The Obstfeld-Rogoff model: Derivation of relative demand and the price index

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These notes provide a derivation of relative goods demand in the Obstfeld-Rogoff model as presented in Walsh (2003, Chapter 6); see also his Appendix 6.7.1. They also provide a derivation of the relevant price index associated with the consumption index.

1 Relative demand

We have that the consumption index of agent j is given by

$$C^j = \left[\int_0^1 c^j(z)^q \, \mathrm{d}z \right]^{\frac{1}{q}}, \qquad 0 < q < 1. \tag{1}$$

To concentrate on the determination of relative demand for individual goods, assume that total nominal expenditures are given by Z. I.e.,

$$\int_0^1 p(z) \, c^j(z) \, \mathrm{d}z = Z \tag{2}$$

The optimal demand for a given good z, is therefore found by maximizing (1) w.r.t $c^j(z)$ subject to (2). The relevant first-order condition is

$$\left[\int_0^1 c^j(z)^q \, \mathrm{d}z \right]^{\frac{1}{q}-1} c^j(z)^{q-1} = \lambda p(z), \tag{3}$$

where λ is the Lagrange-multiplier associated with (2). From this, we get

$$\left[\int_0^1 c^j(z)^q \,\mathrm{d}z\right]^{\frac{1}{q}-1} c^j(z)^q = \lambda p(z)\, c^j(z),$$

and thus (by integrating over all zs)

$$\begin{aligned}\left[\int_0^1 c^j(z)^q \,\mathrm{d}z\right]^{\frac{1}{q}-1}\left[\int_0^1 c^j(z)^q \,\mathrm{d}z\right] &= \lambda \int_0^1 p(z)\, c^j(z)\,\mathrm{d}z,\\ \left[\int_0^1 c^j(z)^q \,\mathrm{d}z\right]^{\frac{1}{q}} &= \lambda \int_0^1 p(z)\, c^j(z)\,\mathrm{d}z\end{aligned}$$

We then find λ by using the definitions of C^j and Z from

$$C^j = \lambda Z,$$

i.e.,

$$\lambda = \frac{C^j}{Z},$$

which is the ratio of real total consumption to total nominal expenditures. Denoting P the price index associated with C^j, one can then write $Z = PC^j$, and it follows that

$$\lambda = \frac{1}{P}. \tag{4}$$

This result is used in (3) to yield

$$\left[\int_0^1 c^j(z)^q \,\mathrm{d}z\right]^{\frac{1}{q}-1} c^j(z)^{q-1} = \frac{p(z)}{P},$$

and thus

$$\left[\int_0^1 c^j(z)^q \,\mathrm{d}z\right]^{\frac{1}{q}} c^j(z)^{-1} = \left[\frac{p(z)}{P}\right]^{\frac{1}{1-q}},$$

which then provides the demand for the individual good z as

$$c^j(z) = \left[\frac{p(z)}{P}\right]^{-\frac{1}{1-q}} C^j. \tag{5}$$

We see that the demand for good z is decreasing in the real price of the good, $p(z)/P$. Note that $1/(1-q)$ is the elasticity of substitution between goods. (Hence, the limiting case of perfect competition applies when $q \to 1$.) Total demand in the two countries of a good z will thus be n times domestic demand, and $1-n$ times foreign demand. A representative foreign consumer's demand function is

$$c^{*j}(z) = \left[\frac{p^*(z)}{P^*}\right]^{-\frac{1}{1-q}} C^{*j}$$

As the nominal exchange rate is S, this is the same as

$$c^{*j}(z) = \left[\frac{Sp^*(z)}{SP^*}\right]^{-\frac{1}{1-q}} C^{*j},$$

and thus, by the law of one price,

$$c^{*j}(z) = \left[\frac{p(z)}{P}\right]^{-\frac{1}{1-q}} C^{*j}$$

Total demand is therefore

$$\begin{aligned} y^d(z) &= n\left[\frac{p(z)}{P}\right]^{-\frac{1}{1-q}} C + (1-n)\left[\frac{p(z)}{P}\right]^{-\frac{1}{1-q}} C^* \\ &= \left[\frac{p(z)}{P}\right]^{-\frac{1}{1-q}} \left[nC + (1-n)C^*\right] \\ &= \left[\frac{p(z)}{P}\right]^{-\frac{1}{1-q}} C^w, \end{aligned}$$

where $C^w \equiv nC + (1-n)C^*$ is total world real consumption. This demand function is the relevant one for producer z, when determining production.

2 Derivation of the price index

Now note that the precise form of the price index P follows from

$$PC^j = \int_0^1 p(z)\, c^j(z)\, \mathrm{d}z$$

and the optimality condition (5). I.e., we get

$$PC^j = \int_0^1 p(z)\left[\frac{p(z)}{P}\right]^{\frac{1}{q-1}} C^j \mathrm{d}z$$

$$P = \int_0^1 p(z)\left[\frac{p(z)}{P}\right]^{\frac{1}{q-1}} \mathrm{d}z$$

$$
\begin{aligned}
P^{1+\frac{1}{q-1}} &= \int_0^1 p(z)^{1+\frac{1}{q-1}} \,\mathrm{d}z \\
P^{\frac{q}{q-1}} &= \int_0^1 p(z)^{\frac{q}{q-1}} \,\mathrm{d}z,
\end{aligned}
$$

and finally

$$
P = \left[\int_0^1 p(z)^{\frac{q}{q-1}} \,\mathrm{d}z\right]^{\frac{q-1}{q}}
$$

which is equation (6.3) in Walsh. Note that the index is therefore not "just a definition," but actually the only specification that is consistent with optimal behavior.

Alternatively, and equivalently, one could have derived P as the index which secures minimization of nominal expenditures subject to real consumption of one unit of the consumption index. I.e.,

$$
\min_{c^j(z)} \; PC^j = \int_0^1 p(z)\, c^j(z) \,\mathrm{d}z
$$

subject to $C^j = 1$. The first-order condition is

$$
\begin{aligned}
p(z) &= \mu \left[\int_0^1 c^j(z)^q \,\mathrm{d}z\right]^{\frac{1}{q}-1} c^j(z)^{q-1} \\
&= \mu \left[C^j\right]^{1-q} c^j(z)^{q-1} \\
&= \mu c^j(z)^{q-1}
\end{aligned}
$$

where μ is the Lagrange multiplier on $C^j = 1$, and where the last line uses that $C^j = 1$ indeed applies. One then gets

$$
c^j(z) = \left[p(z)/\mu\right]^{\frac{1}{q-1}}, \tag{6}
$$

which inserted into $C^j = 1$ gives

$$
\begin{aligned}
\left[\int_0^1 \left[p(z)/\mu\right]^{\frac{q}{q-1}} \,\mathrm{d}z\right]^{\frac{1}{q}} &= 1, \\
\left[\int_0^1 \left[p(z)\right]^{\frac{q}{q-1}} \,\mathrm{d}z\right]^{\frac{1}{q}} \mu^{\frac{1}{1-q}} &= 1,
\end{aligned}
$$

and thus

$$
\mu^{\frac{1}{q-1}} = \left[\int_0^1 \left[p(z)\right]^{\frac{q}{q-1}} \,\mathrm{d}z\right]^{\frac{1}{q}}.
$$

Insert this back into (6):

$$c^j(z) = p(z)^{\frac{1}{q-1}} \left[\int_0^1 [p(z)]^{\frac{q}{q-1}} \, \mathrm{d}z \right]^{-\frac{1}{q}}.$$

Then use

$$\begin{aligned} PC^j &= \int_0^1 p(z) \, c^j(z) \, \mathrm{d}z, \\ P &= \int_0^1 p(z) \, c^j(z) \, \mathrm{d}z, \end{aligned}$$

to find

$$\begin{aligned} P &= \int_0^1 p(z) \, p(z)^{\frac{1}{q-1}} \left[\int_0^1 [p(z)]^{\frac{q}{q-1}} \, \mathrm{d}z \right]^{-\frac{1}{q}} \mathrm{d}z, \\ P &= \int_0^1 p(z)^{\frac{q}{q-1}} \left[\int_0^1 [p(z)]^{\frac{q}{q-1}} \, \mathrm{d}z \right]^{-\frac{1}{q}} \mathrm{d}z, \\ &= \left[\int_0^1 [p(z)]^{\frac{q}{q-1}} \, \mathrm{d}z \right]^{\frac{q-1}{q}}; \end{aligned}$$

which again is equation (6.3) in Walsh.