Equivalence of (9.11) and (9.10) in Walsh (2003) when (9.11) involves linear projections of the shocks on the nominal interest rate

"Monetary Economics: Macro Aspects"
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March 24, 2004

Abstract

This note shows how equation (9.11) implies the optimal value of the coefficient in the extended Poole model with the money base rule as a linear function of the nominal interest rate, when in (9.11) the forecasts of the shocks are linear projections on the interest rate.

We have the resulting interest rate in the model with $b_t = \mu i_t$, and equations

$$y_t = -\alpha i_t + u_t \quad (9,1)$$

$$m_t = -c i_t + y_t + v_t \quad (9.2)$$

$$m_t = b_t + h i_t + \omega_t \quad (9.7)$$

given as

$$i_t = \frac{v_t - \omega_t + u_t}{\alpha + c + \mu + h}. \quad (9.9)$$

The resulting solution for output is given by

$$\begin{aligned} y_t &= -\alpha \frac{v_t - \omega_t + u_t}{\alpha + c + \mu + h} + u_t \\ &= \frac{(c + \mu + h)\, u_t - \alpha\,(v_t - \omega_t)}{\alpha + c + \mu + h}, \end{aligned}$$

with the associated variance

$$\sigma_y^2 = \frac{(c+\mu+h)^2 \sigma_u^2 - \alpha^2 (\sigma_v^2 + \sigma_\omega^2)}{(\alpha+c+\mu+h)^2}.$$

Minimizing this with respect to μ gives the first-order condition

$$\frac{2(c+\mu+h)\sigma_u^2(\alpha+c+\mu+h)^2 - 2(\alpha+c+\mu+h)\left[(c+\mu+h)^2\sigma_u^2 - \alpha^2(\sigma_v^2+\sigma_\omega^2)\right]}{(\alpha+c+\mu+h)^4} = 0,$$

and thus

$$2(c+\mu+h)\sigma_u^2(\alpha+c+\mu+h)^2 - 2(\alpha+c+\mu+h)\left[(c+\mu+h)^2\sigma_u^2 - \alpha^2\left(\sigma_v^2+\sigma_\omega^2\right)\right] = 0,$$

$$(c+\mu+h)\sigma_u^2(\alpha+c+\mu+h) - (c+\mu+h)^2\sigma_u^2 + \alpha^2\left(\sigma_v^2+\sigma_\omega^2\right) = 0,$$

$$(c+\mu+h)(\alpha+c+\mu+h) - (c+\mu+h)^2 + \alpha^2\frac{(\sigma_v^2+\sigma_\omega^2)}{\sigma_u^2} = 0,$$

$$(c+\mu+h)\left[(\alpha+c+\mu+h) - (c+\mu+h)\right] + \alpha^2\frac{(\sigma_v^2+\sigma_\omega^2)}{\sigma_u^2} = 0,$$

and thus

$$\mu^* = -(c+h) + \alpha^2\frac{(\sigma_v^2+\sigma_\omega^2)}{\sigma_u^2}. \tag{9.10}$$

as the **optimal coefficient** in the policy rule.

The alternative is that shocks are observable, so that the base rule can be stated as

$$b_t = \mu_u u_t + \mu_v v_t + \mu_\omega \omega_t.$$

Inserting this into (9.1), (9.2) and (9.7) gives:

$$\begin{aligned} m_t &= \mu_u u_t + \mu_v v_t + \mu_\omega \omega_t + h i_t + \omega_t \\ &= -c i_t + y_t + v_t \end{aligned}$$

Eliminating i by (9.1) yields

$$\mu_u u_t + \mu_v v_t + \mu_\omega \omega_t + h\frac{u_t - y_t}{\alpha} + \omega_t = -c\frac{u_t - y_t}{\alpha} + y_t + v_t$$

from which we get the solution for y_t:

$$y_t\left(1 + \frac{c}{a} + \frac{h}{\alpha}\right) = (\mu_v - 1)v_t + \left(\mu_u + \frac{h}{\alpha} + \frac{c}{\alpha}\right)u_t + (1+\mu_\omega)\omega_t$$

We immediately see that

$$b_t = -\frac{c+h}{\alpha}u_t + v_t - \omega_t$$

completely stabilizes output. However, shocks cannot be observed, so *estimates* of the shocks are made based on the observed interest rate, so that the policy rule becomes

$$b_t = -\frac{c+h}{\alpha}\widehat{u}_t + \widehat{v}_t - \widehat{\omega}_t$$

where

$$\widehat{u}_t = \mathrm{E}\left[u_t|i_t\right], \quad \widehat{v}_t = \mathrm{E}\left[v_t|i_t\right], \quad \widehat{\omega}_t = \mathrm{E}\left[\omega_t|i_t\right].$$

are the estimates of the shocks based on the observed interest rate. We assume that these estimates are made by linear projections on i_t with the aim of minimizing the squared forecast errors.

But what is the nominal interest rate under this rule? From (9.1) and (9.2) one immediately get

$$m_t = -\left(c+\alpha\right)i_t + u_t + v_t$$

Using (9.7) on gets

$$b_t + hi_t + \omega_t = -\left(c+\alpha\right)i_t + u_t + v_t$$

and thus

$$i_t = -\frac{b_t}{\alpha+c+h} + \frac{u_t + v_t - \omega_t}{\alpha+c+h}$$

As the shock forecasts are linear projections on i: $\widehat{u}_t = \mathrm{E}[u_t|i_t] = \widehat{\delta}_u i_t$, $\widehat{v}_t = \mathrm{E}[v_t|i_t] = \widehat{\delta}_v i_t$ and $\widehat{\omega}_t = \mathrm{E}[\omega_t|i_t] = \widehat{\delta}_\omega i_t$, where $\widehat{\delta}_u$, $\widehat{\delta}_v$ and $\widehat{\delta}_\omega$ are estimation coefficients to be determined, we have that

$$b_t = \left(-\frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega\right)i_t \tag{9.11}$$

Therefore we get an expression for the interest rate from

$$i_t\left(\alpha+c+h\right) = -\left(-\frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega\right)i_t + u_t + v_t - \omega_t,$$

so that the solution for the interest rate becomes

$$i_t = \frac{u_t + v_t - \omega_t}{\alpha+c+h-\frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega}.$$

Hence, the estimates are found from

$$\begin{aligned}\widehat{u}_t &= \delta_u i_t \\ &= \delta_u \frac{u_t + v_t - \omega_t}{\alpha + c + h - \frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega},\end{aligned}$$

where δ_u minimizes the squared forecast error. I.e., it solves

$$\begin{aligned}&\min_{\delta_u} \quad \mathrm{E}\left[\widehat{u}_t - u_t\right]^2 \\ =& \min_{\delta_u} \quad \mathrm{E}\left[\delta_u \frac{u_t + v_t - \omega_t}{\alpha + c + h - \frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega} - u_t\right]^2 \\ =& \min_{\delta_u} \quad \left[\delta_u^2 \frac{\sigma_v^2 + \sigma_\omega^2 + \sigma_u^2}{\left(\alpha + c + h - \frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega\right)^2} + \sigma_u^2 - 2\delta_u \frac{\sigma_u^2}{\alpha + c + h - \frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega}\right]\end{aligned}$$

The first-order condition is:

$$2\delta_u \frac{\sigma_v^2 + \sigma_\omega^2 + \sigma_u^2}{\left(\alpha + c + h - \frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega\right)^2} - 2\frac{\sigma_u^2}{\alpha + c + h - \frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega} = 0,$$

or,

$$\delta_u \frac{\sigma_v^2 + \sigma_\omega^2 + \sigma_u^2}{\alpha + c + h - \frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega} - \sigma_u^2 = 0 \qquad (*)$$

We then find the estimation coefficient $\widehat{\delta}_v$ from

$$\begin{aligned}&\min_{\delta_v} \quad \mathrm{E}\left[\widehat{v}_t - v_t\right]^2 \\ =& \min_{\delta_v} \quad \mathrm{E}\left[\delta_v \frac{u_t + v_t - \omega_t}{\alpha + c + h - \frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega} - v_t\right]^2\end{aligned}$$

The first-order condition becomes

$$\delta_v \frac{\sigma_v^2 + \sigma_\omega^2 + \sigma_u^2}{\alpha + c + h - \frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega} - \sigma_v^2 = 0 \qquad (**)$$

Likewise, the first order condition determining δ_ω becomes

$$\delta_\omega \frac{\sigma_v^2 + \sigma_\omega^2 + \sigma_u^2}{\alpha + c + h - \frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega} + \sigma_\omega^2 = 0 \qquad (***)$$

Combining (*), (**) and (***) reveals that the estimates are

$$\begin{aligned}\widehat{\delta}_u &= \alpha, \\ \widehat{\delta}_v &= \frac{\alpha\sigma_v^2}{\sigma_u^2}, \\ \widehat{\delta}_\omega &= -\frac{\alpha\sigma_\omega^2}{\sigma_u^2}.\end{aligned}$$

Hence, the forecast-based rule for b is given by

$$\begin{aligned}b_t &= \left(-\frac{c+h}{\alpha}\widehat{\delta}_u + \widehat{\delta}_v - \widehat{\delta}_\omega\right) i_t \\ &= \left[-(c+h) + \frac{\alpha\left(\sigma_v^2 + \sigma_\omega^2\right)}{\sigma_u^2}\right] i_t \\ &= \mu^* i_t.\end{aligned}$$

I.e., the exact same optimal rule as derived before by (9.11).