May 3, 2004

Finding the solution of the simple New Keynesian model when $k > 0$

Q How does one derive equations (4.3') and (4.4') from the slides of April 26?

A This is a good question, as the equations follow the format of Clarida et al., but as written are a bit imprecise. The message from the equations is that the solution will be then same as when $k = 0$, except for a constant term portraying an inflation bias. Clarida et al. make a simplifying assumption by setting $\beta = 1$, which will make the constant term a simple one. Clearly, $\beta = 1$ should be applied in the coefficients to the u_t shock, but I retained β explicitly, so that you could see that the coefficients on the shocks are the same as in the slides covering the case of $k = 0$. Maybe that is a little confusing. So here are the computations of (4.3') and (4.4') for any value of β:

The central bank's per-period utility function is:

$$-\frac{\alpha}{2}(x_t - k)^2 - \frac{1}{2}\pi_t^2, \qquad k > 0,$$

and the AS curve is

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \lambda x_t + u_t.$$

The relevant first-order condition will be (insert the AS curve into the utility function, maximize w.r.t. x_t and take $\mathrm{E}_t\pi_{t+1}$ as given; just as in the case of $k = 0$):

$$-\alpha(x_t - k) = \lambda \pi_t.$$

This is inserted into the AS curve:

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} - \lambda[(\lambda/\alpha)\pi_t - k] + u_t$$

$$\pi_t\left(1 + \lambda^2/\alpha\right) = \beta \mathrm{E}_t \pi_{t+1} + \lambda k + u_t$$

$$\pi_t = \frac{\beta}{1 + \lambda^2/\alpha}\mathrm{E}_t \pi_{t+1} + \frac{\lambda}{1 + \lambda^2/\alpha}k + \frac{1}{1 + \lambda^2/\alpha}u_t$$

This is a first-order expectational difference equation with one unstable root (as in the case of $k = 0$); so there is a unique non-explosive solution to π_t. In the case with $k = 0$, it was conjectured that $\pi_t = Xu_t$, where X was the undetermined coefficient to be determined. Now, what makes the above difference equation different from the one with $k = 0$?

The presence of a constant term. Hence, a sensible conjecture is one that involves a constant term. I.e., conjecture a solution of the following format:

$$\pi_t = Y + X u_t, \qquad (*)$$

where Y and X are the undetermined coefficients to be determined.

Find the coefficients. Forward (*) one period, and take period t expectations:

$$\begin{aligned} \pi_{t+1} &= Y + X u_{t+1}, \\ \mathrm{E}_t \pi_{t+1} &= Y + X \mathrm{E}_t u_{t+1}, \\ \mathrm{E}_t \pi_{t+1} &= Y + X \rho u_t. \qquad (**) \end{aligned}$$

Insert this into the difference equation:

$$\pi_t = \frac{\beta}{1 + \lambda^2/\alpha} \left[Y + X \rho u_t \right] + \frac{\lambda}{1 + \lambda^2/\alpha} k + \frac{1}{1 + \lambda^2/\alpha} u_t$$

$$\pi_t = \frac{\beta Y + \lambda k}{1 + \lambda^2/\alpha} + \frac{X \beta \rho + 1}{1 + \lambda^2/\alpha} u_t$$

This verifies the form of the conjecture, and identifies the coefficients by the equations

$$\begin{aligned} Y &= \frac{\beta Y + \lambda k}{1 + \lambda^2/\alpha} \\ X &= \frac{X \beta \rho + 1}{1 + \lambda^2/\alpha} \end{aligned}$$

These are solved to get

$$\begin{aligned} Y &= \frac{1}{1 - \beta + \lambda^2/\alpha} \lambda k, \\ X &= \alpha \frac{1}{\lambda^2 + \alpha (1 - \beta \rho)} u_t \end{aligned}$$

Hence,

$$\pi_t = \frac{1}{1 - \beta + \lambda^2/\alpha} \lambda k + \alpha \frac{1}{\lambda^2 + \alpha (1 - \beta \rho)} u_t,$$

which is equation (4.4) in Clarida et al. (1999) and (4.4') in the posted slides on the web (April 26), when one simplifies the constant term by taking Clarida et al.'s simplifying assumption $\beta = 1$ into account (this makes $Y = \alpha/\lambda$).