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CALIBRATING THE LIFE CYCLE MODEL

Technical appendices to paper on

Measuring the deadweight loss from taxation in a small open economy

by

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Abstract: This technical working paper derives the links between the various factor supply elasticities in the standard two period life cycle model with endogenous savings and labour supply. Respecting these links is important when the model is used for quantitative analysis. The paper also derives a formula for the present value of a cohort's capital income relative to the present value of its labour income in the standard multi-period life cycle model. This variable is important when calculating the present value of the marginal deadweight loss from taxation.

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APPENDIX 1
THE LINKS BETWEEN FACTOR SUPPLY ELASTICITIES IN THE LIFE CYCLE MODEL

by Peter Birch Sørensen

This appendix shows how the compensated wage elasticity of saving and the interest elasticity of labour supply are tied to the compensated wage elasticity of labour supply and how the compensated and the uncompensated interest elasticities of saving are linked in the two-period life cycle model with taxes and transfers. The appendix serves to document the results reported in IV.1 in Sørensen (2011) and we consistently follow the notation of that paper.

The compensated wage elasticity of saving

In this subsection we derive the relationship between the compensated wage elasticities of saving and labour supply. Recalling that $w \equiv W\left(1-t^{w}\right)/P$, it follows from (2) and (4) in Sørensen (2011) that the consumer's first-period budget constraint may be written as

$$S = wL + B_1/P - C_1, \tag{A.1}$$

from which we get

$$dS = Ldw + wdL + d\left(B_1/P\right) - dC_1. \tag{A.2}$$

The standard Slutsky decomposition implies

$$\frac{\partial L}{\partial w} = \frac{\partial \tilde{L}}{\partial w} + L\frac{\partial L}{\partial I}, \tag{A.3}$$

where a tilde superscript indicates a compensated effect. In case of a tax-induced compensated change in the marginal after-tax real wage, we have assumed that the compensation takes the form of a change in the first-period transfer B_1/P of the magnitude

$$d\left(B_1/P\right) = -Ldw. \tag{A.4}$$

This compensation will neutralize the income effect in the second term in (A.3), since B_1/P is included in the present value of exogenous lifetime income, *I*. The consumer's lifetime budget constraint $PC_1 + pPC_2 = W\left(1-t^{w}\right)L + B_1 + pB_2$ may be written as

$$C_1 = c\left(wL + \frac{B_1}{P} + p\frac{B_2}{P}\right), \qquad c \equiv \frac{1}{1+p\left(1+g^c\right)}, \qquad g^c \equiv \frac{C_2}{C_1} - 1, \tag{A.5}$$

where g^c is the consumer's optimal growth rate of consumption over the life cycle. If utility is additively separable in consumption and labour supply, g^c will be independent of *L*. Since B_2 is kept constant, it then follows from (A.3) through (A.5) that

$$dC_1 = c\left(Ldw + wdL + d\left(\tfrac{B_1}{P}\right)\right) = c\left(Ldw + wdL - Ldw\right) = cw\frac{d\tilde{L}}{dw}dw. \tag{A.6}$$

Inserting (A.3), (A.4) and (A.6) into (A.2) and rearranging, we get

$$\varepsilon_w^S = \left(1-c\right)\frac{wL}{S}\varepsilon_w^L, \qquad \varepsilon_w^S \equiv \frac{\partial\tilde{S}}{\partial w}\frac{w}{S}, \qquad \varepsilon_w^L \equiv \frac{\partial\tilde{L}}{\partial w}\frac{w}{L}. \tag{A.7}$$

If we substitute (A.5) into (A.1) and use the resulting expression for *S* along with the definition of *w* to derive an expression for $\left(1-c\right)wL/S$, we find from (A.7) that

$$\varepsilon_w^S = \left(\frac{1-t^w}{1-t^w + b_1 - p\left(\frac{c}{1-c}\right)b_2}\right)\varepsilon_w^L, \qquad b_1 \equiv \frac{B_1}{WL}, \qquad b_2 \equiv \frac{B_2}{WL}. \tag{A.8}$$

Using the definition of *c* stated in (A.5) as well as the definition of *p*, equation (A.8) simplifies to the expression for ε_w^S stated in eq. (47) in Sørensen (2011).

The compensated interest elasticity of labour supply

We now turn to the link between the compensated wage and interest elasticities of labour supply. The symmetry properties of the Slutsky matrix imply that

$$-\frac{\partial\tilde{L}}{\partial p} = \frac{\partial\tilde{C}_2}{\partial w} \quad \Leftrightarrow \quad \frac{\partial\tilde{L}}{\partial p}\frac{p}{L} = -\frac{\partial\tilde{C}_2}{\partial w}\frac{p}{L}. \tag{A.9}$$

The consumer's second-period budget constraint $PC_2 = \left[1 + r\left(1-t^r\right)\right]PS + B_2$ implies

$$S = p\left(C_2 - B_2/P\right), \tag{A.10}$$

from which it follows that

$$\frac{\partial\tilde{S}}{\partial w} = p\frac{\partial\tilde{C}_2}{\partial w}, \tag{A.11}$$

since we have assumed that consumers are compensated for a change in *w* via an adjustment of B_1 rather than B_2. Equations (A.7), (A.9) and (A.11) yield

$$\frac{\partial \tilde{L}}{\partial p}\frac{p}{L} = -\frac{\partial \tilde{S}}{\partial w}\frac{1}{L} = -\frac{S}{wL}\varepsilon_w^S = -\left(1-c\right)\varepsilon_w^L. \tag{A.12}$$

The relative price of future consumption is $p \equiv 1/\left(1+r^a\right)$ where $r^a \equiv r\left(1-t^r\right)$, so the change in p caused by a change in the after-tax real interest rate is

$$\frac{dp}{dr^a} = -\frac{1}{\left(1+r^a\right)^2}. \tag{A.13}$$

Using (A.12) and (A.13), we find the compensated net interest elasticity of labour supply to be

$$\varepsilon_r^L \equiv \frac{\partial \tilde{L}}{\partial r^a}\frac{r^a}{L} = \frac{\partial \tilde{L}}{\partial p}\frac{p}{L}\frac{r^a}{p}\frac{dp}{dr^a} = \frac{\partial \tilde{L}}{\partial p}\frac{p}{L}\cdot\left(\frac{-r^a}{1+r^a}\right) \quad \Rightarrow \quad \varepsilon_r^L \equiv \left(\frac{r^a}{1+r^a}\right)\left(1-c\right)\varepsilon_w^L, \tag{A.14}$$

which is identical to eq. (48) in Section IV.1 in Sørensen (2011).

The compensated interest elasticity of saving

Let us finally show how the compensated interest elasticity of saving is related to the corresponding uncompensated elasticity about which we have more empirical knowledge. In the main text we assumed that consumers are compensated for a change in the relative price of future consumption through an adjustment of the present value of pensions, pB_2. According to eq. (9) in Sørensen (2011) the real value of the compensation is thus given by

$$\frac{\partial\left(pB_2\right)}{\partial p}\frac{1}{P} = \frac{\partial E}{\partial p}\frac{1}{P} = C_2. \tag{A.15}$$

From (A.10), (A.13) and (A.15) it follows that

$$\frac{\partial \tilde{S}}{\partial r^a} = -\left(\frac{1}{1+r^a}\right)^2\left[C_2 - \frac{B_2}{P} + p\frac{\partial \tilde{C}_2}{\partial p} - \frac{\partial\left(pB_2\right)}{\partial p}\frac{1}{P}\right] = \left(\frac{1}{1+r^a}\right)^2\left(\frac{B_2}{P} - p\frac{\partial \tilde{C}_2}{\partial p}\right) \quad \Rightarrow$$

$$\frac{\partial \tilde{S}}{\partial r^a}\frac{r^a}{S} = \left(\frac{r^a}{1+r^a}\right)\left(\frac{pB_2/P}{S} - \frac{pC_2}{S}\frac{\partial \tilde{C}_2}{\partial p}\frac{p}{C_2}\right). \tag{A.16}$$

From (A.10) we have $pC_2 = S + pB_2/P$. Inserting this into (A.16) and rearranging, we end up with

$$\varepsilon_r^S \equiv \frac{\partial \tilde{S}}{\partial r^a}\frac{r^a}{S} = \left(\frac{r^a}{1+r^a}\right)\left[b + \varepsilon_p^{C_2}\left(1+b\right)\right], \qquad \varepsilon_p^{C_2} \equiv -\frac{\partial \tilde{C}_2}{\partial p}\frac{p}{C_2}, \qquad b \equiv \frac{B_2/P}{\left(1+r^a\right)S}, \tag{A.17}$$

where ε_r^S is the compensated elasticity of saving with respect to the after-tax real interest rate, $\varepsilon_p^{C_2}$ is the compensated numerical own-price elasticity of demand for future consumption, and b is the amount of old-age consumption financed by public pensions relative to the amount that is financed by previous savings.

When the consumer is *not* compensated for a change in the after-tax real interest rate, it follows from (A.10) and (A.13) that the effect on saving will be

$$\frac{\partial S}{\partial r^a} = -\left(\frac{1}{1+r^a}\right)^2 \left(C_2 - \frac{B_2}{P} + p\frac{\partial C_2}{\partial p}\right) \Rightarrow \frac{\partial S}{\partial r^a}\frac{r^a}{S} = \left(\frac{r^a}{1+r^a}\right)\left(\frac{p\left(B_2/P - C_2\right)}{S} - \frac{pC_2}{S}\frac{\partial C_2}{\partial p}\frac{p}{C_2}\right) \Rightarrow$$

$$\hat{\varepsilon}_r^S = \left(\frac{r^a}{1+r^a}\right)\left[\hat{\varepsilon}_p^{C_2}\left(1+b\right)-1\right], \qquad \hat{\varepsilon}_r^S \equiv \frac{\partial S}{\partial r^a}\frac{r^a}{S}, \qquad \hat{\varepsilon}_p^{C_2} \equiv -\frac{\partial C_2}{\partial p}\frac{p}{C_2}, \tag{A.18}$$

where $\hat{\varepsilon}_r^S$ is the uncompensated net interest elasticity of saving, and $\hat{\varepsilon}_p^{C_2}$ is the uncompensated numerical own-price elasticity of demand for future consumption. We now wish to identify the link between ε_r^S and $\hat{\varepsilon}_r^S$. The Slutsky decomposition implies that

$$\frac{\partial C_2}{\partial p} = \frac{\partial \tilde{C}_2}{\partial p} - C_2\frac{\partial C_2}{\partial I} \quad \Rightarrow \quad \varepsilon_p^{C_2} = \hat{\varepsilon}_p^{C_2} - a\varepsilon_I^C, \qquad \varepsilon_I^{C_2} \equiv \frac{\partial C_2}{\partial I}\frac{I}{C_2}, \qquad a \equiv \frac{pC_2}{I}, \tag{A.19}$$

where $\varepsilon_I^{C_2}$ is the income elasticity of demand for future consumption. From (A.18) and (A.19) it follows that

$$\varepsilon_p^{C_2} = \frac{\left(\frac{1+r^a}{r^a}\right)\hat{\varepsilon}_r^S + 1}{1+b} - a\varepsilon_I^{C_2}. \tag{A.20}$$

Inserting (A.20) into (A.17), we obtain

$$\varepsilon_r^S = \hat{\varepsilon}_r^S + \left(\frac{r^a}{1+r^a}\right)\left(1+b\right)\left(1-a\varepsilon_I^{C_2}\right). \tag{A.21}$$

It is convenient to rewrite (A.21) in terms of the parameters b_1 and b_2. Using the definitions of g^c, b_1 and b_2, we may write the consumer's lifetime budget constraint (6) as

$$PC_2\left[\left(1+g^c\right)^{-1} + p\right] = WL\left(1-t^w+b_1+pb_2\right). \tag{A.22}$$

Inserting (A.22) into the definition $a \equiv pPC_2/\left(B_1+pB_2\right)$, we get

$$a = \left(\frac{p}{p+\left(1+g^c\right)^{-1}}\right)\left(\frac{1-t^w+b_1+pb_2}{b_1+pb_2}\right). \tag{A.23}$$

From the definition of *b* we have

$$b \equiv \frac{pB_2}{PS} = \frac{pB_2}{p\left(PC_2 - B_2\right)} = \frac{B_2}{PC_2 - B_2}. \tag{A.24}$$

Substituting (A.22) into (A.24), we find

$$1 + b = \frac{1 - t^w + b_1 + pb_2}{1 - t^w + b_1 - b_2\left(1 + g^c\right)^{-1}}. \tag{A.25}$$

To get a feel for the likely magnitude of the income elasticity $\varepsilon_I^{C_2}$, let us assume that the lifetime utility function takes the additively separable form

$$U = \frac{C_1^{1-\gamma}}{1-\gamma} + \frac{1}{1+d}\frac{C_2^{1-\gamma}}{1-\gamma} - \frac{L^{1+\eta}}{1+\eta}, \tag{A.26}$$

where d is the utility discount rate, γ is the coefficient of relative risk aversion and η denotes the elasticity of the marginal disutility of work. Maximisation of (A.26) subject to consumer's lifetime budget constraint $PC_1 + pPC_2 = W\left(1 - t^w\right)L + B_1 + pB_2$ can be shown to imply that

$$C_2 = \left(1 + g^c\right)C_1, \qquad g^c = \left(\frac{1 + r^a}{1 + d}\right)^{1/\gamma} - 1, \tag{A.27}$$

$$L = w^{1/\eta} C_1^{-\gamma/\eta}. \tag{A.28}$$

These relationships reveal that, for an optimizing consumer, $1/\gamma$ is the elasticity of intertemporal substitution and $1/\eta$ is the Frisch wage elasticity of labour supply. Inserting (A.27) and (A.28) into the budget constraint (A.22), differentiating with respect to *I*, and using the definition of *I* as well as the budget constraint itself, one finds after some manipulations that

$$\varepsilon_I^{C_2} = \frac{1}{1 + \frac{2\gamma}{\eta} + \left(\frac{\eta + \gamma}{\eta}\right)\left(\frac{1 - t^w}{b_1 + pb_2}\right)}. \tag{A.29}$$

Equation (A.29) provides the useful information that $\varepsilon_I^{C_2}$ must be strictly less than one. It also allows a calibration of $\varepsilon_I^{C_2}$, given existing tax and transfer rates and empirical estimates of the elasticity of intertemporal substitution and the Frisch wage elasticity of labour supply. Using the consumer's lifetime budget constraint, the Slutsky equation (A.3) and the optimality conditions (A.27) and (A.28), one can show that the compensated wage elasticity of labour supply, the Frisch elasticity and the elasticity of intertemporal substitution are linked to each other by equation (51) in Sørensen (2011).

APPENDIX 2
THE TIMING OF CAPITAL INCOME
IN THE LIFE CYCLE MODEL

by Peter Birch Sørensen

I. The problem

The paper by Sørensen (2011) shows that in order to estimate the loss of lifetime welfare imposed on a cohort of individuals when some tax rate goes up, one must calculate the present value of the changes in the amounts of income from labour and capital earned by the cohort over its life cycle. In that context one must account for the fact that the timing of capital income and labour income differs over the life cycle. Sørensen (2011) develops a measure of the marginal deadweight loss from taxation in which this difference in the timing of income is captured by a single parameter θ^s defined as

$$\theta^s \equiv \frac{\text{present value of capital income earned over the cohort's life cycle}}{\text{present value of labour income earned over the cohort's life cycle}}, \tag{1}$$

where the present value is calculated at the beginning of the cohort's active career, i.e., at the time when its members enter the labour market.

The purpose of this note is to derive a formula for the parameter θ^s based on the life cycle model of consumption and labour supply.

II.1. Theoretical framework: a multi-period life cycle model

Our theoretical framework is a standard multi-period life cycle model without bequests. We shall use the following notation:

$C_t =$ consumption in year t

$L_t =$ labour supply (employment) in year t

$W_t =$ pre-tax wage rate in year t

$t^w =$ marginal labour income tax rate (constant over time)

$W_t^a = W_t\left(1-t^w\right) =$ marginal after-tax wage rate in year t

$B_t =$ government transfer received in year t

$P =$ consumer price index (constant over time)

$r =$ pre-tax real interest rate (constant over time)

$r^a =$ after-tax real interest rate (constant over time)

$d =$ utility discount rate (constant over time)

The representative consumer maximises a lifetime utility function of the additive form

$$U = \sum_{t=0}^{T}\left(\frac{1}{1+d}\right)^t\left[u\left(C_t\right)-h_t\cdot v\left(L_t\right)\right], \qquad L_t = 0 \text{ for } t = n, n+1, ..., T, \tag{2}$$

subject to the lifetime budget constraint

$$\sum_{t=0}^{T}\frac{PC_t}{\left(1+r^a\right)^t} = \sum_{t=0}^{T}\frac{W_tL_t\left(1-t^w\right)+B_t}{\left(1+r^a\right)^t}. \tag{3}$$

The per-period utility function $u\left(C_t\right)$ is assumed to be concave and the disutility-of-work function $v\left(L_t\right)$ is taken to be convex. As indicated in (2), the (adult) consumer is active in the labour market until the start of age n when he retires. The length of the adult life is T+1 years and $T+1-n$ is the number of years spent in retirement.

II.2. Solving the consumer's problem

To solve the consumer's problem, we form the Lagrangian

$$\ell = \sum_{t=0}^{T}\left(\frac{1}{1+d}\right)^t\left[u\left(C_t\right)-h_t v\left(L_t\right)\right]-\lambda\left[\sum_{t=0}^{T}\frac{PC_t}{\left(1+r^a\right)^t}-\sum_{t=0}^{T}\frac{W_tL_t\left(1-t^w\right)+B_t}{\left(1+r^a\right)^t}\right], \quad L_t = 0 \text{ for } t = n,..,T,$$

where λ is the Lagrange multiplier associated with the lifetime budget constraint. The first-order conditions for an optimal path of consumption are

$$u'(C_0) = \lambda P,$$

$$\bullet$$

$$\frac{u'(C_t)}{(1+d)^t} = \frac{\lambda P}{(1+r^a)^t},$$

$$\frac{u'(C_{t+1})}{(1+d)^{t+1}} = \frac{\lambda P}{(1+r^a)^{t+1}}, \qquad (4)$$

$$\bullet$$

$$\frac{u'(C_T)}{(1+d)^T} = \frac{\lambda P}{(1+r^a)^T}.$$

From these conditions we obtain the well-known Euler equation

$$u'(C_t) = \left(\frac{1+r^a}{1+d}\right) u'(C_{t+1}), \qquad t = 0,..,T. \qquad (5)$$

For concreteness, let us assume that preferences display constant relative risk aversion so that

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0. \qquad (6)$$

This specification implies that the intertemporal elasticity of substitution in consumption is given by $1/\gamma$. From (5) and (6) we obtain

$$C_{t+1} = (1+g^c)C_t, \qquad g^c \equiv \left(\frac{1+r^a}{1+d}\right)^{1/\gamma} - 1, \qquad (7)$$

where g^c is the annual growth rate of real consumption which is assumed to satisfy the restriction $g^c < r^a$.

The first-order conditions for an optimal labour supply are

$$h_0 v'(L_0) = \lambda W_0 (1-t^w),$$

$$\bullet$$

$$\frac{h_t v'(L_t)}{(1+d)^t} = \frac{\lambda W_t (1-t^w)}{(1+r^a)^t},$$

$$\frac{h_{t+1} v'(L_{t+1})}{(1+d)^{t+1}} = \frac{\lambda W_{t+1} (1-t^w)}{(1+r^a)^{t+1}}, \tag{8}$$

$$\bullet$$

$$\frac{h_{n-1} v'(L_{n-1})}{(1+d)^{n-1}} = \frac{\lambda W_{n-1} (1-t^w)}{(1+r^a)^{n-1}}.$$

From (8) we obtain the condition for an optimal intertemporal allocation of labour:

$$h_t v'(L_t) = \left(\frac{1+r^a}{1+d}\right)\frac{W_t}{W_{t+1}} h_{t+1} v'(L_{t+1}), \qquad t = 0,....,n-2. \tag{9}$$

Due to exogenous productivity growth, the producer real wage is assumed to grow at the constant annual rate $g^w < r^a$, that is:

$$W_t = (1+g^w)^t W_0, \tag{10}$$

Further, the proportionality factor h_t in labour disutility function is taken to grow at the constant annual rate g^h, reflecting that the value of the consumer's non-market activities rises over time. Setting $h_0 = 1$ without loss of generality, we thus have

$$h_t = (1+g^h)^t. \tag{11}$$

Equations (9) through (11) imply that

$$v'(L_t) = \left(\frac{1+r^a}{1+d}\right)\left(\frac{1+g^h}{1+g^w}\right) v'(L_{t+1}), \qquad t = 0,....,n-2. \tag{12}$$

We will focus on the benchmark case where $\left(\frac{1+r^a}{1+d}\right)\left(\frac{1+g^h}{1+g^w}\right) = 1$. It then follows from (12) that labour supply is constant over the consumer's labour market career, i.e.

$$L_t = L_0 \quad \text{for} \quad t = 1,...,n-1, \qquad L_t = 0 \quad \text{for} \quad t = n,...,T. \tag{13}$$

We now wish to derive an expression for the level of optimal consumption in period 0 as a function of the lifetime resources available to the consumer. For this purpose we assume that transfers are indexed to the average level of wage income in the economy such that $B_t / W_t L_t$ is equal to a constant b^w during the consumer's working career and that $B_t / W_t L_t$ equals a constant b^r throughout his retirement period.[1] Inserting these restrictions as well as (7), (10) and (13) into (3), we find that the lifetime budget constraint implies

$$PC_0\left[1+\frac{1}{1+\rho^c}+\left(\frac{1}{1+\rho^c}\right)^2+.....+\left(\frac{1}{1+\rho^c}\right)^T\right]$$

$$=W_0L_0\left(1-t^w+b^w\right)\left[1+\frac{1}{1+\rho^w}+\left(\frac{1}{1+\rho^w}\right)^2+......+\left(\frac{1}{1+\rho^w}\right)^{n-1}\right]$$

$$+\,b^rW_0L_0\left[\left(\frac{1}{1+\rho^w}\right)^n+\left(\frac{1}{1+\rho^w}\right)^{n+1}+.......+\left(\frac{1}{1+\rho^w}\right)^T\right], \tag{14}$$

$$\rho^c \equiv \frac{1+g^c}{1+r^a}-1, \qquad \rho^w \equiv \frac{1+g^w}{1+r^a}-1,$$

where ρ^c and ρ^w are growth-adjusted real interest rates which are positive because of our assumptions that $g^c < r^a$ and $g^w < r^a$. Using the general result that

$$X = 1+a+a^2+....+a^m = \frac{1-a^{m+1}}{1-a}, \tag{15}$$

we can rewrite (14) as

$$PC_0 = \omega\left[\left(1-t^w+b^w\right)H^w + H^r\right], \tag{16}$$

$$\omega \equiv \frac{\rho^c}{1+\rho^c-\left(\frac{1}{1+\rho^c}\right)^T}, \qquad H^w \equiv W_0L_0\left[\frac{1-\left(\frac{1}{1+\rho^w}\right)^n}{1-\left(\frac{1}{1+\rho^w}\right)}\right], \qquad H^r \equiv b^rW_0L_0\left[\frac{\left(\frac{1}{1+\rho^w}\right)^n-\left(\frac{1}{1+\rho^w}\right)^{T+1}}{1-\left(\frac{1}{1+\rho^w}\right)}\right].$$

The parameter ω is the initial propensity to consume wealth, H^w is the present value of pre-tax labour income, and H^r is the present value of transfers received during retirement.

III.1 The present value of capital income earned during working age

Armed with the optimum conditions above, we can now derive the present value of the total capital income earned over the representative consumer's life cycle. We start by considering the capital income earned while the consumer is active in the labour market. Since the life cycle is divided into discrete time intervals (which we think of as years), we must be precise regarding the timing of

[1] Note that since transfers are linked to average wage income, the first-order conditions for optimal labour supply in (8) were derived on the assumption that the representative individual consumer takes B_t as given.

payments. The lifetime budget constraint underlying the consumption function (16) assumes that wages are received and consumption expenditures are incurred at the beginning of each period, whereas interest on the stock of wealth held at the start of the period is not received until the end of the period. The savings undertaken at the start of year zero (when the consumer enters the labour market) are thus given by

$$S_0 = W_0 L_0 \left(1 - t^w + b^w\right) - PC_0. \tag{17}$$

At the end of year zero, the consumer receives an amount of pre-tax interest income equal to rS_0. Seen from the start of the year, this income has the present value

$$R_0 = \frac{rS_0}{1+r^a} = \frac{r\left[W_0 L_0 \left(1 - t^w + b^w\right) - PC_0\right]}{1+r^a}. \tag{18}$$

Note that the relevant discount rate in (18) is the after-tax real interest rate r^a determining the consumer's marginal rate of substitution between present and future consumption. After having paid tax on his interest income, the wealth which the consumer can carry into year 1 is

$$V_1 = \left(1+r^a\right) S_0 = \left(1+r^a\right)\left[W_0 L_0 \left(1 - t^w + b^w\right) - PC_0\right]. \tag{19}$$

In year 1 the consumer's pre-tax wage income has grown to $\left(1+g^w\right)W_0 L_0$, and according to (7) his consumption expenditure has increased to $\left(1+g^c\right)PC_0$. The saving of non-capital income undertaken at the start of year 1 is therefore equal to

$$S_1 = \left(1+g^w\right)W_0 L_0 \left(1 - t^w + b^w\right) - \left(1+g^c\right)PC_0. \tag{20}$$

At the end of year 1 the consumer receives interest on the wealth carried over from year zero, given by (19), and on the new saving given by (20). Seen from the beginning of year zero, the present value of the pre-tax capital income received at the end of year 1 is thus equal to

$$R_1 = \frac{r\left(S_1 + V_1\right)}{\left(1+r^a\right)^2} = \frac{r}{\left(1+r^a\right)^2}\left[\left(1+g^w\right)W_0 L_0 \left(1 - t^w + b^w\right) - \left(1+g^c\right)PC_0\right]$$

$$+ \frac{r}{1+r^a}\left[W_0 L_0 \left(1 - t^w + b^w\right) - PC_0\right] \;\Rightarrow$$

$$R_1 = \frac{r}{1+r^a}\left\{W_0 L_0 \left(1 - t^w + b^w\right)\left[1 + \frac{1}{1+\rho^w}\right] - PC_0\left[1 + \frac{1}{1+\rho^c}\right]\right\}, \tag{21}$$

and from (19) and (20) the wealth carried into year 2 is

$$V_2 = \left(1+r^a\right)\left(S_1 + V_1\right) \quad \Rightarrow$$

$$V_2 = W_0 L_0 \left(1 - t^w + b^w\right)\left[\left(1 + r^a\right)^2 + \left(1 + r^a\right)\left(1 + g^w\right)\right] - PC_0\left[\left(1 + r^a\right)^2 + \left(1 + r^a\right)\left(1 + g^c\right)\right]. \quad (22)$$

Proceeding in a similar way, we find the new saving of non-capital income at the start of year 2 to be

$$S_2 = \left(1 + g^w\right)^2 W_0 L_0 \left(1 - t^w + b^w\right) - \left(1 + g^c\right)^2 PC_0, \quad (23)$$

so from (22) and (23) the present value of capital income received at the end of year 2 is

$$R_2 = \frac{r\left(S_2 + V_2\right)}{\left(1 + r^a\right)^3} \quad \Rightarrow$$

$$R_2 = \frac{r}{1 + r^a}\left\{W_0 L_0 \left(1 - t^w + b^w\right)\left[1 + \frac{1}{1 + \rho^w} + \left(\frac{1}{1 + \rho^w}\right)^2\right] - PC_0\left[1 + \frac{1}{1 + \rho^c} + \left(\frac{1}{1 + \rho^c}\right)^2\right]\right\}, \quad (24)$$

and the wealth carried into year 3 is

$$V_3 = \left(1 + r^a\right)\left(S_2 + V_2\right) \quad \Rightarrow$$

$$\begin{aligned} V_3 = {} & W_0 L_0 \left(1 - t^w + b^w\right)\left[\left(1 + r^a\right)^3 + \left(1 + r^a\right)^2\left(1 + g^w\right) + \left(1 + r^a\right)\left(1 + g^w\right)^2\right] \\ & - PC_0\left[\left(1 + r^a\right)^3 + \left(1 + r^a\right)^2\left(1 + g^c\right) + \left(1 + r^a\right)\left(1 + g^c\right)^2\right]. \end{aligned} \quad (25)$$

We can now see the pattern. Following the above procedure, one can show that the present value of capital income earned during year *t* of the working career is

$$\begin{aligned} R_t = {} & \left(\frac{r}{1 + r^a}\right) W_0 L_0 \left(1 - t^w + b^w\right)\left[1 + \frac{1}{1 + \rho^w} + \left(\frac{1}{1 + \rho^w}\right)^2 + \left(\frac{1}{1 + \rho^w}\right)^3 + \ldots. + \left(\frac{1}{1 + \rho^w}\right)^t\right] \\ & - \left(\frac{r}{1 + r^a}\right) PC_0\left[1 + \frac{1}{1 + \rho^c} + \left(\frac{1}{1 + \rho^c}\right)^2 + \left(\frac{1}{1 + \rho^c}\right)^3 + \ldots. + \left(\frac{1}{1 + \rho^c}\right)^t\right], \end{aligned} \quad (26)$$

where $t \leq n - 1$, and that the wealth carried into the first year of the retirement stage (year *n*) is

$$\begin{aligned} V_n = {} & W_0 L_0 \left(1 - t^w + b^w\right)\Big[\left(1 + r^a\right)^n + \left(1 + r^a\right)^{n-1}\left(1 + g^w\right) + \left(1 + r^a\right)^{n-2}\left(1 + g^w\right)^2 + ..\Big) \\ & \Big(.. + \left(1 + r^a\right)^2\left(1 + g^w\right)^{n-2} + \left(1 + r^a\right)\left(1 + g^w\right)^{n-1}\Big] \\ & - PC_0\Big[\left(1 + r^a\right)^n + \left(1 + r^a\right)^{n-1}\left(1 + g^c\right) + \left(1 + r^a\right)^{n-2}\left(1 + g^c\right)^2 + ..\Big) \\ & \Big(.. + \left(1 + r^a\right)^2\left(1 + g^c\right)^{n-2} + \left(1 + r^a\right)\left(1 + g^c\right)^{n-1}\Big]. \end{aligned} \quad (27)$$

Using the results in (18), (21), (24) and (26), we obtain the following expression for the total present value of all of the consumer's pre-tax capital income earned over his working career:

$$PVR^w = \sum_{t=0}^{n-1} R_t = \left(\frac{r}{1+r^a}\right) W_0 L_0 \left(1-t^w+b^w\right)$$

$$+\left(\frac{r}{1+r^a}\right) W_0 L_0 \left(1-t^w+b^w\right)\left[1+\frac{1}{1+\rho^w}\right]$$

$$+\left(\frac{r}{1+r^a}\right) W_0 L_0 \left(1-t^w+b^w\right)\left[1+\frac{1}{1+\rho^w}+\left(\frac{1}{1+\rho^w}\right)^2\right]$$

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$$+\left(\frac{r}{1+r^a}\right) W_0 L_0 \left(1-t^w+b^w\right)\left[1+\frac{1}{1+\rho^w}+\left(\frac{1}{1+\rho^w}\right)^2+....+\left(\frac{1}{1+\rho^w}\right)^{n-1}\right]$$

$$-\left(\frac{r}{1+r^a}\right) PC_0$$

$$-\left(\frac{r}{1+r^a}\right) PC_0\left[1+\frac{1}{1+\rho^c}\right]$$

$$-\left(\frac{r}{1+r^a}\right) PC_0\left[1+\frac{1}{1+\rho^c}+\left(\frac{1}{1+\rho^c}\right)^2\right]$$

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$$-\left(\frac{r}{1+r^a}\right) PC_0\left[1+\frac{1}{1+\rho^c}+\left(\frac{1}{1+\rho^c}\right)^2+....+\left(\frac{1}{1+\rho^c}\right)^{n-1}\right]. \tag{28}$$

Applyng formula (15) to (28), we find after some manipulations that

$$PVR^w = \left(\frac{rn}{1+r^a}\right)\left(\frac{1+\rho^w}{\rho^w}\right) W_0 L_0 \left(1-t^w+b^w\right)$$

$$-\left(\frac{r}{1+r^a}\right)\left(\frac{1}{\rho^w}\right) W_0 L_0 \left(1-t^w+b^w\right)\left[\frac{1-\left(\frac{1}{1+\rho^w}\right)^n}{1-\left(\frac{1}{1+\rho^w}\right)}\right]$$

$$-\left(\frac{rn}{1+r^a}\right)\left(\frac{1+\rho^c}{\rho^c}\right) PC_0 + \left(\frac{r}{1+r^a}\right)\left(\frac{1}{\rho^c}\right) PC_0\left[\frac{1-\left(\frac{1}{1+\rho^c}\right)^n}{1-\left(\frac{1}{1+\rho^c}\right)}\right]. \tag{29}$$

Inserting the consumption function (16) into (29) and dividing by H^w, we obtain

$$\frac{PVR^w}{H^w}=\left(\frac{r}{1+r^a}\right)\left(1-t^w+b^w\right)\left[\frac{n}{1-\left(\frac{1}{1+\rho^w}\right)^n}-\frac{1}{\rho^w}\right]$$

$$+\left(\frac{r}{1+r^a}\right)\frac{\omega}{\rho^c}\left[1-t^w+b^w+\frac{H^r}{H^w}\right]\left[\frac{1-\left(\frac{1}{1+\rho^w}\right)^n}{1-\left(\frac{1}{1+\rho^w}\right)}-n\left(1+\rho^c\right)\right]. \qquad (30)$$

III.2 The present value of capital income earned during retirement

Consider next the capital income earned during retirement. During the first year of retirement where the consumer has reached the age of *n* he receives a public retirement benefit $\left(1+g^w\right)B_0$ since benefits are indexed to wages. The saving of non-capital income undertaken at the start of year *n* (which may be negative) is therefore equal to

$$S_n=\left(1+g^w\right)^n B_0-\left(1+g^c\right)^n PC_0. \qquad (31)$$

The pre-tax capital income earned at the end of year *n* is $r\left(S_n+V_n\right)$. Using (27) and (31), we find that this income has the following present value at the start of year 0:

$$R_n=\frac{r\left(S_n+V_n\right)}{\left(1+r^a\right)^{n+1}}=\left(\frac{r}{1+r^a}\right)\frac{B_0}{\left(1+\rho^w\right)^n}$$

$$+\left(\frac{r}{1+r^a}\right)W_0L_0\left(1-t^w+b^w\right)\left[1+\frac{1}{1+\rho^w}+\left(\frac{1}{1+\rho^w}\right)^2+....+\left(\frac{1}{1+\rho^w}\right)^{n-1}\right]$$

$$-\left(\frac{r}{1+r^a}\right)PC_0\left[1+\frac{1}{1+\rho^c}+\left(\frac{1}{1+\rho^c}\right)^2+....+\left(\frac{1}{1+\rho^c}\right)^n\right]. \qquad (32)$$

From (27) and (31) it also follows that the wealth carried into year *n*+1 is

$$V_{n+1}=\left(1+r^a\right)\left(S_n+V_n\right)=\left(1+r^a\right)\left(1+g^w\right)^n B_0$$

$$+W_0L_0\left(1-t^w+b^w\right)\left[\left(1+r^a\right)^{n+1}+\left(1+r^a\right)^n\left(1+g^w\right)+\left(1+r^a\right)^{n-1}\left(1+g^w\right)^2+....+\left(1+r^a\right)^2\left(1+g^w\right)^{n-1}\right]$$

$$-PC_0\left[\left(1+r^a\right)^{n+1}+\left(1+r^a\right)^n\left(1+g^c\right)+\left(1+r^a\right)^{n-1}\left(1+g^c\right)^2+....+\left(1+r^a\right)\left(1+g^c\right)^n\right]. \qquad (33)$$

At the beginning of year *n*+1 the consumer's saving from non-capital income is

$$S_{n+1} = \left(1+g^{w}\right)^{n+1} B_0 - \left(1+g^{c}\right)^{n+1} PC_0, \tag{34}$$

and from (33) and (34) the present value of his capital income received at the end of year *n*+1 is found to be

$$R_{n+1} = \frac{r\left(S_{n+1}+V_{n+2}\right)}{\left(1+r^{a}\right)^{n+2}} = \left(\frac{r}{1+r^{a}}\right) B_0 \left[\left(\frac{1}{1+\rho^{w}}\right)^{n} + \left(\frac{1}{1+\rho^{w}}\right)^{n+1}\right]$$

$$+\left(\frac{r}{1+r^{a}}\right) W_0 L_0 \left(1-t^{w}+b^{w}\right)\left[1+\frac{1}{1+\rho^{w}}+\left(\frac{1}{1+\rho^{w}}\right)^{2}+....+\left(\frac{1}{1+\rho^{w}}\right)^{n-1}\right]$$

$$-\left(\frac{r}{1+r^{a}}\right) PC_0 \left[1+\frac{1}{1+\rho^{c}}+\left(\frac{1}{1+\rho^{c}}\right)^{2}+....+\left(\frac{1}{1+\rho^{c}}\right)^{n+1}\right]. \tag{35}$$

Equations (33) and (34) also imply that the wealth carried into year *n*+2 is

$$V_{n+2} = \left(1+r^{a}\right)\left(S_{n+1}+V_{n+1}\right) = \left(1+r^{a}\right)\left(1+g^{w}\right)^{n+1} B_0 + \left(1+r^{a}\right)^{2}\left(1+g^{w}\right)^{n} B_0$$

$$+W_0 L_0 \left(1-t^{w}+b^{w}\right)\left[\left(1+r^{a}\right)^{n+2}+\left(1+r^{a}\right)^{n+1}\left(1+g^{w}\right)+\left(1+r^{a}\right)^{n}\left(1+g^{w}\right)^{2}+....+\left(1+r^{a}\right)^{3}\left(1+g^{w}\right)^{n-1}\right]$$

$$-PC_0\left[\left(1+r^{a}\right)^{n+2}+\left(1+r^{a}\right)^{n+1}\left(1+g^{c}\right)+\left(1+r^{a}\right)^{n}\left(1+g^{c}\right)^{2}+....+\left(1+r^{a}\right)\left(1+g^{c}\right)^{n+1}\right]. \tag{36}$$

Proceeding in this manner, we find that the present value of capital income earned in year *n*+*m* of the retirement stage is

$$R_{n+m} = \frac{r\left(S_{n+m}+V_{n+m}\right)}{\left(1+r^{a}\right)^{n+m+1}} = \left(\frac{r}{1+r^{a}}\right) B_0 \left[\left(\frac{1}{1+\rho^{w}}\right)^{n} + \left(\frac{1}{1+\rho^{w}}\right)^{n+1} +....+\left(\frac{1}{1+\rho^{w}}\right)^{n+m}\right]$$

$$+\left(\frac{r}{1+r^{a}}\right) W_0 L_0 \left(1-t^{w}+b^{w}\right)\left[1+\frac{1}{1+\rho^{w}}+\left(\frac{1}{1+\rho^{w}}\right)^{2}+....+\left(\frac{1}{1+\rho^{w}}\right)^{n-1}\right]$$

$$-\left(\frac{r}{1+r^{a}}\right) PC_0 \left[1+\frac{1}{1+\rho^{c}}+\left(\frac{1}{1+\rho^{c}}\right)^{2}+....+\left(\frac{1}{1+\rho^{c}}\right)^{n+m}\right], \tag{37}$$

where $n+m \leq T-1$. In the last year of life where the consumer has reached the age of *T*, no capital income is earned since wealth is brought down to zero at the beginning of that year (recall that all payments except interest take place at the start of the year). From (32), (35) and (37) it then follows that the present value of the total capital income earned during retirement is

$$PVR^r = \sum_{t=n}^{T-1} R_t = \left(\frac{r}{1+r^a}\right) B_0 \left(\frac{1}{1+\rho^w}\right)^n$$

$$+\left(\frac{r}{1+r^a}\right) B_0 \left[\left(\frac{1}{1+\rho^w}\right)^n + \left(\frac{1}{1+\rho^w}\right)^{n+1}\right]$$

$$+\left(\frac{r}{1+r^a}\right) B_0 \left[\left(\frac{1}{1+\rho^w}\right)^n + \left(\frac{1}{1+\rho^w}\right)^{n+1} + \left(\frac{1}{1+\rho^w}\right)^{n+2}\right]$$

$$+\left(\frac{r}{1+r^a}\right) B_0 \left[\left(\frac{1}{1+\rho^w}\right)^n + \left(\frac{1}{1+\rho^w}\right)^{n+1} + \left(\frac{1}{1+\rho^w}\right)^{n+2} + ... + \left(\frac{1}{1+\rho^w}\right)^{T-1}\right]$$

$$+(T+1-n)\left(\frac{r}{1+r^a}\right) W_0 L_0 \left(1-t^w+b^w\right)\left[1+\frac{1}{1+\rho^w} + \left(\frac{1}{1+\rho^w}\right)^2 + ... + \left(\frac{1}{1+\rho^w}\right)^{n-1}\right]$$

$$-\left(\frac{r}{1+r^a}\right) PC_0 \left[1+\frac{1}{1+\rho^c} + \left(\frac{1}{1+\rho^c}\right)^2 + ... + \left(\frac{1}{1+\rho^w}\right)^{n}\right]$$

$$-\left(\frac{r}{1+r^a}\right) PC_0 \left[1+\frac{1}{1+\rho^c} + \left(\frac{1}{1+\rho^c}\right)^2 + ... + \left(\frac{1}{1+\rho^w}\right)^{n+1}\right]$$

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$$-\left(\frac{r}{1+r^a}\right) PC_0 \left[1+\frac{1}{1+\rho^c} + \left(\frac{1}{1+\rho^c}\right)^2 + ... + \left(\frac{1}{1+\rho^w}\right)^{T-1}\right]. \qquad (38)$$

Using formula (15) and the definition of H^w stated in (16), we find that (38) can be written as

$$PVR^r = (T+1-n)\left(\frac{r}{1+r^a}\right)\left(1-t^w+b^w\right) H^w$$

$$+\left(\frac{r}{1+r^a}\right)\left(\frac{1}{1+\rho^w}\right)^n \left[\frac{B_0}{1-\left(\frac{1}{1+\rho^w}\right)}\right]\left[T+1-n-\left(\frac{1-\left(\frac{1}{1+\rho^w}\right)^{T-n}}{1+\rho^w}\right)\right]$$

$$+\left(\frac{r}{1+r^a}\right)\left[\frac{PC_0}{1-\left(\frac{1}{1+\rho^c}\right)}\right]\left\{\left(\frac{1}{1+\rho^c}\right)^{n+1}\left[\frac{1-\left(\frac{1}{1+\rho^c}\right)^{T-n}}{1-\left(\frac{1}{1+\rho^c}\right)}\right]-(T+1-n)\right\}. \qquad (39)$$

Inserting the consumption function (16) in (39) and dividing through by H^w, we obtain

$$\frac{PVR^r}{H^w} = (T+1-n)\left(\frac{r}{1+r^a}\right)\left(1-t^w+b^w\right)$$

$$+\left(\frac{r}{1+r^a}\right)\left(\frac{1}{1+\rho^w}\right)^n\left[\frac{b^r}{1-\left(\frac{1}{1+\rho^w}\right)^n}\right]\left[T-n+\frac{\rho^w}{1+\rho^w}+\left(\frac{1}{1+\rho^w}\right)^{T+1-n}\right]$$

$$+\left(\frac{r}{1+r^a}\right)\left(\frac{\omega}{1-\left(\frac{1}{1+\rho^c}\right)}\right)\left(1-t^w+b^w+\frac{H^r}{H^w}\right)\left\{\frac{1}{\rho^c}\left[\left(\frac{1}{1+\rho^c}\right)^n-\left(\frac{1}{1+\rho^c}\right)^T\right]-(T+1-n)\right\}. \qquad (40)$$

III.2 The present value of total capital income earned over the life cycle

The parameter θ^s reflecting the timing of capital income relative to labour income over the life cycle is defined as

$$\theta^s = \frac{PVR^w + PVR^r}{H^w}. \qquad (41)$$

Substituting the expressions in (30) and (40) into (41) and collecting terms, we end up with

$$\theta^s = \left(\frac{r}{1+r^a}\right)\left[\left(1-t^w+b^w\right)X_1+\left(\frac{b^r}{\left(1+\rho^w\right)^n-1}\right)X_2\right]$$

$$-\left(\frac{r}{1+r^a}\right)\left(\frac{\omega}{\rho^c}\right)\left(1-t^w+b^w+\frac{H^r}{H^w}\right)X_3, \qquad (42)$$

where

$$X_1 \equiv T+1+\frac{n}{\left(1+\rho^w\right)^n-1}-\frac{1}{\rho^w}, \qquad (43)$$

$$X_2 \equiv T-n+\frac{\rho^w}{1+\rho^w}+\left(\frac{1}{1+\rho^w}\right)^{T+1-n}, \qquad (44)$$

$$X_3 \equiv \frac{\rho^c\left(T+1\right)-1+\left(\frac{1}{1+\rho^c}\right)^T}{1-\left(\frac{1}{1+\rho^c}\right)}. \qquad (45)$$

$$\frac{H^r}{H^w} = \frac{b^r\left[1-\left(\frac{1}{1+\rho^w}\right)^{T+1-n}\right]}{\left(1+\rho^w\right)^n-1}. \qquad (46)$$

The term on the right-hand side of the first line in (42) reflects the contribution of the consumer's income from labour and transfers to his wealth accumulation, while the term in the second line captures the negative impact of consumption on the accumulation of wealth. The result in (46) follows from the definitions stated in (16).

REFERENCE

Sørensen, Peter Birch (2011). Measuring the deadweight loss from taxation in a small open economy – a general framework with an application to Sweden. EPRU Working Paper Series 2011-03, Department of Economics, University of Copenhagen, October 2011.