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On linearizations in chapter 6

Q How does the linearization of equation (6.7) (page 272 in Walsh, 2003) yield equation (6.20) on page 274?

A This is a boring, albeit straightforward exercise, where the trick is to keep track of the steady-state values of the involved variables. Let's jog......

We start out by the equation (6.7) (skipping the j superscripts):

$$\frac{M_t}{P_t} = bC_t \frac{1+i_t}{i_t}$$

A first-order Taylor approximation of both sides around steady state (upper bars denote steady states) yields

$$\begin{aligned} & \frac{\overline{M}}{\overline{P}} + \frac{1}{\overline{P}}\left(M_t - \overline{M}\right) - \frac{\overline{M}}{\overline{P}^2}\left(P_t - \overline{P}\right) \\ = \; & b\overline{C}\frac{1+\overline{i}}{\overline{i}} + b\frac{1+\overline{i}}{\overline{i}}\left(C_t - \overline{C}\right) - \frac{b\overline{C}}{\overline{i}^2}\left(i_t - \overline{i}\right) \end{aligned}$$

or,

$$\frac{1}{\overline{P}}\left(M_t - \overline{M}\right) - \frac{\overline{M}}{\overline{P}^2}\left(P_t - \overline{P}\right) = b\frac{1+\overline{i}}{\overline{i}}\left(C_t - \overline{C}\right) - \frac{b\overline{C}}{\overline{i}^2}\left(i_t - \overline{i}\right)$$

This is re-written as

$$\begin{aligned} & \frac{\overline{M}}{\overline{P}}\frac{M_t - \overline{M}}{\overline{M}} - \frac{\overline{M}}{\overline{P}}\frac{P_t - \overline{P}}{\overline{P}} \\ = \; & b\overline{C}\frac{1+\overline{i}}{\overline{i}}\frac{C_t - \overline{C}}{\overline{C}} - \frac{b\overline{C}}{\overline{i}}\frac{\left(i_t - \overline{i}\right)}{\overline{i}} \end{aligned}$$

Letting lower-case letters denote percentage deviations from steady state, which under a first-order approximation equal log deviations, we get

$$\frac{\overline{M}}{\overline{P}}m_t - \frac{\overline{M}}{\overline{P}}p_t = b\overline{C}\frac{1+\overline{i}}{\overline{i}}c_t - \frac{b\overline{C}}{\overline{i}}\widehat{i}_t$$

where $\widehat{i}_t$ denotes the percentage deviation of the nominal interest rate from steady state. Dividing the left-hand side by $\frac{\overline{M}}{\overline{P}}$ and the right-hand-side by $b\overline{C}\frac{1+\overline{i}}{\overline{i}}$, we get

$$m_t - p_t = c_t - \frac{1}{1+\overline{i}}\widehat{i}_t \qquad (*)$$

Equation (*) is the linearized money demand relation expressed as a function of the *nominal* interest rate.

To express this in terms of inflation and the real interest rate, we use the Fisher equation

$$R_t = \frac{1+i_t}{1+\pi_{t+1}}$$

This is approximated by

$$R_t - \overline{R} = \frac{1}{1+\overline{\pi}}\left(i_t - \overline{i}\right) - \frac{1+\overline{i}}{\left(1+\overline{\pi}\right)^2}\left(\pi_{t+1} - \overline{\pi}\right)$$

Divide the left-hand side by $\overline{R}$ and the right-hand-side by $\left(1+\overline{i}\right)/\left(1+\overline{\pi}\right)$ to get

$$\frac{R_t - \overline{R}}{\overline{R}} = \frac{\overline{i}}{1+\overline{i}}\frac{i_t - \overline{i}}{\overline{i}} - \frac{\pi_{t+1} - \overline{\pi}}{1+\overline{\pi}}$$

Letting $\overline{\Pi} \equiv 1+\overline{\pi}$, we can rewrite this as

$$r_t = \frac{\overline{i}}{1+\overline{i}}\widehat{i}_t - \widehat{\pi}_{t+1}$$

where $\widehat{\pi}_{t+1}$ is the deviation of (gross) inflation from steady state. Hence,

$$\widehat{i}_t = \frac{1+\overline{i}}{\overline{i}}\left(r_t + \widehat{\pi}_{t+1}\right)$$

Insert this into (*) to get:

$$m_t - p_t = c_t - \frac{1}{\overline{i}}\left(r_t + \widehat{\pi}_{t+1}\right) \qquad (**)$$

We are almost finished now. We know from the consumption Euler equation that the steady-state gross real interest rate is $1/\beta$. Hence, by the steady-state version of the Fisher equation we get:

$$\frac{1}{\beta} = \frac{1+\overline{i}}{\overline{\Pi}}$$

From this we get the steady state nominal interest rate as

$$\overline{i} = \frac{\overline{\Pi}}{\beta} - 1$$

Using this in (**) we get

$$m_t - p_t = c_t - \frac{\beta}{\overline{\Pi} - \beta}\left(r_t + \widehat{\pi}_{t+1}\right),$$

or in Walsh's notation:

$$m_t - p_t = c_t - \delta\left(r_t + \widehat{\pi}_{t+1}\right), \qquad (***)$$

where

$$\delta \equiv \frac{\beta}{\overline{\Pi} - \beta}.$$

Equation (***) is Walsh's equation (6.20), with the only difference being that I have used a "hat notation" to distinguish log deviations in gross inflation from non-linearized inflation (Walsh does not make that distinction explicit).