

Plan for today:

1. Money in the utility function (start)

- a. Short on the Tobin effect mentioned in Chapter 2's introduction
- b. The basic money in the utility function model
- c. Optimal behavior and steady-state equilibrium properties:
Long-run superneutrality of money

Literature: Walsh (2003, Chapter 2, pp. 43-59)

Introductory remarks

- The standard model for (exogenous) economic growth is the simple Solow model featuring a fixed savings rate and a law of motion for physical capital accumulation
- When extended with optimizing savings behavior, we get the Ramsey model: The foundation for the Real Business Cycle research program
- Models have no role for money and monetary policy
- Purpose of models/analyses in coming lectures is to introduce a role for money in these type of models
- Money is introduced in various ways: Often short-cuts
 - Short-cuts are helpful for understanding simple features, and the more robust results are to particular short cut, of course, the better
- The so-called Tobin model extends the Solow model by *postulating* a demand for money (the short cut)
 - Highlights the implications of inflation for the choice between investment in physical and financial assets
- The Money-in-the-Utility function model extends the Ramsey model by *postulating that money gives utility* (the short cut)
 - Highlights the importance of microfoundations and optimizing private-sector behavior (absent in the Tobin model)

The Tobin Effect

- Like Solow model, output is produced by a CRS production function:

$$Y_t = F(K_{t-1}, N_t),$$

which in “intensive” form is:

$$y_t = f(k_{t-1}), \quad y_t \equiv Y_t/N_t, \quad k_{t-1} \equiv K_{t-1}/N_{t-1}$$

(population grows at a constant rate $N_t/N_{t-1} = 1 + n$)

- Households invest in capital or money, M_t (of which one unit buys $1/P_t$ goods). Real per capita wealth:

$$a_t \equiv k_t + m_t, \quad m_t \equiv (M_t/P_t)/N_t$$

- Government makes real lump-sum transfers (or withdrawals) to households in the form of money supply changes:

$$\tau_t = \frac{\Delta M_t}{P_t N_t}$$

- Economy-wide household budget constraint is (no depreciation of capital)

$$Y_t + \tau_t N_t + \frac{M_{t-1}}{P_t} = C_t + \Delta K_t + \frac{M_t}{P_t}$$

or

$$Y_t + \tau_t N_t + \frac{M_{t-1}}{(1 + \pi_t) P_{t-1}} = C_t + \Delta K_t + \frac{M_t}{P_t}$$

Note how inflation $\pi_t \equiv (P_t/P_{t-1}) - 1$ *erodes* initial resources available for consumption, investment in capital and future real money holdings

- Rewrite budget constraint so it depicts resources available for consumption, for investment in capital and accumulation of real money holdings (subtract M_{t-1}/P_{t-1} from both sides):

$$Y_t + \tau_t N_t + \frac{M_{t-1}}{(1 + \pi_t) P_{t-1}} - \frac{M_{t-1}}{P_{t-1}} = C_t + \Delta K_t + \frac{\Delta M_t}{P_t}$$

$$Y_t + \tau_t N_t - \frac{\pi_t M_{t-1}}{(1 + \pi_t) P_{t-1}} = C_t + \Delta K_t + \frac{\Delta M_t}{P_t}$$

Note that inflation *erodes* available resources

- As in Solow model, $0 < s < 1$ is saved, $1 - s$ is consumed. \implies

$$\Delta K_t + \frac{\Delta M_t}{P_t} = s \left(Y_t + \tau_t N_t - \frac{\pi_t M_{t-1}}{(1 + \pi_t) P_{t-1}} \right)$$

- In per capita terms:

$$\Delta k_t + \Delta \left(\frac{M_t}{P_t} \right) \frac{1}{N_t} = s \left(y_t + \tau_t - \frac{\pi_t}{(1 + \pi_t)(1 + n)} m_{t-1} \right) - \frac{n}{1 + n} k_{t-1}$$

- Expressed exclusively as a physical capital accumulation expression:

$$\Delta k_t = s \left(y_t + \tau_t - \frac{\pi_t}{(1 + \pi_t)(1 + n)} m_{t-1} \right) - \frac{n}{1 + n} k_{t-1}$$

$$- \frac{\theta_t - \pi_t}{(1 + \pi_t)(1 + n)} m_{t-1}$$

– Per capita physical capital changes with *total* savings, net of “depreciation” due to population growth, and net of changes in real per capital money

– $\theta_t \equiv \Delta M_t/M_{t-1}$ is the growth rate of the nominal money supply

- Use the definition of transfers

$$\tau_t = \frac{\Delta M_t}{P_t N_t} = \frac{\theta_t}{(1 + \pi_t)(1 + n)} m_{t-1},$$

and we get

$$\Delta k_t = sf(k_{t-1}) - (1 - s) \frac{\theta_t - \pi_t}{(1 + \pi_t)(1 + n)} m_{t-1} - \frac{n}{1 + n} k_{t-1}$$

- Increases in real money per capita

$$\Delta \left(\frac{M_t}{P_t} \right) / N_t = \frac{\theta_t - \pi_t}{(1 + \pi_t)(1 + n)} m_{t-1}$$

have two effects:

- a) It crowds out Δk_t *completely* for given total savings
- b) It increases savings by a fraction $s < 1$

- The net effect is *negative*: Increases in real money per capita divert resources away from physical capital accumulation

- **Steady state** ($\Delta m_t = \Delta k_t = 0$)

– $\Delta m_t = 0$ implies $\Delta m_t / m_{t-1} = 0$ and therefore

$$(1 + \theta) = (1 + \pi)(1 + n)$$

$$\pi \approx \theta - n$$

– $\Delta k_t = 0$ implies

$$sf(k^{ss}) = (1 - s) \frac{\theta - \pi}{(1 + \pi)(1 + n)} m^{ss} + \frac{n}{1 + n} k^{ss}$$

$$sf(k^{ss}) = (1 - s) \frac{1 + \theta - (1 + \pi)}{(1 + \theta)} m^{ss} + \frac{n}{1 + n} k^{ss}$$

$$sf(k^{ss}) = (1 - s) \frac{n}{(1 + n)} m^{ss} + \frac{n}{1 + n} k^{ss},$$

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condensed as

$$sf(k^{ss}) = [(1 - s) \phi^{ss} + 1] \bar{n} k^{ss}$$

$$\phi^{ss} \equiv \frac{m^{ss}}{k^{ss}}, \quad \bar{n} \equiv \frac{n}{1 + n}$$

- Note that ϕ^{ss} will affect the steady-state value of physical capital and thus per-capita output
- In a standard Solow diagram, ϕ^{ss} is an additional part in investment needed to maintain a constant value of capital per labor
- Hence, *higher ϕ^{ss} decreases k^{ss}* .
- But what determines the steady-state real money to physical capital-ratio?

– Assumption: Capital and money's relative real yields

* Physical capital yields $f_k(k)$ (the real interest rate, r)

* Real money “yields” $-\pi / (1 + \pi)$ (as seen before; inflation erodes the value)

– I.e., $\phi \equiv \phi(r, \pi) = \phi(f_k(k), \pi)$; both partial derivatives negative

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- Steady-state relationship then becomes

$$s f(k^{ss}) = [(1-s)\phi(f_k(k^{ss}), \pi^{ss}) + 1] \bar{n} k^{ss}$$

- Total differentiation yields

$$dk^{ss} = \frac{(1-s)\phi_{\pi} \bar{n} k^{ss}}{s f_k - \bar{n} [1 + (1-s)\phi + (1-s)\phi_r f_{kk}]} d\pi$$

- One can show

$$\frac{dk^{ss}}{d\pi} > 0$$

- The (Mundell-) Tobin Effect: Higher inflation decreases m_t^{ss}/k_t^{ss} implying a substitution away from real money towards physical capital
- Result: Higher money growth and inflation causes higher output

Discussion

- Neutrality of money holds in model (different *levels* of M_t has no real effects — only effects on P_t)
- One-for-one long run relationship between nominal money growth and inflation as in data
- Positive long-run relationship between money growth and output — monetary superneutrality of money fails
 - Particular relationship not in data
 - Hence, *if* effect is important, other effects must neutralize it in real life
- Why is money in the model? It hurts! (is the short cut a good one?)
- All behavior is postulated; is this reasonable? What would optimizing agents do?

Money in the utility function (start)

- This model amends the standard Ramsey model (optimizing household) with money
 - I.e., a qualification of the Tobin model like the Ramsey model qualifies the Solow model
- The short-cut here for introducing a role for money is that real money *provides utility* per se
 - One interpretation: Money facilitates transactions on the market and reduces “shopping time” (money as such is an otherwise useless commodity....)
- Hence, it is assumed that the per-period utility function of household is

$$U_t = u(c_t, m_t)$$

with u being increasing and concave in both arguments. (It is, of course, *real* money that enters in u , i.e., M_t 's value relative to what it can buy at price P_t .)
- Often, to ensure existence of an equilibrium where money is held (a “monetary equilibrium”), it is assumed that for some \bar{m} , $u_{m_n}(c_t, \bar{m}) = 0$ and $u_{m_n}(c_t, m_t) < 0$ for $m_t > \bar{m}$
- Aim of representative household is to maximize:

$$W = \sum_{t=0}^{\infty} \beta^t u(c_t, m_t), \quad 0 < \beta < 1. \quad (2.1)$$

- What is the relevant constraint?
- Assuming the physical capital depreciates at rate $0 < \delta < 1$, the economy-wide budget constraint is (*ignoring* for simplicity financial asset holdings B_t used in Walsh)

$$Y_t + \tau_t N_t + (1 - \delta) K_{t-1} + \frac{M_{t-1}}{P_t} = C_t + K_t + \frac{M_t}{P_t}, \quad (2.2')$$

The left-hand side is available resources at t for consumption, the capital stock and real money
- In per-capita version, *assuming no population growth as in Walsh* (set $n = 0$), we get

$$y_t + \tau_t + (1 - \delta) k_{t-1} + \frac{1}{1 + \pi_t} m_{t-1} = c_t + k_t + m_t$$
- Assuming again $y_t = f(k_{t-1})$, we get

$$\omega_t = f(k_{t-1}) + \tau_t + (1 - \delta) k_{t-1} + \frac{1}{1 + \pi_t} m_{t-1} = c_t + k_t + m_t \quad (2.4')$$
- Hence, ω_t is the total available resources, treated *as given* at t by the households.
 - It is the relevant *state variable* when choosing the optimal paths of c , k , and m at date t

- Household's optimization problem is solved by dynamic *programming* (could be done with Lagrangian methods as well)
- Involves use of the *value function* — the maximal value of W given optimal behavior of the household, and given the current state (ω)

- The value function V must therefore satisfy

$$V(\omega_t) = \max \{u(c_t, m_t) + \beta V(\omega_{t+1})\}$$

Maximization is over c_t, m_t, k_t subject to the budget constraint and the definition of ω_{t+1} (available resources one period ahead)

- To make it simple, one substitutes out ω_{t+1} and substitutes out k_t as $k_t = \omega_t - c_t - m_t$. One then maximizes (unconstrained) over c_t and m_t :

$$\max \left\{ \begin{array}{l} u(c_t, m_t) \\ + \beta V \left(\underbrace{f(k_t) + \tau_t + (1 - \delta)k_t + \frac{1}{1 + \pi_{t+1}}m_t}_{\omega_{t+1}} \right) \end{array} \right\}$$

and thus

$$\max \left\{ \begin{array}{l} + \beta V \left(\begin{array}{l} u(c_t, m_t) \\ f(\omega_t - c_t - m_t) + \tau_t \\ + (1 - \delta)(\omega_t - c_t - m_t) + \frac{1}{1 + \pi_{t+1}}m_t \end{array} \right) \end{array} \right\}$$

- First-order condition concerning choice of c_t :

$$\begin{aligned} u_c(c_t, m_t) + \beta V_\omega(\omega_{t+1}) \frac{\partial \omega_{t+1}}{\partial c_t} &= 0 \\ u_c(c_t, m_t) - \beta V_\omega(\omega_{t+1}) [f_k(k_t) + 1 - \delta] &= 0 \end{aligned} \quad (2.6')$$

Marginal utility of period t consumption equals its marginal loss (in terms of the marginal value of less period $t + 1$ capital)

- First-order condition concerning choice of m_t :

$$\begin{aligned} u_m(c_t, m_t) + \beta V_\omega(\omega_{t+1}) \frac{\partial \omega_{t+1}}{\partial m_t} &= 0 \\ u_m(c_t, m_t) + \beta V_\omega(\omega_{t+1}) \left[\frac{1}{1 + \pi_{t+1}} - (f_k(k_t) + 1 - \delta) \right] &= 0 \end{aligned} \quad (2.8')$$

Marginal utility period t real money (in terms of direct utility plus marginal value of more real money resources in period $t + 1$) equals marginal loss (in terms of the marginal value of less period $t + 1$ capital)

- Furthermore, transversality conditions must be satisfied:

$$\begin{aligned} \lim_{t \rightarrow \infty} \beta^t u_c(c_t, m_t) k_t &= 0 \\ \lim_{t \rightarrow \infty} \beta^t u_c(c_t, m_t) m_t &= 0 \end{aligned}$$

(otherwise over-accumulation of k and m is taking place — life-time utility could be improved through higher consumption by accumulating less)

- In the first-order conditions one can eliminate the value function V by use of the so-called *Envelope Theorem*:
- Note: optimal consumption and money holding choices will be functions of ω_t
- Define these as $c_t = c(\omega_t)$ and $m_t = m(\omega_t)$, respectively
- The value function is therefore *by definition* characterized as

$$V(\omega_t) = u(c(\omega_t), m(\omega_t)) + \beta V(\omega_{t+1}). \quad (*)$$

As (*) holds for *any* value of ω_t it follows that

$$V_\omega(\omega_t) = u_c(c(\omega_t), m(\omega_t))c'(\omega_t) + u_m(c(\omega_t), m(\omega_t))m'(\omega_t) + \beta V_\omega(\omega_{t+1})\frac{\partial \omega_{t+1}}{\partial \omega_t}. \quad (**)$$

- Now, find $\partial \omega_{t+1}/\partial \omega_t$ when $c_t = c(\omega_t)$ and $m_t = m(\omega_t)$ applies. One gets

$$\frac{\partial \omega_{t+1}}{\partial \omega_t} = [f_k(k_t) + 1 - \delta](1 - c'(\omega_t) - m'(\omega_t)) + \frac{1}{1 + \pi_{t+1}}m'(\omega_t)$$

- Combining this with (**):

$$V_\omega(\omega_t) = u_c(c(\omega_t), m(\omega_t))c'(\omega_t) + u_m(c(\omega_t), m(\omega_t))m'(\omega_t) + \beta V_\omega(\omega_{t+1}) \left\{ \begin{aligned} & [f_k(k_t) + 1 - \delta](1 - c'(\omega_t) - m'(\omega_t)) \\ & + \frac{1}{1 + \pi_{t+1}}m'(\omega_t) \end{aligned} \right\}.$$

- Collect the $c'(\omega_t)$ and $m'(\omega_t)$ terms:

$$V_\omega(\omega_t) = [u_c(c(\omega_t), m(\omega_t)) - \beta V_\omega(\omega_{t+1})(f_k(k_t) + 1 - \delta)]c'(\omega_t) + \left[\begin{aligned} & u_m(c(\omega_t), m(\omega_t)) + \beta V_\omega(\omega_{t+1})\frac{1}{1 + \pi_{t+1}} \\ & - \beta V_\omega(\omega_{t+1})[f_k(k_t) + 1 - \delta] \end{aligned} \right] m'(\omega_t) + \beta V_\omega(\omega_{t+1})[f_k(k_t) + 1 - \delta]$$

- NOTE:

$$V_\omega(\omega_t) = \underbrace{\left[u_c(c(\omega_t), m(\omega_t)) - \beta V_\omega(\omega_{t+1})(f_k(k_t) + 1 - \delta) \right]}_{=0 \text{ by (2.6')}} c'(\omega_t) + \left[\begin{aligned} & u_m(c(\omega_t), m(\omega_t)) + \beta V_\omega(\omega_{t+1})\frac{1}{1 + \pi_{t+1}} \\ & - \beta V_\omega(\omega_{t+1}) \underbrace{[f_k(k_t) + 1 - \delta]}_{=0 \text{ by (2.8')}} \end{aligned} \right] m'(\omega_t) + \beta V_\omega(\omega_{t+1})[f_k(k_t) + 1 - \delta]$$

- All terms in front of $c'(\omega_t)$ and $m'(\omega_t)$ are zero by the first-order conditions.

– This makes sense, as these terms indeed capture the marginal values of c and $m \dots \dots$ **in an optimum these must, of course, be . . . zero**

- Therefore, (**) reduces immediately to

$$V_{\omega}(\omega_t) = \beta V_{\omega}(\omega_{t+1}) [f_k(k_t) + 1 - \delta]$$

- Then use the first-order condition for consumption choice,

$$u_c(c_t, m_t) - \beta V_{\omega}(\omega_{t+1}) [f_k(k_t) + 1 - \delta] = 0,$$

to obtain Walsh's expression (which is technically the envelope theorem):

$$V_{\omega}(\omega_t) = u_c(c_t, m_t). \quad (2.10)$$

- Marginal utility of consumption equals marginal value of wealth
- A familiar restatement of optimal intratemporal consumption choice!

- With this expression, the first-order conditions can be rewritten as

$$u_c(c_t, m_t) = \beta u_c(c_{t+1}, m_{t+1}) [f_k(k_t) + 1 - \delta]$$

(a discrete-time, money version of the standard Keynes-Ramsey rule), and

$$u_m(c_t, m_t) + \beta \frac{u_c(c_{t+1}, m_{t+1})}{1 + \pi_{t+1}} = u_c(c_t, m_t)$$

(marginal gain of m_t equal to the marginal loss in terms of lower capital in period $t + 1$ — equal to the marginal utility of c_t by the “Keynes-Ramsey rule”)

- These conditions, together with the budget constraint characterizes the optimal paths of c , k , and m
- We will, for now, concentrate on the long-run properties of the model; i.e., a steady state with $\Delta c_t = \Delta k_t = \Delta m_t = 0$.

- First, from “Keynes-Ramsey rule” it follows that in steady state

$$1 = \beta [f_k(k^{ss}) + 1 - \delta],$$

or,

$$f_k(k^{ss}) + 1 - \delta = \frac{1}{\beta} \quad (2.18')$$

- This — **independently of any monetary factors** — defines the steady-state capital per capita (and thus output per capita).

- Strong contrast with Tobin model
- Difference is because this model envisions optimal behavior.
 - If, e.g., $k_t < k^{ss}$ the current marginal product of capital is relatively high (as $f_{kk} < 0$) \implies optimal to postpone consumption to later \implies capital is accumulated until $f_k(k^{ss}) + 1 - \delta = 1/\beta$ holds again
 - If one imagined that a Tobin effect was there; one would be self-contradictory:
 - * Assume higher inflation increases capital above steady state
 - * Then the marginal product decreases, and households would want to consume now rather than later \implies they **endogenously** save less and capital decreases until k^{ss} is reached again!
 - * Inflation has no steady-state effect on capital. Possible in Tobin model, where individuals are modelled as having an **exogenous** savings rate

- What about consumption in steady state?
 - The budget constraint in steady state is

$$f(k^{ss}) + \tau^{ss} + (1 - \delta)k^{ss} + \frac{1}{1 + \pi}m^{ss} = c^{ss} + k^{ss} + m^{ss}$$
 - Transfers are

$$\begin{aligned} \tau_t &= (M_t - M_{t-1})/P_t, \\ &= \theta_t M_{t-1}/P_t, \\ &= \frac{\theta_t}{1 + \pi_t} m_{t-1}, \end{aligned}$$
 - so in steady state:

$$\tau^{ss} = \frac{\theta}{1 + \pi} m^{ss}$$
 - Since a constant m implies $\pi = \theta$, as in the Tobin model, one gets

$$f(k^{ss}) - \delta k^{ss} = c^{ss}$$
- This is simply the economy's resource constraint (national account): Output less gross investment equals consumption
- Implication is that c^{ss} is determined exclusively by k^{ss} ; and thus also independent of money growth
 - Model exhibits *superneutrality of money*

- Does money growth and inflation not affect anything?
- Yes, the opportunity cost of holding real money, and thus the steady-state value of real money
- To see this, note relative demand for consumption versus real money is given by [use first-order cond. for money holdings and divide through by $u_c(c_t, m_t)$]

$$\begin{aligned} \frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} &= \frac{u_c(c_t, m_t)}{u_c(c_t, m_t)} - \frac{1}{1 + \pi_{t+1}} \frac{\beta u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)} \\ &= 1 - \frac{1}{1 + \pi_{t+1}} \frac{1}{f_k(k_t) + 1 - \delta} \\ &= 1 - \frac{1}{(1 + \pi_{t+1})(1 + r_t)} \end{aligned}$$

with $r_t \equiv f_k(k_t) - \delta$ being the real interest rate

- Note that the real interest rate is the nominal rate net of expected inflation:

$$1 + r_t = (1 + i_t) / (1 + \pi_{t+1}), \quad (r_t \approx i_t - \pi_{t+1})$$

- Hence,

$$\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{i_t}{1 + i_t} \equiv I_t \quad (2.12)$$

So, as nominal interest rate is determined by the *Fisher relation*-*ship*, $i_t \approx r_t + \pi_{t+1}$, higher inflation leads to a higher nominal interest rate, and for given c_t, m_t is likely to fall (as $u_{mm} < 0$).

- Will a unique steady-state value for m exist? Must solve $u_m(c^{ss}, m^{ss}) = I^{ss} u_c(c^{ss}, m^{ss})$. Not necessarily unique.....
- Stability properties? For separable utility, $u(c_t, m_t) = v(c_t) + \gamma \phi(m_t)$, resulting difference equation (from first-order condition) will imply a saddle-point stable $m^{ss} > 0$ (m' in Figure 2.1, page 56)

- Problem: One cannot necessarily rule out the paths with falling m below steady-state (“speculative” hyperinflations), leading to $m^{ss} = 0$. (Typically, however, we will just do it.....)

Summary

- Tobin model and MTU models both have one for one relationships between inflation and money growth
- Main difference is that the Tobin model does not have long-run superneutrality
- Reason is the postulated and policy invariant private-sector behavior.
- This difference highlights importance of microfoundations to avoid Lucas critique
- Still, the MTU approach *is* a short-cut

Plan for next lecture

Wednesday, Feb. 11

1. Money in the utility function (continued)
 - a. Welfare costs of inflation
 - b. Potential non-supernutrality of money
 - c. Dynamics and calibration

Literature: Walsh (2003, Chapter 2, pp. 59-80, but check the Appendix as well)