Deriving (9.21) in Walsh (2003)

"Monetary Economics: Macro Aspects"
Institute of Economics, University of Copenhagen
Henrik Jensen

March 24, 2004

Abstract

This note shows how to derive the nominal interest rate securing that the actual money supply always is equal to the value securing the average inflation target. I.e., the nominal interest rate that uses the actual money supply as an intermediate target.

The model is given by

$$
\begin{aligned}
y_t &= a\left(\pi_t - E_{t-1}\pi_t\right) + z_t \\
y_t &= -\alpha\left(i_t - E_t\pi_{t+1}\right) + u_t \\
m_t - p_t &= m_t - \pi_t - p_{t-1} = y_t - ci_t + v_t
\end{aligned}
$$

What money supply would give an inflation target of π^*? The trick is to acknowledge that with the strict inflation targeting preferences, in expected value we have inflation on target. I.e., $\mathrm{E}_{t-1}\pi_t = \mathrm{E}_t\pi_{t+1} = \pi^*$. Hence the model is rewritten as

$$
\begin{aligned}
y_t &= a\left(\pi_t - \pi^*\right) + z_t \\
y_t &= -\alpha\left(i_t - \pi^*\right) + u_t \\
m_t - p_t &= m_t - \pi_t - p_{t-1} = y_t - ci_t + v_t
\end{aligned}
$$

Now, the LM curve is inserted into the IS curve to eliminate i_t:

$$
y_t = -\alpha\frac{y_t + v_t - m_t + \pi_t + p_{t-1}}{c} + \alpha\pi^* + u_t,
$$

and

$$y_t\left(1+\frac{\alpha}{c}\right)=\alpha\frac{-v_t+m_t-\pi_t-p_{t-1}}{c}+\alpha\pi^*+u_t,$$

$$y_t\frac{\alpha+c}{c}=\alpha\frac{-v_t+m_t-\pi_t-p_{t-1}}{c}+\alpha\pi^*+u_t,$$

$$y_t=\frac{\alpha}{\alpha+c}\left(m_t-\pi_t-p_{t-1}-v_t\right)+\frac{c}{\alpha+c}\left(\alpha\pi^*+u_t\right).$$

We then find the *actual* inflation rate by combining this expression with the "modified" Lucas supply schedule:

$$\frac{\alpha}{\alpha+c}\left(m_t-\pi_t-p_{t-1}-v_t\right)+\frac{c}{\alpha+c}\left(\alpha\pi^*+u_t\right)=a\left(\pi_t-\pi^*\right)+z_t,$$

from which we get the solution for the inflation rate for a given money supply:

$$\frac{\alpha}{\alpha+c}\left(m_t-\pi_t-p_{t-1}-v_t\right)+\frac{c}{\alpha+c}\left(\alpha\pi^*+u_t\right)=a\left(\pi_t-\pi^*\right)+z_t,$$

$$\pi_t\left(a+\frac{\alpha}{\alpha+c}\right)=\frac{\alpha}{\alpha+c}\left(m_t-p_{t-1}-v_t\right)+\frac{c}{\alpha+c}\left(\alpha\pi^*+u_t\right)+a\pi^*-z_t$$

$$\pi_t\frac{a\left(\alpha+c\right)+\alpha}{\alpha+c}=\frac{\alpha}{\alpha+c}\left(m_t-p_{t-1}-v_t\right)+\frac{c}{\alpha+c}\left(\alpha\pi^*+u_t\right)+a\pi^*-z_t$$

and therefore

$$\pi_t=\frac{\alpha}{a\left(\alpha+c\right)+\alpha}\left(m_t-p_{t-1}-v_t\right)+\frac{c}{a\left(\alpha+c\right)+\alpha}\left(\alpha\pi^*+u_t\right)+\frac{a\left(\alpha+c\right)\pi^*-\left(\alpha+c\right)z_t}{a\left(\alpha+c\right)+\alpha}$$

We then solve for the value of m_t that secures $\pi_t=\pi^*$. I.e., this value must satisfy

$$\pi^*=\frac{\alpha}{a\left(\alpha+c\right)+\alpha}\left(m_t-p_{t-1}-v_t\right)+\frac{c}{a\left(\alpha+c\right)+\alpha}\left(\alpha\pi^*+u_t\right)+\frac{a\left(\alpha+c\right)\pi^*-\left(\alpha+c\right)z_t}{a\left(\alpha+c\right)+\alpha},$$

from which we get

$$\pi^*\left[1-\frac{c\alpha+a\left(\alpha+c\right)}{a\left(\alpha+c\right)+\alpha}\right]=\frac{\alpha}{a\left(\alpha+c\right)+\alpha}\left(m_t-p_{t-1}-v_t\right)+\frac{c}{a\left(\alpha+c\right)+\alpha}u_t-\frac{\left(\alpha+c\right)}{a\left(\alpha+c\right)+\alpha}z_t,$$

and

$$\pi^*\frac{a\left(\alpha+c\right)+\alpha-c\alpha-a\left(\alpha+c\right)}{a\left(\alpha+c\right)+\alpha}=\frac{\alpha}{a\left(\alpha+c\right)+\alpha}\left(m_t-p_{t-1}-v_t\right)+\frac{c}{a\left(\alpha+c\right)+\alpha}u_t-\frac{\left(\alpha+c\right)}{a\left(\alpha+c\right)+\alpha}z_t,$$

$$\pi^* \frac{\alpha(1-c)}{a(\alpha+c)+\alpha} = \frac{\alpha}{a(\alpha+c)+\alpha}(m_t - p_{t-1} - v_t) + \frac{c}{a(\alpha+c)+\alpha} u_t - \frac{(\alpha+c)}{a(\alpha+c)+\alpha} z_t,$$

and finally

$$m_t = p_{t-1} + v_t + (1-c)\pi^* - \frac{c}{\alpha} u_t + \frac{\alpha+c}{\alpha} z_t.$$

As shocks are unobservable, the optimal target of m_t is given by

$$\widehat{m}_t = p_{t-1} + \rho_v v_{t-1} + (1-c)\pi^* - \frac{c}{\alpha}\rho_u u_{t-1} + \frac{\alpha+c}{\alpha}\rho_z z_{t-1} \tag{9.19}$$

The actual money supply, for a given interest rate $\widehat{i}_t$, follows from the LM curve as

$$m_t\left(\widehat{i}_t\right) = \pi_t\left(\widehat{i}_t\right) + p_{t-1} + y_t\left(\widehat{i}_t\right) - c\widehat{i}_t + v_t. \tag{*}$$

Note that we have that

$$\widehat{i}_t = \pi^* + \frac{1}{\alpha}(\rho_u u_{t-1} - \rho_z z_{t-1}) \tag{9.17}$$

and

$$\pi_t\left(\widehat{i}_t\right) = \pi^* + \frac{\varphi_t - e_t}{a} \tag{9.18}$$

We can then find $y_t\left(\widehat{i}_t\right)$ by inserting $\pi_t\left(\widehat{i}_t\right)$ into the Lucas supply schedule:

$$\begin{aligned} y_t\left(\widehat{i}_t\right) &= a\left(\pi^* + \frac{\varphi_t - e_t}{a} - \pi^*\right) + z_t \\ &= \varphi_t - e_t + z_t \\ &= \varphi_t + \rho_z z_{t-1} \end{aligned}$$

Then insert the found expressions for $\pi_t\left(\widehat{i}_t\right)$, $y_t\left(\widehat{i}_t\right)$ and $\widehat{i}_t$ into (*):

$$\begin{aligned} m_t\left(\widehat{i}_t\right) &= \pi^* + \frac{\varphi_t - e_t}{a} + p_{t-1} + \varphi_t + \rho_z z_{t-1} \\ &\quad -c\left[\pi^* + \frac{1}{\alpha}(\rho_u u_{t-1} - \rho_z z_{t-1})\right] + v_t \\ &= (1-c)\pi^* + p_{t-1} + v_t + \frac{1+a}{a}\varphi_t - \frac{c}{\alpha}\rho_u u_{t-1} - \frac{1}{a}e_t + \frac{\alpha+c}{\alpha}\rho_z z_{t-1} \end{aligned} \tag{**}$$

From (9.19) note that

$$\widehat{m}_t - \rho_v v_{t-1} = p_{t-1} + (1-c)\pi^* - \frac{c}{\alpha}\rho_u u_{t-1} + \frac{\alpha+c}{\alpha}\rho_z z_{t-1},$$

which applied on (**) yields

$$m_t\left(\widehat{i_t}\right) = \widehat{m}_t - \rho_v v_{t-1} + v_t + \frac{1+a}{a}\varphi_t - \frac{1}{a}e_t \qquad ((9.20))$$
$$= \widehat{m}_t + \psi_t + \frac{1+a}{a}\varphi_t - \frac{1}{a}e_t$$

Now, when actual m_t conditional on $\widehat{i_t}$ changes relative to $\widehat{m}_t$, it is time to change i_t such that $m_t = \widehat{m}_t$ again. What value of the interest rate will accomplish that? I.e., how do we derive equation (9.21) on page 443 in Walsh (2003)?

The trick is to solve the model for m_t as a function of *any* value of the interest rate, and then find the interest rate that delivers $m_t = \widehat{m}_t$. This can be accomplished by the central bank, as it observes m_t even though it doesn't observe the various period-t disturbances.

As the model is

$$y_t = a(\pi_t - \pi^*) + z_t$$
$$y_t = -\alpha(i_t - \pi^*) + u_t$$
$$m_t - p_t = m_t - \pi_t - p_{t-1} = y_t - ci_t + v_t$$

we first combine the AS and IS curve to find inflation as a function of the interest rate:

$$a(\pi_t - \pi^*) + z_t = -\alpha(i_t - \pi^*) + u_t$$

and thus

$$\pi_t = \frac{a+\alpha}{a}\pi^* - \frac{\alpha}{a}i_t + \frac{1}{a}(u_t - z_t)$$

We have output a function of the interest rate directly from the IS curve:

$$y_t = -\alpha(i_t - \pi^*) + u_t$$

We can the use this in the LM relationship to find

$$m_t = \frac{a+\alpha}{a}\pi^* - \frac{\alpha}{a}i_t + \frac{1}{a}(u_t - z_t) + p_{t-1} - \alpha(i_t - \pi^*) + u_t - ci_t + v_t$$
$$= -\frac{\alpha(1+a)+ca}{a}i_t + \frac{a+\alpha+\alpha a}{a}\pi^* + p_{t-1} + \frac{1+a}{a}u_t + v_t - \frac{1}{a}z_t$$

Securing that $m_t = \widehat{m}_t$ requires that we use (9.19) and find the value of i_t that secures this:

$$-\frac{\alpha(1+a)+ca}{a}i_t + \frac{a+\alpha+\alpha a}{a}\pi^* + p_{t-1} + \frac{1+a}{a}u_t + v_t - \frac{1}{a}z_t$$
$$= p_{t-1} + \rho_v v_{t-1} + (1-c)\pi^* - \frac{c}{\alpha}\rho_u u_{t-1} + \frac{\alpha+c}{\alpha}\rho_z z_{t-1},$$

or,

$$\begin{aligned} & -\frac{\alpha\left(1+a\right)+ca}{a}i_t+\frac{a+\alpha+\alpha a}{a}\pi^*+\frac{1+a}{a}u_t+v_t-\frac{1}{a}z_t \\ = & \rho_v v_{t-1}+\left(1-c\right)\pi^*-\frac{c}{\alpha}\rho_u u_{t-1}+\frac{\alpha+c}{\alpha}\rho_z z_{t-1}, \end{aligned}$$

$$\begin{aligned} & -\frac{\alpha\left(1+a\right)+ca}{a}i_t+\frac{a+\alpha+\alpha a}{a}\pi^*+\frac{1+a}{a}\rho_u u_{t-1} \\ & +\frac{1+a}{a}\varphi_t+\psi_t-\frac{1}{a}\rho_z z_{t-1}-\frac{1}{a}e_t \\ = & \left(1-c\right)\pi^*-\frac{c}{\alpha}\rho_u u_{t-1}+\frac{\alpha+c}{\alpha}\rho_z z_{t-1}, \end{aligned}$$

$$\begin{aligned} & -\frac{\alpha\left(1+a\right)+ca}{a}i_t+\frac{a+\alpha+\alpha a}{a}\pi^*+\left[\frac{1+a}{a}+\frac{c}{\alpha}\right]\rho_u u_{t-1} \\ & +\frac{1+a}{a}\varphi_t+\psi_t-\left(\frac{1}{a}+\frac{\alpha+c}{\alpha}\right)\rho_z z_{t-1}-\frac{1}{a}e_t \\ = & \left(1-c\right)\pi^* \end{aligned}$$

$$\begin{aligned} & \left(\frac{a+\alpha+\alpha a}{a}+c-1\right)\pi^*+\left[\frac{1+a}{a}+\frac{c}{\alpha}\right]\rho_u u_{t-1} \\ & +\frac{1+a}{a}\varphi_t+\psi_t-\left(\frac{1}{a}+\frac{\alpha+c}{\alpha}\right)\rho_z z_{t-1}-\frac{1}{a}e_t \\ = & \frac{\alpha\left(1+a\right)+ca}{a}i_t \end{aligned}$$

$$\begin{aligned} & \left(\frac{a+\alpha+\alpha a}{a}+c-1\right)\pi^*+\frac{\alpha\left(1+a\right)+ca}{a\alpha}\rho_u u_{t-1} \\ & +\frac{1+a}{a}\varphi_t+\psi_t-\frac{\alpha+a\left(\alpha+c\right)}{a\alpha}\rho_z z_{t-1}-\frac{1}{a}e_t \\ = & \frac{\alpha\left(1+a\right)+ca}{a}i_t \end{aligned}$$

An thus

$$\begin{aligned} i_t\frac{\alpha\left(1+a\right)+ca}{a} = & \frac{\alpha\left(1+a\right)+ca}{a}\pi^* \\ & +\frac{\alpha\left(1+a\right)+ca}{a\alpha}\left(\rho_u u_{t-1}-\rho_z z_{t-1}\right) \\ & +\frac{1+a}{a}\varphi_t+\psi_t-\frac{1}{a}e_t, \end{aligned}$$

which finally gives

$$\begin{aligned} i_t &= \pi^* + \frac{1}{\alpha}\left(\rho_u u_{t-1} - \rho_z z_{t-1}\right) \\ &\quad + \frac{(1+a)\,\varphi_t - e_t + a\psi_t}{\alpha\,(1+a) + ca}. \end{aligned}$$

Using the result for $\widehat{i_t}$, equation (9.17), this readily reduces to

$$i_t = \widehat{i_t} + \frac{(1+a)\,\varphi_t - e_t + a\psi_t}{\alpha\,(1+a) + ca} \equiv i_t^T$$

which is equation (9.21) in Walsh (2003).