

Technical notes to Walsh (2003), Section 5.3.2

"Monetary Economics: Macro Aspects", Spring 2004
University of Copenhagen

Henrik Jensen
March 8, 2004

Abstract

This note presents some derivations of expressions in Walsh (2003), Section 5.3.1.

1 Derivation of the expression for P_t following equation (5.22) on p. 218

Profits are zero if revenues equals total costs, i.e., if

$$P_t Y_t = \int_0^1 P_t(i) Y_t(i) di.$$

Then replace $Y_t(i)$ with the demand function, (5.22) to get

$$P_t Y_t = \int_0^1 P_t(i) \left[\frac{P_t}{P_t(i)} \right]^{\frac{1}{1-q}} Y_t di.$$

We immediately see that Y_t drops out such that

$$P_t = \int_0^1 P_t(i) \left[\frac{P_t}{P_t(i)} \right]^{\frac{1}{1-q}} di.$$

The condition is then rewritten as

$$P_t^{1-\frac{1}{1-q}} = \int_0^1 P_t(i)^{1-\frac{1}{1-q}} di,$$

$$P_t^{\frac{q}{q-1}} = \int_0^1 P_t(i)^{\frac{q}{q-1}} di,$$

and finally

$$P_t = \left[\int_0^1 P_t(i)^{\frac{q}{q-1}} di \right]^{\frac{q-1}{q}},$$

which is the expression on p. 218.

2 Derivation of expression (5.30)

The expression is an approximation of (5.29) where discounting is ignored (i.e., $R_t = 1$). One therefore has to take a first-order Taylor approximation of the expression

$$\bar{P}_t = \frac{\mathbb{E}_t [P_t^\theta V_t Y_t + P_{t+1}^\theta V_{t+1} Y_{t+1}]}{q \mathbb{E}_t \left[P_t^{\frac{1}{1-q}} Y_t + P_{t+1}^{\frac{1}{1-q}} Y_{t+1} \right]},$$

around the steady state (where variables are constants; i.e., the price level is constant and thus there is assumed no inflation). The left-hand side needs no approximation. Now, let an undated variable indicate its steady-state value. A first-order Taylor expansion of the right-hand side then yields:

$$\begin{aligned} & \frac{\mathbb{E}_t [P_t^\theta V_t Y_t + P_{t+1}^\theta V_{t+1} Y_{t+1}]}{q \mathbb{E}_t \left[P_t^{\frac{1}{1-q}} Y_t + P_{t+1}^{\frac{1}{1-q}} Y_{t+1} \right]} \\ \approx & \frac{2P^\theta V Y}{q 2P^{\frac{1}{1-q}} Y} + \frac{\theta P^{\theta-1} V Y \left(2q P^{\frac{1}{1-q}} Y \right) - q^{\frac{1}{1-q}} P^{\frac{1}{1-q}-1} Y \left(2P^\theta V Y \right)}{\left(q 2P^{\frac{1}{1-q}} Y \right)^2} (\bar{P}_t - P) \\ & + \frac{\theta P^{\theta-1} V Y \left(2q P^{\frac{1}{1-q}} Y \right) - q^{\frac{1}{1-q}} P^{\frac{1}{1-q}-1} Y \left(2P^\theta V Y \right)}{\left(q 2P^{\frac{1}{1-q}} Y \right)^2} \mathbb{E}_t (\bar{P}_{t+1} - P) \\ & + \frac{P^\theta Y}{q \left(2P^{\frac{1}{1-q}} Y \right)} (V_t - V) + \frac{P^\theta Y}{q \left(2P^{\frac{1}{1-q}} Y \right)} \mathbb{E}_t (V_{t+1} - V) \\ & + \frac{P^\theta V \left(2q P^{\frac{1}{1-q}} Y \right) - q P^{\frac{1}{1-q}} \left(2P^\theta V Y \right)}{q \left(2P^{\frac{1}{1-q}} Y \right)^2} (Y_t - Y) \\ & + \frac{P^\theta V \left(2q P^{\frac{1}{1-q}} Y \right) - q P^{\frac{1}{1-q}} \left(2P^\theta V Y \right)}{q \left(2P^{\frac{1}{1-q}} Y \right)^2} \mathbb{E}_t (Y_{t+1} - Y). \end{aligned}$$

This expression is reduced to

$$\begin{aligned}
& \frac{\mathbf{E}_t [P_t^\theta V_t Y_t + P_{t+1}^\theta V_{t+1} Y_{t+1}]}{\mathbf{E}_t \left[P_t^{\frac{1}{1-q}} Y_t + P_{t+1}^{\frac{1}{1-q}} Y_{t+1} \right]} \\
& \approx \frac{1}{q} P^{\theta - \frac{1}{1-q}} V + 2qV \frac{\theta P^{\theta-1} \left(P^{\frac{1}{1-q}} \right) - \frac{1}{1-q} P^{\frac{1}{1-q}-1} (P^\theta)}{\left(q2P^{\frac{1}{1-q}} \right)^2} (\bar{P}_t - P) \\
& \quad + 2qV \frac{\theta P^{\theta-1} \left(P^{\frac{1}{1-q}} \right) - \frac{1}{1-q} P^{\frac{1}{1-q}-1} (P^\theta)}{\left(q2P^{\frac{1}{1-q}} \right)^2} \mathbf{E}_t (\bar{P}_{t+1} - P) \\
& \quad + \frac{1}{2q} P^{\theta - \frac{1}{1-q}} (V_t - V) + \frac{1}{2q} P^{\theta - \frac{1}{1-q}} \mathbf{E}_t (V_{t+1} - V)
\end{aligned}$$

Now use that $\theta = (2 - q) / (1 - q)$ in order to get

$$\begin{aligned}
& \frac{\mathbf{E}_t [P_t^\theta V_t Y_t + P_{t+1}^\theta V_{t+1} Y_{t+1}]}{\mathbf{E}_t \left[P_t^{\frac{1}{1-q}} Y_t + P_{t+1}^{\frac{1}{1-q}} Y_{t+1} \right]} \\
& \approx \frac{1}{q} P^{\theta - \frac{1}{1-q}} V + 2qV \frac{\frac{2-q}{1-q} P^{\frac{1}{1-q}} \left(P^{\frac{1}{1-q}} \right) - \frac{1}{1-q} P^{\frac{1}{1-q}-1} P^{\frac{2-q}{1-q}}}{\left(q2P^{\frac{1}{1-q}} \right)^2} (\bar{P}_t - P) \\
& \quad + 2qV \frac{\frac{2-q}{1-q} P^{\frac{1}{1-q}} \left(P^{\frac{1}{1-q}} \right) - \frac{1}{1-q} P^{\frac{1}{1-q}-1} P^{\frac{2-q}{1-q}}}{\left(q2P^{\frac{1}{1-q}} \right)^2} \mathbf{E}_t (\bar{P}_{t+1} - P) \\
& \quad + \frac{1}{2q} P^{\theta - \frac{1}{1-q}} (V_t - V) + \frac{1}{2q} P^{\theta - \frac{1}{1-q}} \mathbf{E}_t (V_{t+1} - V),
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{\mathbf{E}_t [P_t^\theta V_t Y_t + P_{t+1}^\theta V_{t+1} Y_{t+1}]}{\mathbf{E}_t \left[P_t^{\frac{1}{1-q}} Y_t + P_{t+1}^{\frac{1}{1-q}} Y_{t+1} \right]} & \approx \frac{1}{q} P^{\theta - \frac{1}{1-q}} V + \frac{1}{2q} V (\bar{P}_t - P) \\
& \quad + \frac{1}{2q} V \mathbf{E}_t (\bar{P}_{t+1} - P) \\
& \quad + \frac{1}{2q} P^{\theta - \frac{1}{1-q}} (V_t - V) + \frac{1}{2q} P^{\theta - \frac{1}{1-q}} \mathbf{E}_t (V_{t+1} - V),
\end{aligned}$$

We thus have the approximated relationship

$$\begin{aligned}
\bar{P}_t & = \frac{1}{q} P^{\theta - \frac{1}{1-q}} V + \frac{1}{2q} V (\bar{P}_t - P) + \frac{1}{2q} V \mathbf{E}_t (\bar{P}_{t+1} - P) \\
& \quad + \frac{1}{2q} P^{\theta - \frac{1}{1-q}} (V_t - V) + \frac{1}{2q} P^{\theta - \frac{1}{1-q}} \mathbf{E}_t (V_{t+1} - V),
\end{aligned}$$

or as $\bar{P} = \frac{1}{q}P^{\theta - \frac{1}{1-q}}V = \frac{1}{q}PV$:

$$\begin{aligned}\bar{P}_t - \bar{P} &= \frac{1}{2q}V(\bar{P}_t - P) + \frac{1}{2q}V\mathbf{E}_t(\bar{P}_{t+1} - P) \\ &\quad + \frac{1}{2q}P(V_t - V) + \frac{1}{2q}P\mathbf{E}_t(V_{t+1} - V),\end{aligned}$$

Divide the left-hand side by \bar{P} and the right-hand side by $\frac{1}{q}PV$, and the expression becomes

$$\begin{aligned}\frac{\bar{P}_t - \bar{P}}{\bar{P}} &= \frac{1}{2}\frac{\bar{P}_t - P}{P} + \frac{1}{2}\mathbf{E}_t\left(\frac{\bar{P}_{t+1} - P}{P}\right) \\ &\quad + \frac{1}{2}\frac{V_t - V}{V} + \frac{1}{2}\mathbf{E}_t\left(\frac{V_{t+1} - V}{V}\right).\end{aligned}$$

Letting lower-case letters denote percentage deviations from steady state one recovers

$$\bar{p}_t = \frac{1}{2}(p_t + \mathbf{E}_t p_{t+1}) + \frac{1}{2}(v_t + \mathbf{E}_t v_{t+1}),$$

which is equation (5.30).

3 Derivation of the last part of expression (5.31)

The second line in (5.31) states

$$\bar{p}_t = \frac{1}{2}\bar{p}_{t-1} + \frac{1}{2}\mathbf{E}_t\bar{p}_{t+1} + \gamma[m_t - p_t + \mathbf{E}_t m_{t+1} - \mathbf{E}_t p_{t+1}].$$

One then uses that the aggregate price level is an average of prices set in the current and last period; i.e.,

$$p_t = \frac{1}{2}(\bar{p}_{t-1} + \bar{p}_t).$$

This is inserted into the equation above:

$$\begin{aligned}\bar{p}_t &= \frac{1}{2}\bar{p}_{t-1} + \frac{1}{2}\mathbf{E}_t\bar{p}_{t+1} \\ &\quad + \gamma\left[m_t - \frac{1}{2}(\bar{p}_{t-1} + \bar{p}_t) + \mathbf{E}_t m_{t+1} - \mathbf{E}_t \frac{1}{2}(\bar{p}_t + \bar{p}_{t+1})\right]\end{aligned}$$

Solving for \bar{p}_t yields:

$$\begin{aligned}\bar{p}_t\left(1 + \frac{\gamma}{2} + \frac{\gamma}{2}\right) &= \frac{1}{2}\bar{p}_{t-1} + \frac{1}{2}\mathbf{E}_t\bar{p}_{t+1} \\ &\quad + \gamma\left[m_t - \frac{1}{2}\bar{p}_{t-1} + \mathbf{E}_t m_{t+1} - \mathbf{E}_t \frac{1}{2}\bar{p}_{t+1}\right]\end{aligned}$$

This immediately yields

$$\bar{p}_t = \frac{1}{2}\frac{1-\gamma}{1+\gamma}[\bar{p}_{t-1} + \mathbf{E}_t\bar{p}_{t+1}] + \frac{\gamma}{1+\gamma}[m_t + \mathbf{E}_t m_{t+1}],$$

which is the last line of equation (5.31).

4 Application of the method of undetermined coefficients in derivation of (5.32)

One makes the conjecture that the solution of (5.31) takes the form

$$\bar{p}_t = a\bar{p}_{t-1} + bm_t.$$

Forward this one period and take period t expectations:

$$\mathbf{E}_t \bar{p}_{t+1} = a\bar{p}_t + bm_t,$$

where it has been used that money supply follows a random walk ($\mathbf{E}_t m_{t+1} = m_t$). Insert this expression in (5.31) (and use again that $\mathbf{E}_t m_{t+1} = m_t$):

$$\bar{p}_t = \frac{1}{2} \frac{1-\gamma}{1+\gamma} [\bar{p}_{t-1} + a\bar{p}_t + bm_t] + \frac{\gamma}{1+\gamma} 2m_t,$$

and solve for \bar{p}_t :

$$\begin{aligned} \bar{p}_t \left[1 - \frac{a}{2} \frac{1-\gamma}{1+\gamma} \right] &= \frac{1}{2} \frac{1-\gamma}{1+\gamma} [\bar{p}_{t-1} + bm_t] + \frac{\gamma}{1+\gamma} 2m_t, \\ \bar{p}_t &= \frac{1-\gamma}{2(1+\gamma) - a(1-\gamma)} \bar{p}_{t-1} + \frac{(1-\gamma)b + 4\gamma}{2(1+\gamma) - a(1-\gamma)} m_t. \end{aligned}$$

This verifies the form of the conjecture, and shows that the undetermined coefficients must satisfy

$$\begin{aligned} a &= \frac{1-\gamma}{2(1+\gamma) - a(1-\gamma)}, \\ b &= \frac{(1-\gamma)b + 4\gamma}{2(1+\gamma) - a(1-\gamma)}. \end{aligned}$$

The first determines a from

$$-a^2(1-\gamma) + 2(1+\gamma)a - (1-\gamma) = 0,$$

or from

$$a^2 - \frac{2(1+\gamma)}{(1-\gamma)}a + 1 = 0,$$

as stated by Walsh. The coefficient b can then be determined subsequently. Indeed, it is easy to verify that $b = 1 - a$.