Coordination on saddle path solutions : the eductive viewpoint.
2 - linear multivariate models.

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PRELIMINARY AND INCOMPLETE

1 Introduction

This paper comes as a continuation of a previous paper entitled : "Coordination on saddle path solutions : the eductive viewpoint. Linear univariate models", by G. Evans and R. Guesnerie (1999). The purpose of both papers is to revisit the justifications of the saddle path stable solution by taking the somewhat more basic perspective of "eductive learning," which refers to considerations that have a game-theoretical flavour and explicitly refer to Common Knowledge considerations.

Specifically, the viewpoint we take, the "Strong Rationality viewpoint"[1], proceeds as follows. We start from restrictions on the possible paths of the system, which themselves reflect restrictions on individual strategies. These restrictions, tentatively supposed to be Common Knowledge (from now on CK) trigger a mental process, which, when rationality is itself "commonly known," mimicks the process of determination of rationalizable strategies (from the initial set of restricted strategies). When such a process converges to the candidate equilibrium, the equilibrium is said to be *Strongly Rational.* Actually, as in the following, the CK initial restrictions will always be taken

[1] This could be called the "local unique rationalizability" viewpoint, in the terminology of Bernheim (1984) or Pearce (1984), or the "local dominance solvability"viewpoint in the terminology of Farqhason (1969) and Moulin (1979).

locally, so that we shall only be concerned with a weaker variant of the test that selects *Locally* Strongly Rational Equilibria. The question treated in this paper, as well as in the companion paper, can then be more compactly reformulated: when is it the case that the *saddle path stable solution* of a dynamical system *is* a good candidate for expectational coordination, in the sense just introduced of being *Locally Strongly Rational*, for restrictions to be made precise?

At this stage, two different points are in order.

First, it is useful to stress the relevance of the question: economic modelling routinely assumes that saddle-path like stable solutions, or stable manifold solutions provide the appropriate "rational expectations solutions" even when they compete with many others. Convenience pleads in favour of such a practice but it is not as such a fully convincing intellectual argument. Determinacy considerations, which point out that such solutions are "locally isolated" rational expectations equilibria, even when the broader concept of sunspot equilibria is envisaged, provide better intellectual arguments for serious foundations but do not exhaust the question. The present approach proposes an alternative, and in our view more basic, view on the problem.

As the reader will easily guess, our methodology to approach the question, as briefly sketched above, is almost meaningless if we refer to standard reduced forms of dynamical systems. In order to make sense of the question we raise concerning Common Knowledge, we must, as we did in Evans-Guesnerie (1993) in a different context, imbed the model in a framework where agents and their strategies are well defined. This is indeed what we did in the previous paper and we repeat this set-up here in next Section.

2 The framework.

2.1 Dynamic expectations models

We are interested in models of the following kind:

$$Q(y_{t-1}, y_t, y^e_{t+1}) = O,$$

where t is a time index, y is a finite dimensional vector, and Q is a temporary equilibrium map that relates y_t to its lagged values and to expectations. The quantity y^e_{t+1} denotes the expectation of y_{t+1} formed by agents at time t. In this formulation we assume that agents are able to observe y_t when forming their expectations or, if not, that they can condition their actions on the values y_t that are realized.

We need to be more precise on the strategic aspects of the coordination problem. To do so we will adopt a very simple strategic interpretation of the model which makes explicit the decision theoretic aspects of the model and the aggregation of these decisions into a temporary equilibrium map.

2.2 Strategic expectations model.

2.2.1 The basic structure

We now embed the dynamic model in a dynamic game, along lines that are somewhat similar to those of Evans-Guesnerie (1993). We assume that, at each period t, there exists a continuum of agents, a part of whose strategies are not reactive to expectations (in an OLG context, these are the agents, who are at the last period of their lives), and a part of which "react to expectations". The latter agents are denoted ω_t and belong to a convex segment of R, endowed with Lebesgue measure $d\omega_t$. It is assumed that an agent of period t is different from any other agent of period $t', t' \neq t$. [2] More precisely, agent ω_t has a (possibly indirect) utility function that depends upon

1) his own strategy $s(\omega_t)$,

2) sufficient statistics of the strategies played by others i.e. on $y_t = F(\Pi_{\omega_t} \{s(\omega_t)\}, *)$, where F in turn depends first, upon the strategies of all agents who at time t react to expectations, and second, upon $(*)$, which is here supposed to be sufficient statistics of the strategies played by those who do not react to expectations, and that includes but is not necessarily identified with – see below – y_{t-1},

3) finally upon the sufficient statistics for time $t+1$, as perceived at time t: i.e. on $y_{t+1}(\omega_t)$, which *may be random* and, now directly, upon the sufficient statistics y_{t-1}.

We assume that the strategies played at time t can be made conditional on the equilibrium value of the of the t sufficient statistics y_t. Now, let $(\bullet)$ denotes both (the product of) $y_{t-1}$and the probability distribution of the random variable $y_{t+1}(\omega_t)$, (the random expectation held by ω_t of y_{t+1}). Let

[2] This means either that each agent is "physically" different or that the agents have strategies that are independent from period to period. In an OLG interpretation of the model, each agent lives for two periods but only reacts to expectations in the first period of his life.

then $G(\omega_t, y_t, \bullet)$ be the best response function of agent ω_t. Under these assumptions, the sufficient statistics for the strategies of agents who do not react to expectations is $(*) = (y_{t-1}, y_t)$.

The equilibrium equations at time t are written:

$$y_t = F\left[\Pi_{\omega_t}\left\{G(\omega_t, y_t, y_{t-1}, \tilde{y}_{t+1}(\omega_t))\right\}, y_{t-1}, y_t\right]. \tag{1}$$

Note that when all agents have the same point expectations denoted y^e_{t+1}, the equilibrium equations determine what we called earlier the temporary equilibrium mapping

$$Q(y_{t-1}, y_t, y^e_{t+1}) = y_t - F\left[\Pi_{\omega_t}\left\{G(\omega_t, y_t, y_{t-1}, y^e_{t+1})\right\}, y_{t-1}, y_t\right].$$

2.2.2 Linearization

The right hand side of (1) is a rather complex term, but under regularity assumptions[3], it has, through two different channels, derivatives with respect to y_t, and with respect to y_{t-1}. Also assuming that all $\tilde{y}_{t+1}$ have a very small common support "around" some given y^e_{t+1}, decision theory suggests that G, to the first order, depends on the expectation[4] of the random variable $\tilde{y}_{t+1}(\omega_t)$ that is denoted $y^e_{t+1}(\omega_t)$ (and is close to y^e_{t+1})

Taking into account the previous remark, the heterogeneity of expectations across agents, and assuming again the existence of adequate derivatives, it is reasonable to linearize, around any initially given situation, denoted (0), the above expression as follows[5]:

$$y_t - U(0)y_t + V(0)y_{t-1} + \int W(0, \omega_t) y^e_{t+1}(\omega_t) d\omega_t,$$

where $y_t, y_{t-1}, y^e_{t+1}(\omega_t)$ now denote small deviations from the initial values of y_t, y_{t-1}, y^e_{t+1}, and $U(0), V(0), W(0, \omega_t)$ are $n \times n$ square matrices.

Such a linearizarion is valid everywhere, but we will consider it only around a steady state of the system. Hereafter, y_t, y_{t-1}, etc denote deviations from the steady state and $U(0), V(0), W(0, \omega_t)$ are simply $U, V, W(\omega_t)$.

[3] For a less sketchy discussion, see Evans-Guesnerie (1993), p.637.

[4] This could be formalized along lines similar to those taken in Chiappori-Guesnerie (1989), who also argue that the property is general in economic models that adopt the Bayesian view of uncertainty.

[5] As in Guesnerie (1999), this can be viewed as an "axiom", whose field of validity is very large.

Supposing $I - U$ is invertible, we have :

$$y_t = ((I - U)^{-1}V)y_{t-1} + (I - U)^{-1} \int W(\omega_t) y^e_{t+1}(\omega_t) d\omega_t.$$

When expectations are homogenous, $y^e_{t+1}(\omega_t) = y^e_{t+1}$, the system becomes

$$y_t = By^e_{t+1} + Dy_{t-1}, \text{ with } B = (I - U)^{-1}W, \text{ where } W = \int W(\omega_t)d\omega_t, \quad (2)$$

and when y is one-dimensional, using the corresponding small Greek letters, we write the system as $y_t = \beta y^e_{t+1} + \delta y_{t-1}$.

With the new notation, assuming W invertible, the initial system can also be written

$$y_t = Dy_{t-1} + BW^{-1} \int W(\omega_t) y^e_{t+1}(\omega_t) d\omega_t.$$

Putting $Z(\omega_t) = W^{-1}W(\omega_t)$, we rewrite this as

$$y_t = Dy_{t-1} + B \int Z(\omega_t) y^e_{t+1}(\omega_t) d\omega_t, \quad (3)$$

where $\int Z(\omega_t)d\omega_t = I$. This will be the basic equation of our study. We assume that (3) holds for $t = 1, 2, 3, \ldots$, and that initial conditions y_0 are given.

3 Perfect Foresight.

3.1 Perfect Foresight Paths.

A Perfect Foresight path, is a sequence of n-dimensional vectors, y_t, $t = 1, ... + \infty$, starting from y_0, and such that :

$$y_t = By_{t+1} + Dy_{t-1}. \quad (4)$$

We remind the reader briefly of the standard methodology of the study of such paths. We assume thoughout that B is nonsingular.

Defining $X(t) = \left\{ \begin{array}{c} y_t \\ y_{t-1} \end{array} \right\}$, we write

$$X(t+1) = \Phi X(t),$$

where Φ is the $2n \times 2n$ matrix

$$\Phi = \begin{bmatrix} B^{-1} & -B^{-1}D \\ I & O \end{bmatrix}.$$

Such a matrix has $2n$ eigenvalues, λ_i, $i = 1, ...2n$, associated with eigenvectors of the form $\left\{ \begin{array}{c} \lambda_i x_i \\ x_i \end{array} \right\}$.

We always assume here that the matrix has distinct eigenvalues, so that the matrix is semi-simple and diagonalizable. We rank all the eigenvalues in the order of increasing modulus, such that $i \leq j, \Leftrightarrow |\lambda_i| \leq |\lambda_j|$. Calling P, the matrix whose columns are the coordinates of the eigenvectors in the canonical basis of R^{2n}, we have

$$\Phi = P\Lambda P^{-1},$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{2n})$ is the diagonal matrix with the eigenvalues on the diagonal[6], so that

$$X(t+1) = P\Lambda^t P^{-1} X(1).$$

Partitioning P, and calling $P^{-1}X(1) = \left\{ \begin{array}{c} I_1 \\ I_2 \end{array} \right\}$, the dynamics of the system can be written, with straightforward notation,

$$\left\{ \begin{array}{c} y_{t+1} \\ y_t \end{array} \right\} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} \Lambda_1^t & 0 \\ 0 & \Lambda_2^t \end{pmatrix} \left\{ \begin{array}{c} I_1 \\ I_2 \end{array} \right\}.$$

Here the submatrices P_{ij} and Λ_i are $n \times n$ and note that $P_{11} = P_{21}\Lambda_1$, and $P_{12} = P_{22}\Lambda_2$. It follows, in particular, that

$$y_t = P_{21}\Lambda_1^t I_1 + P_{22}\Lambda_2^t I_2. \tag{5}$$

We assume that P_{21} is nonsingular.

Let X^S denote the (n dimensional) subspace of R^{2n} generated by the eigenvectors associated with the n eigenvalues of smallest modulus, $\lambda_1, ...\lambda_n$.

[6]If Φ has nonreal eigenvalues then the real canonical form can be used. For each pair of complex eigenvalues $a \pm ib$, with eigenvectors $u \pm iv$, the corresponding diagonal entries in Λ are replaced by the 2×2 block $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, and the corresponding columns in P are replaced by v and u.

X^S is a solution subspace, i.e. $X(t-1) \in X^S$ and $X(t) = \Phi X(t-1)$ implies $X(t) \in X^S$. A vector $X(t) = (y_t', y_{t-1}')'$ belongs to X^S if and only if, in the basis of eigenvectors, it can be written as $\left\{ \begin{array}{c} \eta \\ 0 \end{array} \right\}$, i.e. in the canonical basis it is of the form $\left\{ \begin{array}{c} P_{11}\eta \\ P_{21}\eta \end{array} \right\}$. Hence $X(t) \in X^S$ if and only if $y_t = P_{11}(P_{21})^{-1}y_{t-1}$, i.e.

$$y_t = Sy_{t-1}, \text{ where } S = P_{21}\Lambda_1(P_{21})^{-1}. \tag{6}$$

This solution corresponds to (5) with $I_2 = 0$, i.e. $y_t = P_{21}\Lambda_1^t I_1$.

It follows that given any $y_0 \in R^n$ there exists a unique solution to (4) such that $X(t)$ lies in X^S for all t, and this solution takes the form (6). This result, of course, requires the assumption that P_{21} is nonsingular. If P_{21} is singular then the solution space X^S has dimension less than n. In this case solutions in X^S still take the form $y_t = P_{21}\Lambda_1^t I_1$, but such solutions only exist for initial y_0 which lie in the column space of P_{21}. Throughout the remainder of the paper we make the assumption that P_{21} has full rank.

In the following, when we will assume that the n eigenvalues of smallest modulus, $\lambda_1, \ldots, \lambda_n$, have modulus smaller than one, and that the others have modulus larger than one, the above equations are the equations of the stable subspace of the system. (We refer to it later as S_*). In this case, called the "saddlepoint stable case," for every initial y_0 there exists a unique "nonexplosive" solution and it takes the form (6). This follows directly from the foregoing discussion. For example, any other solution satisfies (5) with $P_{22}\Lambda_2^t I_2 \neq 0$, which implies that at least one component of y_t tends to $\pm\infty$ as $t \to \infty$.

3.2 Perfect Foresight in "Extended Growth Rates".

Suppose that

$$y_t = S_t y_{t-1}$$

for some given $n \times n$ invertible matrix S_t. Then the perfect foresight "follower" of y_t is the y_{t+1} that satisfies, assuming B invertible,

$$y_{t+1} = B^{-1}(I - DS_{t-1}^{-1})y_t.$$

Writing $y_{t+1} = S_{t+1}y_t$ it follows that

$$S_{t+1} = B^{-1}(I - DS_t^{-1}). \tag{7}$$

Choosing S_1 arbitrarily, (7) generates an infinite sequence of matrices, S_t, $t = 1, 2, 3, \ldots$ with the property that for arbitrary y_0 the sequence $y_1 = S_1 y_0, \ldots, y_t = S_t y_{t-1}, \ldots$ is a perfect foresight path.

Following Gauthier (2000), one may call the sequence S_t a sequence of "extended growth rates,"[7] or an "EGR sequence." A limit point S of a sequence of extended growth rates must satisfy

$$S = B^{-1}(I - DS^{-1}), \tag{8}$$

which can be rewriten as the matrix quadratic equation

$$S^2 - B^{-1}S + B^{-1}D = O.$$

Consider a subset of n vectors of R^n, $\left(\;.\; x_i, \;\; . \;\; x_{j,} \;.\right)$ where the n vectors $x_i, x_j, \ldots$ are the n-dimensional restriction of any subset K of the $2n$ eigenvectors of Φ, (which are of the form $\left\{ \begin{array}{c} \lambda_i x_i \\ x_i \end{array} \right\}$). Consider the case where the n vectors under consideration form a basis of R^n. A matrix, that will be denoted S_K, which transforms x_i to $\lambda_i x_i, \forall i \in K$, is a fixed point of (8). Indeed, in the basis consisting of the $x_i, i \in K$,a vector $\left\{ \begin{array}{c} y_t \\ y_{t-1} \end{array} \right\} = \left\{ \begin{array}{c} \Lambda\alpha \\ \alpha \end{array} \right\}$ is transformed into $y_{t+1} = \Lambda^2\alpha$, so that $y_{t+1} = \Lambda y_t$.

In the canonical basis of R^n,

$$S_K = P_K \Lambda_K P_K^{-1},$$

where we have now factored $\Phi = P\Lambda P^{-1}$ as

$$P = \begin{pmatrix} P_K\Lambda_K & P_L\Lambda_L \\ P_K & P_L \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} \Lambda_K & 0 \\ 0 & \Lambda_L \end{pmatrix}$$

The solution of the previous section, $S_* = P_{21}\Lambda_1 P_{21}^{-1}$, corresponds to the choice $\Lambda_K = \Lambda_1$. We remark that there can be up to C_n^{2n} distinct solutions S_K, with exactly C_n^{2n} such solutions when all roots are real and all subsets of n vectors $\left(\;.\; x_i, \;\; . \;\; x_{j,} \;.\right)$ yield linearly independent sets.

[7] cf the one dimensional case for the terminology

It is straightforward to verify algebraically that, in the canonical basis of R^n, $S_K = P_K \Lambda_K P_K^{-1}$ satisfies the fixed point equation (8). This follows immediately from partitioned matrix multiplication of the equation $\Phi P = P\Lambda$, using the above partition. Therefore $y_t = S_K y_{t-1}$ is a solution for any initial condition y_0. The converse can also be shown, i.e. every S that provides a solution of the form $y_t = S y_{t-1}$ for every initial condition y_0 can be expressed as $P_K \Lambda_K P_K^{-1}$. To see this note that $X(t+1) = P\Lambda^t P^{-1} X(1) = P\Lambda^t I$ can be written

$$\begin{pmatrix} y_{t+1} \\ y_t \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{2n} I(i)\lambda_i^{t+1} x_i \\ \sum_{i=1}^{2n} I(i)\lambda_i^{t} x_i \end{pmatrix},$$

Then $y_{t+1} = S y_t$ for all t implies $\sum_{i=1}^{2n} I(i)\lambda_i^t (S x_i - \lambda_i x_i) = 0$ for all t, which implies

$$I(i)(S x_i - \lambda_i x_i) = 0 \text{ for } i = 1, \ldots, 2n.$$

and the initial conditions imply that

$$y_0 = \sum_{i=1}^{2n} I(i) x_i.$$

Let K be the set of indices such that $S x_i = \lambda_i x_i$ for $i \in K$, so that $I(i) = 0$ for $i \notin K$. If K has fewer than n elements then the initial conditions cannot be satified for all choices y_0. Since S is $n \times n$ there must be exactly n elements in K and to meet the initial conditions for all possible y_0 the columns of $P_K = (x_t)_{i \in K}$ must be linearly independent. Thus $SP_K = P_K \Lambda_K$, and the result follows.

Using also the last result of the previous section, we have demonstrated the following

Proposition 1 *Solutions to the perfect foresight equation (4) taking the form $y_t = \bar{S} y_{t-1}$, and holding for all arbitrary initial conditions y_0, satisfy $\bar{S} = P_K \Lambda_K P_K^{-1}$, where P_K is nonsingular. In the saddlepoint case $|\lambda_n| < 1 < |\lambda_{n+1}|$ there exists a unique solution which is nonexplosive and it is given by $y_t = S_* y_{t-1}$, with $S_* = P_{21} \Lambda_1 P_{21}^{-1}$.*

Now, given a fixed point $\bar{S}$ of the "extended growth rates", one says that it is *determinate* if and only if there is no sequence S_t, starting from any $S_0 \neq \bar{S}$, that converges locally to $\overline{S}$. This is the standard definition except that we here apply it to extended growth rates rather than to the state variable y_t.

A fixed point $\bar{S}$ is determinate under (7) if the linear mapping tangent at $\bar{S}$ to the mapping $S \rightarrow B^{-1}(I - DS^{-1})$ is a source, i.e. has all roots outside the unit circle. We next turn to the connection between determinateness and learning for the saddlepoint case.

4 Eductive Learning.

4.1 Iterative Expectational Stability.

In developing the Strong Rationality conditions we will examine their connection to the conditions for Iterative Expectational Stability (or IE-Stability). IE-Stability can be viewed as a process, taking place in virtual or notional time τ, that works as follows (see, for example, Evans (1985)). Economic agents posit a conjectured law of motion in which y_t, for all t, evolves at an arbitrary fixed extended growth rate S_τ. From this perceived law of motion S_τ, which we suppose is homogeneous across agents, one can obtain the actual law of motion and show that the actual dynamics follow an extended growth rate path, though in general at a rate $T(S_\tau)$ different from S_τ. IE-stability then considers the iterative revision, in notional time τ, given by $S_{\tau+1} = T(S_\tau)$. If this sequence converges to a fixed point $\bar{S} = T(\bar{S})$, from all initial S in a neighborhood of $\bar{S}$, then we say that $\bar{S}$ is locally IE-stable (or LIE-stable). The sequence $S_{\tau+1} = T(S_\tau)$ can be thought of as a stylized notional time learning rule in which forecasted growth rates are updated to the growth rates implied by those forecasts.

Concretely, for the case at hand, consider the conjectured law of motion

$$y^e_{t+1} = S_\tau y_t, \forall t.$$

Inserting these (homogeneous) expectations into the model (2) the actual dynamics between $t-1$ and t will be $y_t = Dy_{t-1} + BS_\tau y_t$, so that

$$y_t = (I - BS_\tau)^{-1} D y_{t-1}, \forall t,$$

provided $I - BS_\tau$ is invertible. Thus constant expected extended growth rates S_τ would lead to constant actual extended growth rates $T(S_\tau) = (I - BS_\tau)^{-1}D$. The IE-Stability dynamics (or "IE-learning process") are therefore given by

$$S_{\tau+1} = (I - BS_\tau)^{-1} D. \tag{9}$$

Fixed points of the IE learning process satisfy $S = (I - BS)^{-1}D$, which is equivalent to the fixed points of the extended growth rate dynamics (7) $S = B^{-1}(I - DS^{-1})$, provided S is invertible. Indeed, provided $S_{\tau+1}$ is invertible, (9) can be equivalently written as

$$S_\tau = B^{-1}(I - DS_{\tau+1}^{-1}),$$

which has the same form as (7). From the preceding section we know that the fixed points are given by $S_K = P_K \Lambda_K P_K^{-1}$.

From inspection of these equations it is clear that if, starting from any S_0 close enough to a fixed point $\bar{S}$, the sequence of the IE-learning process converges to $\bar{S}$, then any perfect foresight EGR sequence, starting from the same neighbourhood, diverges locally from $\bar{S}$. ($\bar{S}$ is a sink for the mapping (9) if and only if it is a source for the inverse mapping given by (7)). It follows that an Equilibrium EGR is determinate in the extended growth rate dynamics if and only if it is LIE-Stable.

This is the form taken here by what Gauthier calls the *equivalence principle,* a property of broader scope than the present model (see Gauthier S. (1999)).

We can then prove the following :

Proposition 2 *Assume that we are in the saddle point case :* $|\lambda_n| < 1 < |\lambda_{n+1}|$ *and consider the equilibrium EGR associated with the stable subspace (here the stable saddle point solution), given by* $S_* = P_{21}\Lambda_1 P_{21}^{-1}$.

Then this solution is determinate in EGR dynamics and is LIE-Stable.

Proof. From the equivalence principle above, it is enough to show that the equilibrium EGR is locally determinate i.e that the dynamics of the perfect foresight EGR is locally divergent.

Assume the contrary. Then in every neighborhood of S_* there exists initial $S_2 \neq S_*$ such that $S_t \to S_*$ under the perfect foresight extended growth rate dynamics. Then take $y_0 \neq 0$ and $y_1 \neq S_* y_0$, so that $(y_1', y_0')'$ does not belong to the stable subspace. The sequence $y_2 = S_2 y_1, \ldots, y_t = \left(\prod_{i=2}^t S_i\right) y_1, \ldots$ is a perfect foresight sequence in states y_t. But $\prod_{i=2}^t S_i \to 0$ as $t \to \infty$, since $S_t \to S_*$ and all eigenvalues of S_* are smaller than one in modulus. Hence $y_t \to 0$. This is a contradiction since we know that there does not exist a perfect foresight sequence in states that starts outside the stable subspace and converges to zero. *Q.E.D.*

Remark: The result is equivalent to the fact that all eigenvalues of the linear mapping tangent (at S_*) to the mapping $S \to (I - BS)^{-1}D$ are smaller that one in modulus. This can also be shown algebraically[8], but the above proof is more direct.

4.2 Strong Rationality.

We now develop the eductive learning argument and the criterion, called in Guesnerie (1992) Strong Rationality, under which eductive learning will lead to coordination on an equilibrium path. We therefore return to the "strategic reduced form" of the model (3) developed in Section 2.2.2 and reproduced here for convenience:

$$y_t = Dy_{t-1} + B \int Z(\omega_t) y^e_{t+1}(\omega_t) d\omega_t.$$

We apply the SR test to an equilibrium EGR, i.e to an $\bar{S}$ that satisfies $\bar{S} = (I - B\bar{S})^{-1}D$. Along the lines of the general procedure recalled in Section 1, the initial CK (Common Knowledge) restriction will be stated as follows:

CK Assumption: *It is CK that, for all t, $y_t = S_t y_{t-1}$ with $S_t \in V(\bar{S})$, where V is a small (enough) neighbourhood of $\bar{S}$.*

Local Strong Rationality (LSR) will obtain if and only if the initial CK restriction triggers the conclusion : It is CK that, for all t, $y_t = \bar{S} y_{t-1}$.

We develop the argument as follows.

(i) We first note that the subjective expected value $y^e_{t+1}(\omega_t)$ of agent ω_t can be viewed as $S(\omega_t)y_t$ where $S(\omega_t)$ is the "expected" matrix S that agent ω_t's subjective probability distribution on $V(\bar{S})$ generates. Then, for this state of beliefs the value of y_t is given by

$$y_t = Dy_{t-1} + \left(B \int Z(\omega_t) S(\omega_t) d\omega_t \right) y_t.$$

This is our basic relationship, which we rewrite :

$$y_t = \left(I - B \int Z(\omega_t) S(\omega_t) d\omega_t \right)^{-1} Dy_{t-1}$$

[8] This point was brought to our attention by S. Gauthier, who has provided us with such a proof in a private correspondence.

(ii) Without loss of generality, we conduct the argument by taking a neighbourhood $V(\bar{S})$ of the form:

$$\left\|S-\bar{S}\right\| \leq \epsilon,$$

where $\|.\|$ is a vector norm (to be defined later) on R^{n^2}, which is identified with the space of $n \times n$ matrices.

(iii) Now consider the map

$$\prod_{\omega_t}(\Delta(S(\omega_t))) \rightarrow \Delta(\left[I - B\int Z(\omega_t)S(\omega_t)d\omega_t\right]^{-1}D),$$

where Δ denotes deviations from $\bar{S}$, e.g. $\Delta(S(\omega_t)) = S(\omega_t) - \bar{S}$. The linear approximation to this map, which we call Γ, can be written

$$\prod_{\omega_t}(\Delta(S(\omega_t))) \rightarrow \int L(\omega_t)\Delta(S(\omega_t))d\omega_t$$

where $L(\omega_t)$ is a linear mapping from R^{n^2} into R^{n^2}, given explicitly later below.

Hence $\left\|\Delta([I - B\int Z(\omega_t)S(\omega_t)d\omega_t]^{-1}D)\right\|$ is approximately equal to

$$\left\|\int L(\omega_t)\Delta(S(\omega_t))d\omega_t\right\| \leq \int \|L(\omega_t)\Delta(S(\omega_t)\|\, d\omega_t.$$

Let $\rho(\omega_t) = \|L(\omega_t)\|$ be the norm induced[9] (on linear mappings from $R^{n^2} to$ R^{n^2}) by the initial vector norm (on R^{n^2}). Then

$$\|\Delta(S(\omega_t))\| = \left\|(S(\omega_t)) - \bar{S}\right\| \leq \epsilon \ \forall\omega_t$$

implies that to a first approximation

$$\left\|\Delta(\left[I - B\int Z(\omega_t)S(\omega_t)d\omega_t\right]^{-1}D)\right\| \leq (\int \rho(\omega_t)d\omega_t)\epsilon.$$

We can also define $\|\Gamma\|$ as the norm of the linear mapping $\Gamma\left(\Pi_{\omega_t}\left(\Delta(S(\omega_t))\right)\right)$ induced by the norm on $\Pi_{\omega_t}\left(\Delta(S(\omega_t))\right)$, which we take to be $\sup_{\omega_t}\|\Delta(S(\omega_t))\|$. Then

$$\|\Gamma\| \leq \int \rho(\omega_t)d\omega_t.$$

[9]For the theory of matrix norms, see Horn and Johnson (1985). We shall refer here only to induced matrix norms.

Given a vector norm $|.|$, the matrix norm $\|.\|$ induced by the vector norm is defined as follows :$\|A\| = \max_{|x|=1}|Ax|$

(iv) Consider now the homogenous expectations case and call γ the linear map tangent, at $\bar{S}$, to the map (from R^{n^2} to R^{n^2})

$$S \rightarrow (I - BS)^{-1} D.$$

Note, that, since $\int Z(\omega_t) d\omega_t = I$ whenever $(S(\omega_t) - \bar{S}) = \Delta S$ independently of ω_t, the map Γ, acting on $\Pi_{\omega_t}(\Delta S\ )$, takes the same value as the map γ acting on ΔS. Hence, it must be the case that

$$\|\gamma\| \leq \|\Gamma\|\,,$$

where $\|\gamma\|$ is the norm, that from now on we denote $n\gamma$, induced by the vector norm in R^{n^2} previously introduced.

Finally, this implies, to a first order approximation, that

$$(n\gamma)\epsilon \leq \sup_{\|\Delta(S(\omega_t)\| \leq \epsilon} \left\| \Delta(\left[I - B \int Z(\omega_t) S(\omega_t) d\omega_t \right]^{-1} D) \right\| \leq \left(\int \rho(\omega_t) d\omega_t \right) \epsilon$$

It follows that the condition $\int \rho(\omega_t) d\omega_t < 1$ is sufficient for LSR of $\bar{S}$. Given the CK assumption $\|\Delta S_t\| = \left\| S_t - \bar{S} \right\| \leq \epsilon$ the expectations of all agents ω_t must satisfy $\|\Delta(S(\omega_t)\| \leq \epsilon$ for all t. The above arguments show that this implies that $\|\Delta S_t\| \leq (\int \rho(\omega_t) d\omega_t)\, \epsilon$ for all t, which tightens the CK assumption when $\int \rho(\omega_t) d\omega_t < 1$. The above process, relating actual extended growth rates to beliefs, can then be iterated and, after n stages it is CK that $\|\Delta S(\omega_t)\| \leq (\int \rho(\omega_t) d\omega_t)^n\, \epsilon$ for all t. Since this holds for all positive integers n it follows that it is CK that $\|\Delta S(\omega_t)\| = 0$ for all t and hence that $y_t = \bar{S} y_{t-1}$ for all t.

We can translate this analysis into a formal proposition that will provide a basic reference for further reflection.

Theorem 3 *i) LSR $\Longrightarrow$ LIE-Stability*

ii) If agents are homogeneous, LSR is identical to LIE-Stability

iii) A sufficient condition for LSR is

$$\int \rho(\omega_t) d\omega_t < 1,$$

where $\rho(\omega_t)$ is the norm of $L(\omega_t)$, induced by a vector norm on R^{n^2}, and where $L(\omega_t)$ describes the approximate change on the aggregate state variable, triggered by a change in expected EGR of agent ω_t.

Proof. i) $n\gamma$ is greater than the eigenvalue of maximal modulus of γ, and $n\gamma \leq \|\Gamma\|$, which is smaller than one because of LSR. That is enough for LIE-Stability, which requires that the maximal modulus of γ be less than one.[10]

ii)Take $Z(\omega_t) = \sigma(\omega_t)I, \sigma(\omega_t) \geq 0$, and $\int \sigma(\omega_t)d\omega_t = 1$. Then Γ "coincides" with γ, and $n\gamma$ can be made arbitrarily close to the maximal modulus of γ, for an appropriate choice of norms, (See Horn and Johnson (1991)).

iii) already shown.

We remark that this proposition implies the corollary that in the saddle point case, if agents are homogeneous then the stable Equilibrium EGR (EEGR), S_*, is LSR since it is LIE-Stable by Proposition 2. In fact, as can be seen from the proof of ii), some deviations from complete homogeneity leave this result intact, while iii) shows that sufficient heterogeneity can render the saddlepoint solution no longer LSR.[11]

In the Proposition, iii) captures the idea of heterogeneity: it is powerful but abstract. In order to make it more intuitve, we have to specialize the statement by choosing some special vector norms in R^{n^2}.

First Specialization:

Consider the mapping γ, the derivative of $(I - BS)^{-1}D$, at S_*. This can be written, in matrix form[12]:

$$\gamma : \Delta S \rightarrow (I - BS_*)^{-1}B(\Delta S)(I - BS_*)^{-1}D = S_*D^{-1}B(\Delta S)S_*.$$

or, after "vectorization,"

$$\gamma : \text{vec}\,\Delta S \rightarrow (S_*^t \otimes S_*D^{-1}B)(\text{vec}\,\Delta S)$$

where $\otimes$ designates the Kronecker product, S^t is the transpose of S and vec(ΔS) denotes the vector obtained by stacking in order the columns of ΔS. We have here used the relationship $\text{vec}(ABC) = (C^t \otimes A)\,\text{vec}(B)$.

[10] The result is a priori obvious, but it is worth deriving from our inequalities.

[11] Evans and Guesnerie (1999) provide an explicit calculation of the effect of heterogeneity for the univariate case.

[12] Taking the differential of $(I - BS)^{-1}(I - BS) = I$ it follows that the differential of $(I - BS)^{-1}$ is given by $d(I - BS)^{-1} = (I - BS)^{-1}B(dS)(I - BS)^{-1}$. See Magnus and Neudecker (1988) for these and related matrix results.

Similarly the mappings $L(\omega_t)$ can be written in matrix form as[13]

$$L(\omega_t) : \Delta S \to S_* D^{-1} B Z(\omega_t)(\Delta S) S_*, \text{ or}$$

$$L(\omega_t) : \operatorname{vec} \Delta S \to (S_*^t \otimes S_* D^{-1} B)(Z(\omega_t))(\operatorname{vec} \Delta S).$$

Now, take as vector norm for R^{n^2}, the Euclidean norm in the eigenvector basis of γ.[14] For this vector norm, the induced matrix norm for $S_*^t \otimes S_* D^{-1} B$ is the modulus of its eigenvalue of highest modulus.

Now $\rho(\omega_t)$ is the norm of the matrix $(S_*^t \otimes S_* D^{-1} B)(Z(\omega_t))$ induced by the just defined Euclidean norm. It must be at least as large as $\lambda(\omega_t)$, the modulus of the eigenvalue of highest modulus of the considered matrix. Hence :

Theorem 4 *A sufficient condition for the saddle path stable EGR, S_*, to be LSR is that :*

$$\int \rho(\omega_t) d\omega_t < 1$$

where $\rho(\omega_t)$ is the norm of the matrix $S_*^t \otimes (S_* D^{-1} B)(Z(\omega_t))$ induced by the Euclidean norm of the eigenvector basis of the matrix $(S_*^t \otimes S_* D^{-1} B)$.

A necessary condition is that

$$\alpha \leq \int \lambda(\omega_t) d\omega_t < 1,$$

where α,(resp.$\lambda(\omega_t)$) is (are) the eigenvalue(s) of highest modulus of the matrix(ces) $(S_*^t \otimes S_* D^{-1} B)$, (resp. $S_*^t \otimes (S_* D^{-1} B)(Z(\omega_t))$).

Note that what has been done applies to the stable EEGR, if we are in the saddle point case of "determinacy" of the long run equilibrium zero, but also to other EEGRs, if $|\lambda_{n+1}| < 1$.

The next Theorem gives alternative sufficient conditions (the proof is similar):

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[13] The differential of the map $\left(I - B \int Z(\omega_t) S(\omega_t) d\omega_t\right)^{-1} D$ at $S(\omega_t) = \bar{S}\ \forall \omega_t$ is $d\left(I - B \int Z(\omega_t) S(\omega_t) d\omega_t\right)^{-1} D = (I - B\bar{S})^{-1} B \left(d\left(\int Z(\omega_t) S(\omega_t) d\omega_t\right)\right)(I - B\bar{S})^{-1} D$
$= \bar{S} D^{-1} B \left(d\left(\int Z(\omega_t) S(\omega_t) d\omega_t\right)\right)\bar{S} = \int \bar{S} D^{-1} B Z(\omega_t)(dS(\omega_t))(I - B\bar{S})^{-1} D d\omega_t$
$= \int L(\omega_t)(dS(\omega_t)) d\omega_t$, with $L(\omega_t)$ given as stated and $\bar{S} = S_*$.

[14] which exists, if γ is semi-simple, as we assume here.

Theorem 5 *A sufficient condition for the saddle path stable EGR, S_*, to be LSR is that:*

$$\int \rho(\omega_t)d\omega_t < 1$$

where $\rho(\omega_t)$ is the norm of the matrix $S_*^t \otimes (S_* D^{-1} B)(Z(\omega_t))$ induced by any vector norm, and in particular the standard Euclidean norm, on R^{n^2}.

In other words, alternative sufficient conditions may be obtained by alternative choices of norms. This may prove useful in applications.

4.3 Discussion

The results of this section provide a defense, from basic principles, of the saddle path solution in the "saddlepoint stable case." For the case of homogeneous agents, or if the degree of structural heterogeneity is not too large, we have shown that the saddlepath solution is always LSR, so that a process of eductive reasoning, begining with a local CK restriction leads ineluctably to coordination on this solution. Proposition 3 and Theorem 4 show that when there is sufficient heterogeneity, the saddlepoint solution will no longer invariably be LSR and we provide both convenient sufficient conditions and convenient necessary conditions for LSR of the saddlepoint solution. Thus these results highlight the importance of heterogeneity, which might be overlooked in "representative agent" or reduced form models.

Although the class of models we consider, multivariate one-step ahead one-step memory multivariate models, is quite general, it is of course not fully general.[15] In particular, altering the information assumptions so that not all time t information is available, when time t decisions are taken, can lead to more restrictive LSR conditions that may not be met in the saddle point case, even with homogeneous agents, as our earlier work has shown (Guesnerie (1992), Evans and Guesnerie (1993,1999)). It is therefore striking that for the framework at hand we obtain such strong results.

5 Application

6 Conclusions

[15] An important generalization will be to consider the model (3) with B singular.

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