$ is strictly increasing in ϕ and $\Pr[F] = 2p(1-p)(1-\phi) + (1-p)^2$ is strictly decreasing in ϕ.

Finally, we consider an easy test. This case is the symmetric opposite of the tough test case. Using Lemmas 3 and 2, a fail starts an immediate cascade on rejection as $q_1^* < 1-p$, while a pass sends a positive signal weaker than the one sent out when a neutral test is passed, so $q_1^* \in (\frac{1}{2}, p)$, giving:

$$\Pi\left[\phi \in (\tfrac{1}{2}, 1]\right] = \left\{p^2 + 2p(1-p)\phi\right\} \pi_{q_1^* \in (\frac{1}{2}, p)} \tag{4.3}$$

Comparing the different expected number of endorsements, we find the following proposition.

Proposition 1. *For any choice of three test types $\phi \in \left\{\phi_T, \frac{1}{2}, \phi_E\right\}$ such that $\phi_T \in [0, \frac{1}{2})$ and $\phi_E \in (\frac{1}{2}, 1]$, the good type of principal strictly prefers the tough type of test ϕ_T to the neutral test type $\phi = \frac{1}{2}$ and to the easy test type ϕ_E.*

Proof. See Appendix. □

Despite the fact that tough tests are less likely to be passed, the good principal prefers any tough test to any neutral or easy one. She prefers tough tests because of the strong impact on agents' decisions from a pass, which leads to a cascade on endorsement, while a fail in a tough test is not too costly as the test is known to be tough, diluting the impact of failure on agents' beliefs.

Next we define the concept of ϵ-optimization.

Definition 4. *To ϵ-optimize Π over a range of ϕ values, $\Omega \subseteq [0, 1]$, means to select a $\phi \in \Omega$ such that $\Pi > \sup_{\phi \in \Omega} \Pi - \epsilon$.*

The following lemma, which makes use of this definition, follows from the fact that $\frac{d\left(\Pi\left[\phi \in [0,\frac{1}{2})\right]\right)}{d\phi} > 0$, so $\sup_{\phi < \frac{1}{2}} \Pi = p\pi_{q_1^*>p} + (1-p)\pi_{q_1^* \in (1-p,\frac{1}{2})}$.

Lemma 7. *For any $\epsilon > 0$ the principal can ϵ-optimize over $\phi \in [0, \frac{1}{2})$ by selecting ϕ sufficiently close to $\frac{1}{2}$, thus achieving $\Pi > \sup_{\phi < \frac{1}{2}} \Pi - \epsilon$.*

Informally, the lemma says that the optimal tough test is one whose toughness is arbitrarily mild (i.e., with ϕ strictly below but arbitrarily close to $\frac{1}{2}$).

Proposition 1 tells us that any tough type of test beats the neutral test or any easy type of test, which implies that $\sup_{\phi\in[0,1]} \Pi = \sup_{\phi<\frac{1}{2}} \Pi$. Together with Lemma 7 this gives:

Proposition 2. *For any $\epsilon > 0$ the principal can ϵ-optimize over $\phi \in [0,1]$ by selecting ϕ sufficiently close to $\frac{1}{2}$, thus achieving* $\Pi > \sup_{\phi\in[0,1]} \Pi - \epsilon$.

Informally, the optimal test type is a tough test which has arbitrarily mild toughness. A pass in *any* tough type of test leads to an immediate cascade on endorsement, so the principal prefers a tough test that is as close as possible to the neutral test type so as to maximize the probability of a pass.

Finally, we find that the neutral test is strictly preferred to no test at all. The signal arising from the neutral test allows some information transmission through the sequence of agents which, in expectation, is valuable to the good type of principal.

Proposition 3. *The choice of taking no test is strictly worse than taking the neutral type of test $\phi = \frac{1}{2}$ and a fortiori strictly worse than a tough test type.*

Proof. See Appendix. □

The following two figures illustrate the size of the increase in the expected number of endorsements from opting to face a tough test. Figure 1 shows the percentage increase in the expected number of endorsements from taking the toughest test over a neutral test, which peaks at 25%, while Figure 2 shows the percentage increase from using the tough test type which is almost neutral, which peaks at 50%.

Figure 1: $\frac{\Pi[\phi=0]-\Pi\left[\phi=\frac{1}{2}\right]}{\Pi\left[\phi=\frac{1}{2}\right]}$

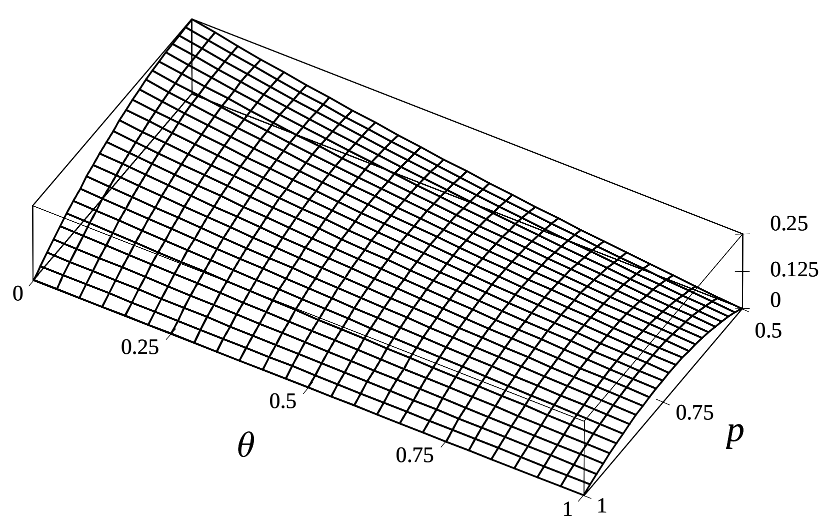
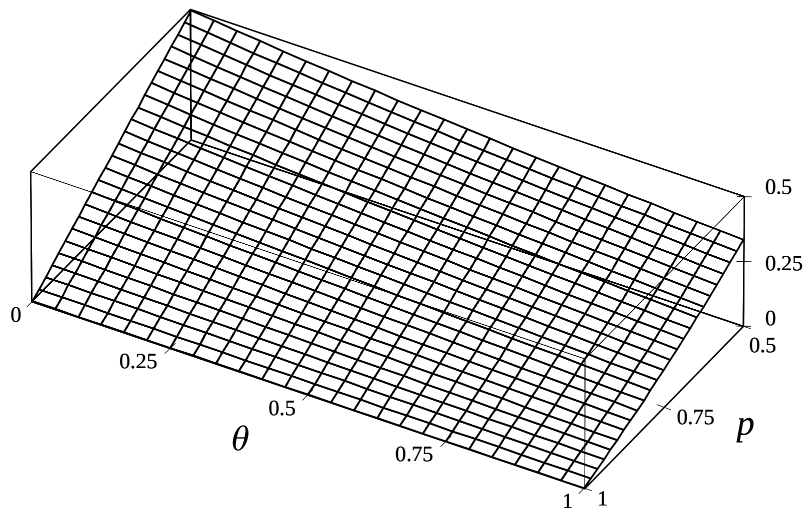


Figure 2: $\frac{\sup_{\phi<\frac{1}{2}} \Pi - \Pi\left[\phi=\frac{1}{2}\right]}{\Pi\left[\phi=\frac{1}{2}\right]}$



5. Conclusion

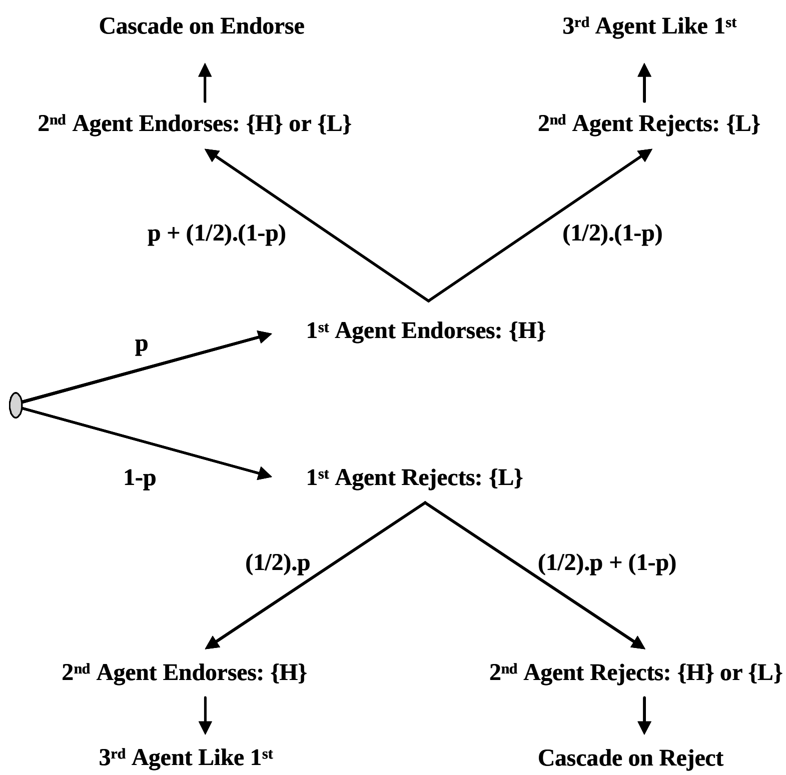
Within the model presented in which test toughness is tightly defined, as are the available sources of information for agents, we find that a principal of type unknown to a sequence of agents should seek to face a public test if this is possible. Furthermore, that principal should seek out the mildest form of tough test available. If there are only very tough tests, then those are the ones that should be selected. Tests that are relatively easy are not optimal, as they provide too damaging a signal in the event of a fail, and too little gain in the event of a pass. Converting these results into practical normative advice, job applicants might consider resisting the temptation to approach relatively soft referees. Firms should avoid "yes men" reviewers for their products. A politician should consider opting to select where to be interviewed, or where to leak new policies, based on the simple premise of first ruling out optimistic or positively biased journalists and then selecting the mildest of those who are intrinsically biased against the politician and his policies. At a descriptive level, we have an explanation for the existence of tough tests, biased newspapers, tough referees, etc. that have a well known harsh or overly critical style, and yet are regularly chosen. When selecting a test to take, a reviewer to observe your product, a referee to provide a letter of recommendation or an interviewer to face, the old Roman proverb is perhaps the best summary of our findings: fortune favors the brave.

Appendix

Proof of Lemma 4. Given $q_1^* = \frac{1}{2}$, the first agent will follow his signal, i.e., will endorse iff $X_1 = H$. Suppose the first agent endorses. This reveals his signal to be H to the second agent. If the second agent also gets a H signal, he therefore also endorses, but if he gets a L signal the H and L signals cancel, so he is indifferent and flips a coin. If the third agent observes two endorse decisions, a cascade on endorse starts. If he gets $X_3 = L$, his signal and that of the first agent cancel, but because the second agent endorsed, he is more likely to have observed $X_2 = H$ than $X_2 = L$. Formally, for the third agent $\frac{\Pr[V=1|I_3]}{\Pr[V=-1|I_3]} = \frac{p+\frac{1}{2}(1-p)}{(1-p)+\frac{1}{2}p} > 1$. Thus the third agent endorses if he receives a bad signal, and *a fortiori* endorses with a good signal, so a cascade on endorse has started.

Suppose instead that the first agent endorses, but the second agent rejects. Then the third agent can infer $\{X_1 = H,\, X_2 = L\}$. These two signals cancel, so the third agent is in exactly the same situation as the first agent before he received a signal.

The case where the first agent rejects is the symmetric opposite. If the second agent also rejects, a cascade on reject starts. If the second agent endorses, then the third agent is back to exactly the same situation as the first agent. All this can be illustrated in the following decision tree:[8]



We can use this tree to calculate the expected number of endorsements for the principal in this case by finding the expected endorsements down various branches of the tree and multiplying by the probability

[8]Remember that the principal knows it is a good type. The branch probabilities are predicated upon this assumption.

of the relevant branch. Note that we have a recursive structure, whereby the expected number of endorsements from various points further down the tree are equivalent to those from points higher up in the tree. Letting $\pi_{q_1^*=\frac{1}{2}}$ be the expected number of endorsements at the beginning of the tree, we get:

$$\pi_{q_1^*=\frac{1}{2}} = p\left[p + \tfrac{1}{2}(1-p)\right]\left(\tfrac{1}{1-\theta}\right) + p\left[\tfrac{1}{2}(1-p)\right]\left(1 + \theta^2 \pi_{q_1^*=\frac{1}{2}}\right) + (1-p)(\tfrac{1}{2}p)\left(\theta + \theta^2 \pi_{q_1^*=\frac{1}{2}}\right)$$

which solves to give the value for $\pi_{q_1^*=\frac{1}{2}}$ in the lemma. □

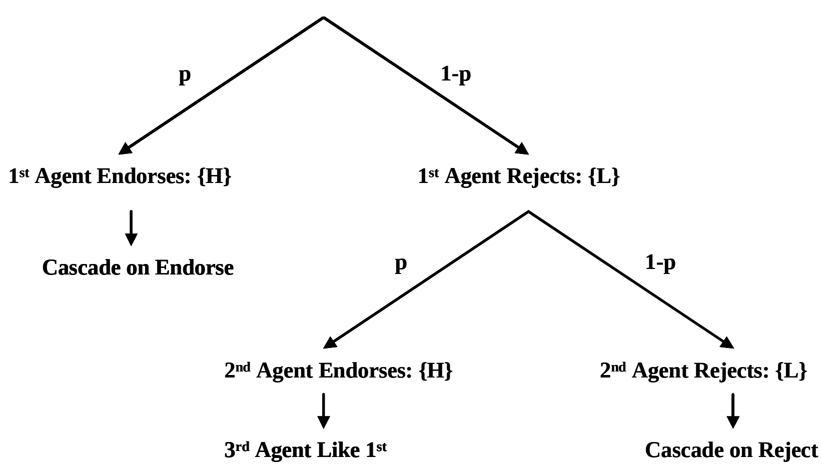
Proof of Lemma 5. From Lemma 1 the first agent will endorse iff $X_1 = H$. Following endorsement by the first agent, a cascade on endorse starts. The second agent can infer that the first one got a H signal. Thus, $q_2^* = \frac{pq_1^*}{pq_1^*+(1-p)(1-q_1^*)} > p$ as $q_1^* > pq_1^* + 1 - p - q_1^* + pq_1^*$ iff $2q_1^*(1-p) > 1-p$ or $q_1^* > \frac{1}{2}$, which of course we have assumed. Hence by Lemma 2 a cascade on endorse starts.

If the first agent rejects, then the second agent endorses iff $X_2 = H$. The second agent can infer $X_1 = L$, so if $X_2 = H$, the two signals cancel, and hence the second agent endorses as $q_1^* > \frac{1}{2}$. If $X_2 = L$, then the agent has effectively seen two negative signals, and so rejects given the first agent with a single negative signal does so.

Following rejection by the first agent, and an endorsement by the second, the third agent can infer a L and a H signal, which cancel leaving him in exactly the same position as the first agent before he received a signal.

Following rejection by the first two agents, a cascade on reject starts. If the third agent receives a H signal, this cancels one of the two inferred L signals, so the agent is left with just one L signal which, just as for the first agent with L, leads to rejection. *A fortiori*, he also rejects if he receives a L signal.

All this information can be summarized in the following decision tree, with branch probabilities conditional on the principal being the good type.



Thus, for any specific $q_1^* \in (\frac{1}{2}, p)$, we can find expected number of endorsements for the good type of principal,

$$\pi_{q_1^*\in(\frac{1}{2},p)} = p\left(\frac{1}{1-\theta}\right) + (1-p)p\left[\theta + \theta^2\pi_{q_1^*\in(\frac{1}{2},p)}\right]$$

which solves to give the value for $\pi_{q_1^*\in(\frac{1}{2},p)}$ in the lemma. □

Proof of Lemma 6. The shape of the decision tree, which is determined by agents who do not know the type of principal, will be the symmetric opposite of the one in the proof of Lemma 5 for the $q_1^* \in (\frac{1}{2}, p)$ case. A L signal starts a cascade on reject, just like before a H signal started a cascade on endorse, while two H signals start a cascade on endorse, just like before two L signals started a cascade on reject. If the first agent endorses but the second does not, the inferred signals cancel. Thus, for any specific $q_1^* \in (1-p, \frac{1}{2})$,

$$\pi_{q_1^*\in(1-p,\frac{1}{2})} = p^2\left(\frac{1}{1-\theta}\right) + p(1-p)\left[1 + \theta^2\pi_{q_1^*\in(1-p,\frac{1}{2})}\right]$$

which solves to give the value for $\pi_{q_1^*\in(\frac{1}{2},p)}$ in the lemma. □

Proof of Proposition 1. To show that any $\phi_T \in [0, \frac{1}{2})$ gives a larger number of expected endorsements than $\phi = \frac{1}{2}$ we simply need to show that $\Pi[\phi = 0] > \Pi\left[\phi = \frac{1}{2}\right]$, given $\frac{d\left(\Pi\left[\phi\in[0,\frac{1}{2})\right]\right)}{d\phi} > 0$. Using (4.2), (3.7) and Lemma 2, $\Pi[\phi = 0] = p^2\left(\frac{1}{1-\theta}\right) + \left(1-p^2\right)\frac{p[p+(1-p)(1-\theta)]}{\left[1-p(1-p)\theta^2\right](1-\theta)}$, which simplifies to:

$$\Pi[\phi = 0] = \frac{p\left[1+p(1-p)+p^2(1-p)\theta(1-\theta)-(1-p)\theta\right]}{\left[1-p(1-p)\theta^2\right](1-\theta)} \tag{A.1}$$

From (4.1), $\Pi\left[\phi = \frac{1}{2}\right] = p\pi_{q_1^*=p} + (1-p)\pi_{q_1^*=1-p}$. Thus, using (3.4), (3.3) and (3.5):

$$\Pi\left[\phi = \tfrac{1}{2}\right] = p\left[p + \tfrac{1}{2}(1-p)\right]\left(\frac{1}{1-\theta}\right) + p\tfrac{1}{2}(1-p)\theta\frac{p\left[2-(1-p)\theta^2\right]}{2\left[1-p(1-p)\theta^2\right](1-\theta)} + (1-p)\tfrac{1}{2}p\left(1 + \theta\frac{p\left[2-(1-p)\theta^2\right]}{2\left[1-p(1-p)\theta^2\right](1-\theta)}\right)$$

which simplifies to

$$\Pi\left[\phi = \tfrac{1}{2}\right] = \frac{p[2+2p(1-p)\theta(1-\theta)-(1-p)\theta]}{2\left[1-p(1-p)\theta^2\right](1-\theta)} \tag{A.2}$$

From (A.1) and (A.2),

$$\Pi[\phi = 0] - \Pi\left[\phi = \tfrac{1}{2}\right] = \frac{p[2+2p(1-p)+2p^2(1-p)\theta(1-\theta)-2(1-p)\theta]-p[2+2p(1-p)\theta(1-\theta)-(1-p)\theta]}{2\left[1-p(1-p)\theta^2\right](1-\theta)}$$

$$\therefore \Pi[\phi = 0] - \Pi\left[\phi = \tfrac{1}{2}\right] = \frac{p(1-p)\left[2p+2p^2\theta(1-\theta)-\theta-2p\theta(1-\theta)\right]}{2\left[1-p(1-p)\theta^2\right](1-\theta)}$$

The denominator is strictly positive, as is $p(1-p)$, so to show $\Pi[\phi = 0] > \Pi\left[\phi = \frac{1}{2}\right]$ we just need to show that $2p + 2p^2\theta(1-\theta) - \theta - 2p\theta(1-\theta) > 0$. Thus, a sufficient condition is that $2p - \theta - 2p\theta(1-\theta) > 0$. This hold iff $2p[1-\theta(1-\theta)] > \theta$. But $2p > 1$, so a further sufficient condition is that $1 - \theta(1-\theta) > \theta$, or $(\theta-1)^2 > 0$, which is clearly true.

Our final task is to show that $\phi_T \in [0, \frac{1}{2})$ gives a larger number of expected endorsements than $\phi_E \in (\frac{1}{2}, 1]$. From (4.3), $\Pi\left[\phi \in (\frac{1}{2}, 1]\right] = \left\{p^2 + 2p(1-p)\phi\right\}\pi_{q_1^*\in(\frac{1}{2},p)}$. Now, $\frac{d\left(\Pi\left[\phi\in(\frac{1}{2},1]\right]\right)}{d\phi} > 0$ as $\Pr[P] =$

$\left\{p^2+2p(1-p)\phi\right\}$ is strictly increasing in ϕ and from (3.6), $\pi_{q_1^*\in(\frac{1}{2},p)}$ is constant in ϕ. Thus, we simply need to show that $\Pi\left[\phi=0\right]>\Pi\left[\phi=1\right]$. Using (3.6):

$$\Pi\left[\phi=1\right]=\left[p^2+2p(1-p)\right]\frac{p[1+(1-p)\theta(1-\theta)]}{\left[1-p(1-p)\theta^2\right](1-\theta)} \tag{A.3}$$

Using (A.1) and (A.3), we can derive

$$\Pi\left[\phi=0\right]-\Pi\left[\phi=1\right]=\frac{p(1-p)(1-\theta)[1-2p(1-p)\theta]}{\left[1-p(1-p)\theta^2\right](1-\theta)}$$

The denominator is strictly positive, as is $p(1-p)(1-\theta)$, so the sign of $\Pi\left[\phi=0\right]-\Pi\left[\phi=1\right]$ and $1-2p(1-p)\theta$ must be the same. Now, since $p>\frac{1}{2}$, it must be that $p(1-p)<\frac{1}{4}$, so $2p(1-p)\theta$ must always remain smaller than a half. Thus, $1-2p(1-p)\theta>0$. $\square$

Proof of Proposition 3. With no test, the first agent's q_1^* equals the prior belief $\frac{1}{2}$. Therefore, $\Pi\left[\text{No Test}\right]=\pi_{q_1^*=\frac{1}{2}}$. Thus, using (3.3) and (A.2), $\Pi\left[\phi=\frac{1}{2}\right]>\Pi\left[\text{No Test}\right]$ iff:

$$2+2p(1-p)\theta(1-\theta)-(1-p)\theta>2-(1-p)\theta^2\Leftrightarrow 2p(1-\theta)-1>-\theta\Leftrightarrow(2p-1)(1-\theta)>0$$

which holds. $\square$

References

Albano, G.L., Lizzeri, A., 2001. Strategic Certification and Provision of Quality. Int. Econ. Rev. 42, 267-283.

Banerjee, A.V., 1992. A Simple Model of Herd Behavior. Quart. J. Econ. 107, 797-817.

Bikhchandani, S., Hirshleifer, D., Welch. I., 1992. A Theory of Fads, Fashion, Custom and Cultural Change as Informational Cascades. J. Polit. Economy 100, 992-1026.

Calvert, R. L., 1985. The Value of Biased Information: A Rational Choice Model of Political Advice. Journal of Politics 47, 530-555.

Chamley, C.P., 2004. Rational Herds: Economic Models of Social Learning. Cambridge University Press.

Gill, D., Sgroi, D., 2004. The Superiority of Biased Reviewers in a Model of Simultaneous Sales. Department of Economics Discussion Paper 206, University of Oxford.

Lerner, J., Tirole, J., 2004. A Model of Forum Shopping, with Special Reference to Standard Setting Organizations. Harvard NOM Research Paper No. 04-31.

Ottaviani, M., 1999. Monopoly Pricing and Social Learning. Mimeo, University College London.

Ottaviani, M., Pratt, A., 2001. The Value of Public Information in Monopoly. Econometrica 69, 1673-1683.

Sgroi, D., 2002. Optimizing Information in the Herd: Guinea Pigs, Profits and Welfare. Games Econ. Behav. 39, 137-166.

Smith, L., Sorensen, P., 2000. Pathological Outcomes of Observational Learning. Econometrica 68, 371-398.

Taylor, C.R., 1999. Time-on-the-Market as a Sign of Quality. Rev. Econ. Stud. 66, 555-578.