Controlling inflation in a cointegrated vector autoregressive model with an application to US data

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Abstract

Based on the cointegrated VAR model we discuss how to control a nonstationary target variable to become stationary using another variable as an instrument. We give conditions under which this is feasible and show that in general a necessary condition is non-neutrality between instrument and target expressed as a non-zero element in the long-run impact matrix. An application to US data covering the Greenspan period shows a significant (but positive) long-run impact from the federal funds rate to inflation rate. Thus, inflation can be made stationary using an appropriate control rule and we demonstrate how. The estimated long-run impact of shocks to the federal funds rate (or the target rate) on the market interest rates were all insignificant implying that the latter cannot be directly controlled. Thus, based on this information set we do not find support for the widely held belief that the Federal Reserve Bank can bring US CPI inflation down in the long run by raising the federal funds rate nor that the short-term market rates would qualify as intermediate targets.

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1 Introduction

The last decade has witnessed a burgeoning of the literature on monetary policy reaction rules. The original idea, first suggested by Taylor (1993), was to describe a central bank's policy decisions by a simple rule that associated changes in the monetary instrument with the discrepancy between a target variable and its desired value and with the state of the economy summarized by some key macroeconomic variables. Further developments have included different models and approaches (forward looking rational expectations versus adaptive expectations, micro foundations versus no such foundations), see for example the collected papers in Taylor (1999), Clarida, Gali and Gertler (1998) and references therein. Most of the applications contain the following elements: a formulation of a monetary policy rule for the central bank, estimation of a model for the economy (usually a VAR model), derivation of the implications of the proposed rule for the variance of the target variable, and finally an investigation of whether monetary authorities have followed the suggested rule.

This literature does not, in general, examine the conditions on the parameters of the VAR model that would render the ultimate (nonstationary) goal variables controllable given the available instruments. Furthermore, most applications either assume that the variables are stationary or allow for the variables to be nonstationary but do not consider whether this may have consequences for the controllability of the goal variable.

The purpose of the present paper is to address these questions. In doing so we rely on the following basic assumptions: (1) the VAR model is capable of satisfactorily describing the dynamics of the data, (2) the central bank is free to change the value of its instruments, (3) the central bank changes the value of its instruments based on a linear control rule which takes into account the target variable and the state of the economy, and (4) the 'economy' variables, i.e. the non-instrument variables, take into account the central bank intervention implying that the equations for these variables change as a result of the intervention.

The time it takes for a policy intervention to influence the final target is often considered long and difficult to predict, whereas the response of an intermediate target is more immediate and, therefore, easier to estimate empirically. Moreover, for example when inflation rate is the goal variable, the interventions take place on a daily basis, but the inflation rate is only measured on a monthly basis. Hence, the direct evaluation of the final impact of a monetary intervention on the goal variable is often difficult. Because of this the assessment of monetary policy has often been based on models containing intermediate targets and goal variables, but not necessarily the instrument variables. The present paper shows that this can be a valid procedure provided the intermediate target can be controlled by the central bank authorities and provided it cointegrates with the final target.

In the first part of the paper we derive the theoretical conditions on the parameters of the VAR model under which a goal variable is controllable. For a nonstationary variable we find that a necessary condition for controllability is that there exists a significant long-run impact of shocks to the instrument variable on the target variable. Given controllability, we derive a suitable control rule for the instrument variable(s) with the following property: when applying the control rule at all points in time, a nonstationary target variable will become stationary with a desired mean and a stationary target variable will remain stationary and obtain the desired mean.

Having derived the conditions for controllability the second part of the paper applies these results to US data on real money, output, the inflation rate, the 10 year bond rate, the 3 and 6 month treasury bill rates, the federal funds rate, and the federal funds target rate. The idea is to investigate whether the Federal Reserve Bank during the Greenspan period had the ability to render the inflation rate stationary based on the estimated relationship between the instrument and the intermediate target and the transmission mechanisms of the economy. We find a significant (though positive) long-run impact from the federal funds rate to the inflation rate implying controllability of the inflation rate. Using the feasible control rule we show how the inflation rate would have become stationary by application of this rule. However, the estimated long-run impacts of shocks to the federal funds rate (or the target rate) on the three market interest rates were all insignificant. This surprising, but empirically robust result, suggests that the market rates have not been directly controllable in this period and, therefore, would not have qualified as intermediate targets.

Based on the present information set we do not seem to find support for the widely held belief that the Federal Reserve Bank had the ability to reduce the US CPI inflation rate in the long run by *raising* the federal funds rate. However, by including more information, for example from the labor market, it is possible that this result would change. It should also be pointed out that our results say nothing about whether the central bank was able to control the variation of the market interest rates and thus, possibly the variation of the inflation rate. The highly volatile short-term interest rates during the period of money targeting in the eighties as compared to the much more stable rates in the subsequent period of interest rate targeting are evidence in favor of this view. How to control the *variation* of a target variable is, however, outside the scope of this paper.

The structure of the paper is as follows: For ease of exposition Section 2 illustrates the concepts and the results using a simple policy control rule for the simple case of a VAR(1) model. Section 3 gives the general formulation of the control problem, finds conditions for controllability of a (non)stationary variable in a VAR model, derives a control rule by which a given variable is controllable, and shows that the application of the control rule introduces a new cointegrating relation in the system. Section 4 reports the empirical application, discusses the cointegration results, the estimated long-run impact matrix, and simulates the new process in which the inflation rate is stationary. Section 5 concludes. Appendix A provides some proofs and Appendix B contains various graphs of the data.

2 Definition of the control problem in the VAR(1) model

This section introduces the problems and the general methodology in a simple cointegrated VAR model with one lag and no trend and leaves the treatment of the general model with several lags and a trend for Section 3.

2.1 Properties of the VAR(1) model

We consider the $p-$dimensional VAR(1) model

$$\Delta x_{t+1} = \alpha(\beta' x_t - \mu) + \varepsilon_{t+1}, \tag{1}$$

where ε_t are i.i.d. $(0, \Omega)$ and α and β are $p \times r$ matrices and μ is $r \times 1$. It is well known that (1) defines a cointegrated $I(1)$ process with r cointegrating relations β, if and only if

$$\rho(I_r + \beta'\alpha) < 1, \tag{2}$$

where $\rho(M)$ is the largest absolute value of an eigenvalue of M, i.e.

$$\rho(M) = \max\{|\lambda| \text{ so that } det(\lambda I - M) = 0\}.$$

The condition implies that $\beta'\alpha$ is non-singular so that the process is represented as

$$x_t = C\sum_{i=1}^{t}\varepsilon_i + y_t + A + \alpha(\beta'\alpha)^{-1}\mu,$$

where A depends on initial values ($\beta' A = 0$) and y_t is stationary with mean zero. Hence $\beta' x_t = \beta' y_t + \mu$ is stationary with mean μ and the long-run impact matrix is

$$C = \beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp' = I_p - \alpha(\beta'\alpha)^{-1}\beta', \tag{3}$$

where $\alpha_\perp$ is a $p \times (p-r)$ matrix of full rank, so that $\alpha'\alpha_\perp = 0$. If $\alpha = 0$ (i.e. no error correction) then $\alpha_\perp = I_p$ (i.e. p autonomous stochastic trends in the data). If $\alpha = I_p$ (i.e. $x_t \sim I(0)$) then $\alpha_\perp = 0$ (no stochastic trends in the data). Condition (2) implies that $\beta'\alpha$ has full rank and, hence, that $|\alpha_\perp'\beta_\perp| \neq 0$, so that C is well defined. In this case $\beta' x_t$ is a VAR(1) process with coefficient matrix $(I_r + \beta'\alpha)$ and condition (2) rules out unit roots and explosive roots.

Figure 1 illustrates the adjustment dynamics of the process. For the simple VAR(1) the adjustment vectors $\pm\alpha$ point towards the attractor set

$$\{x|\beta' x = \mu\} = \alpha(\beta'\alpha)^{-1}\mu + sp(\beta_\perp), \tag{4}$$

and the process is pulled towards (4) with a force that depends on the magnitude of the distance $\beta' x_t - \mu$. Thus, for points on the attractor set the force is zero and there is no tendency to move away. Such points are called equilibrium or steady state positions. The common trends $\alpha_\perp' \sum_{i=1}^{t}\varepsilon_i$ push the process along the attractor set and generate the nonstationary behavior of the process.

The following result defines the long-run value of the process, that is, the value it would converge towards if the errors were switched off:

Lemma 1 *For the $I(1)$ process x_t given by (1) the expectation $E(x_{t+h}|x_t)$ converges to the long-run expected value defined by*

$$x_{t,\infty} = \lim_{h\to\infty} E(x_{t+h}|x_t) = Cx_t + \alpha(\beta'\alpha)^{-1}\mu,$$

which is a point in the attractor set.

We say that x_t aims at $x_{t,\infty}$, but, since the variance diverges, we do not say that x_{t+h} converges to the long-run value.

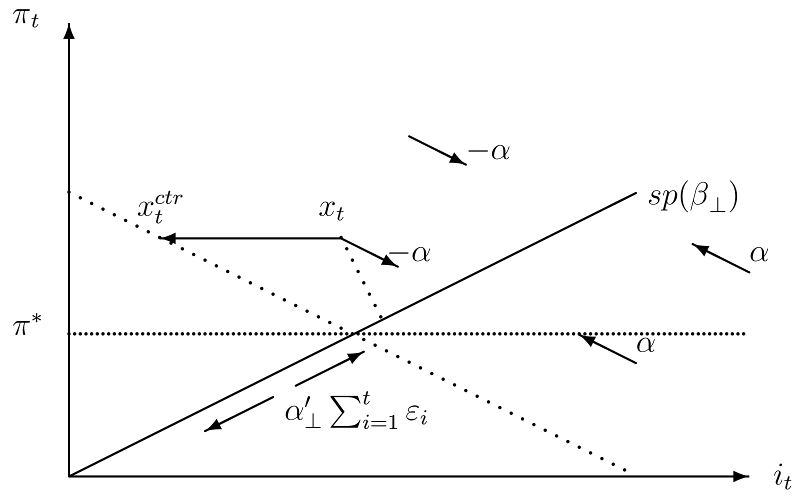

Figure 1: At the point $x_t = (i_t, \pi_t)$ the process is moved to x_t^{ctr} by changing the nominal interest rate i_t. If the errors were switched off, the process would move in a straight line from this position along the direction of $-\alpha$ to a point where inflation $\pi_t = \pi^*$.

2.2 The control problem

We discuss here a policy control situation where the central bank sets the value of the central bank instrument as a reaction to the current state of the economy in order to make a nonstationary target variable like the inflation rate stationary around a given value. Our aim is to formalize the impact on the dynamics of the system, when the central bank applies such a policy control rule at all points of time.

We let the central bank instrument be given by $a'X_t$ for a suitable unit vector $a \in R^p$ and define a linear control rule by

$$x_t^{ctr} = x_t + a(\kappa' x_t - \kappa^*),$$

where $\kappa \in R^p$ and $\kappa^* \in R$. Thus the control rule is linear in the deviation of $\kappa' x_t$ from a given value κ^*, measuring the current state of the economy. Note that $a'x_t^{ctr} = a'x_t + \kappa' x_t^{ctr} - \kappa^*$ shows that the deviation $\kappa' x_t^{ctr} - \kappa^*$ is added to the instrument variable at all time points. Because $a'_\perp x_t^{ctr} = a'_\perp x_t$, we also note that the remaining variables are unchanged.

After the bank has changed x_t to x_t^{ctr} the market is assumed to react on the controlled value. Equation (1), which describes the market behavior up to the adoption of the policy control rule, is assumed to remain unchanged, except that the market now reacts on x_t^{ctr} instead of x_t thereby generating new values of the process x_t^{new}. The steps describing the actions of the central bank and the actions of the market are modelled recursively:

- Based on the observation of the process generated by the market, x_t^{new}, the central bank makes an intervention at time t consistent with the control rule

$$x_t^{ctr} = x_t^{new} + a(\kappa' x_t^{new} - \kappa^*) = (I_p + a\kappa')x_t^{new} - a\kappa^*. \quad (5)$$

- The market on the other hand, in view of the value set by the bank, x_t^{ctr}, generates the next value of the new process, x_{t+1}^{new}, based on (1)

$$x_{t+1}^{new} = x_t^{ctr} + \alpha(\beta' x_t^{ctr} - \mu) + \varepsilon_{t+1} = (I_p + \alpha\beta')x_t^{ctr} - \alpha\mu + \varepsilon_{t+1}. \quad (6)$$

Figure 2 uses a simple bivariate process $x_t' = [x_{1,t}, x_{2,t}]$ to illustrate the steps. At time t the market produces x_t by taking into account the information x_{t-1}. At this point the bank intervenes and replaces x_{2t} by x_{2t}^{ctr}. At time $t+1$ the market produces the new values of the process x_{t+1} by taking into account the information (x_{1t}, x_{2t}^{ctr}). We define $x_{t+1}^{ctr} = (x_{1t+1}, x_{2t+1}^{ctr})$ and $x_{t+1}^{new} = (x_{1t+1}, x_{2t+1})$.

By eliminating x_t^{ctr} in (6), the equations for the observed process x_t^{new} become:

$$x_{t+1}^{new} = (I_p + \alpha\beta')(I_p + a\kappa')x_t^{new} - \alpha\mu - (I_p + \alpha\beta')a\kappa^* + \varepsilon_{t+1}. \quad (7)$$

The intuition behind the choice of a control rule of the form (5) is related to the general error correction property of the cointegrated VAR model (1). In this model the error correcting behavior of the market produces the stationarity of $\beta' x_t$ around its mean μ. In a policy control situation the error correcting behavior of the central bank to the discrepancy $\kappa' x_t^{ctr} - \kappa^*$ should, therefore, make $\kappa' x_t^{ctr}$ stationary around κ^*. For example, for $\kappa' x_t$ proportional to the inflation rate, κ^* would correspond to the desired target rate, and the control rule

$$i_t^{ctr} = i_t + \lambda(\pi^* - \pi_t)$$

would lead to a stationary inflation rate around the target rate.

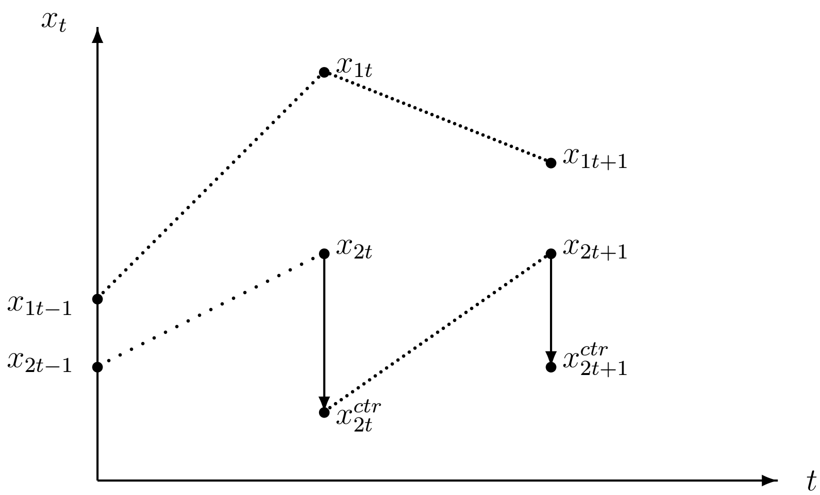

Figure 2: The timing of the interventions and the defintion of x_t^{new} and x_t^{ctr}. The variable x_2 is an instrument variable and x_1 the 'economy' variable. In the period $(t-1, t)$ the variables are moved by the market from x_{t-1} to x_t. At this point the bank intervenes and sets the value of x_{2t} to $x_{2,t}^{ctr}$. Then the market moves the variables from (x_{1t}, x_{2t}^{ctr}) to x_{t+1}, and again the bank intervenes and moves x_{2t+1} to x_{2t+1}^{ctr}. The process x_{t+1}^{ctr} is defined as $(x_{1t+1}, x_{2t+1}^{ctr})$ and x_{t+1}^{new} is defined as (x_{1t+1}, x_{2t+1}).

A major assumption in this paper is that the parameters of (6) are unchanged by the intervention and, thus, that the market reacts to the controlled value in the same way as without an intervention. However, the coefficients of the equations (7) for the observed process x_t^{new} are not the same as in the original process (1) implying that the interventions by the central bank change the properties of this process

Thus, by applying the policy rule the properties of the process x_t change, possibly in a destabilizing way. Therefore, we need to answer the following questions related to the dynamic properties of the new process:

- What are the implications of applying the control rule (5) at all time points for x_t^{new} defined by (6)? Under which conditions on the parameters of the VAR model does the new process x_t^{new} remain $I(1)$ (instead of explosive, say) with $\kappa' x_t^{new}$ as a stationary cointegration vector?

- Given a linear combination $b'x_t$ defining the (nonstationary) target variable, under which conditions is it possible to choose a control rule (κ, κ^*) which will make $b'x_t$ stationary with mean b^*, the target value?

Section 3 discusses how the properties of the new process depend on the relation between α and a and between β and κ and gives the answers to the above questions in Theorems 2, 5, and 6. In the subsections below we illustrate how to choose a simple control rule defined in terms of the long-run value of the process.

2.3 Illustrating a simple control rule

The problem considered here is how to control a nonstationary target variable $b'x_t$ using $a'x_t$ as an instrument. The idea, illustrated in Figure 1, is to move the process to a controlled position $x_t^{ctr} = x_t + v$, by choosing the intervention v such that, if the errors were switched off, the process $b'x_{t+h}$, $h = 1, 2, \ldots$, starting at $b'x_t^{ctr}$, would continue towards $b'x_{t,\infty} = b^*$. This implies that we choose the intervention so that

$$b^* = b'(Cx_t^{ctr} + \alpha(\beta'\alpha)^{-1}\mu) = b'(C(x_t + av) + \alpha(\beta'\alpha)^{-1}\mu), \tag{8}$$

see Lemma 1. For the equations to have a solution for v, the following condition

$$b'Ca \neq 0 \tag{9}$$

has to be satisfied. This defines a necessary condition for controllability, see Theorem 2, where the precise result is formulated. If $b'Ca \neq 0$, equation (8) can be solved for v which, using (3), gives us the control rule

$$\begin{aligned} x_t^{ctr} &= x_t + av \\ &= x_t + a(b'Ca)^{-1}(b^* - b'\alpha(\beta'\alpha)^{-1}\mu - b'Cx_t) \\ &= x_t + a(b'Ca)^{-1}[(b^* - b'x_t) + b'\alpha(\beta'\alpha)^{-1}(\beta'x_t - \mu)]. \end{aligned} \tag{10}$$

The intervention v in (10) needed for the process to be on the right track depends on

- $b'x_t - b^*$, which is the observed discrepancy between the value of the target variable $b'x_t$ and its desired target value b^*.
- $\beta'x_t - \mu$, which measures the deviation from the steady state value at the time of the intervention.

If the economy is in steady state, as defined by the attractor set $\{\beta'x = \mu\}$, then the discrepancy between the observed and the desired target value determines the magnitude of the necessary intervention. But if the economy is away from steady state, then the magnitude of the equilibrium error also affects the size of the intervention.

In this example a condition for controllability is defined by a condition on the elements of the long-run impact matrix C, defined by the orthogonal complements of α and β, see (3). Therefore, a stationary variable, being a linear combination of $\beta'x_t$, cannot be controlled by this rule, but Theorem 5 gives a result that can be applied in this case.

When there is only one target and one instrument, condition (9) requires that the long-run impact of a shock (an intervention) to the instrument variable on the target variable is nonzero. Therefore, controllability by this rule is inconsistent with long-run neutrality of target to instrument.

To summarize, we have seen that if $b'Ca \neq 0$, it is possible to define a natural control rule as:

$$x_t^{ctr} = x_t + a(b'Ca)^{-1}[(b^* - b'x_t) + b'\alpha(\beta'\alpha)^{-1}(\beta'x_t - \mu)].$$

The rule has the property that the position $b'x_t^{ctr}$ aims at the target value b^*, that is:

$$b'(Cx_t^{ctr} + \alpha(\beta'\alpha)^{-1}\mu) = b^*,$$

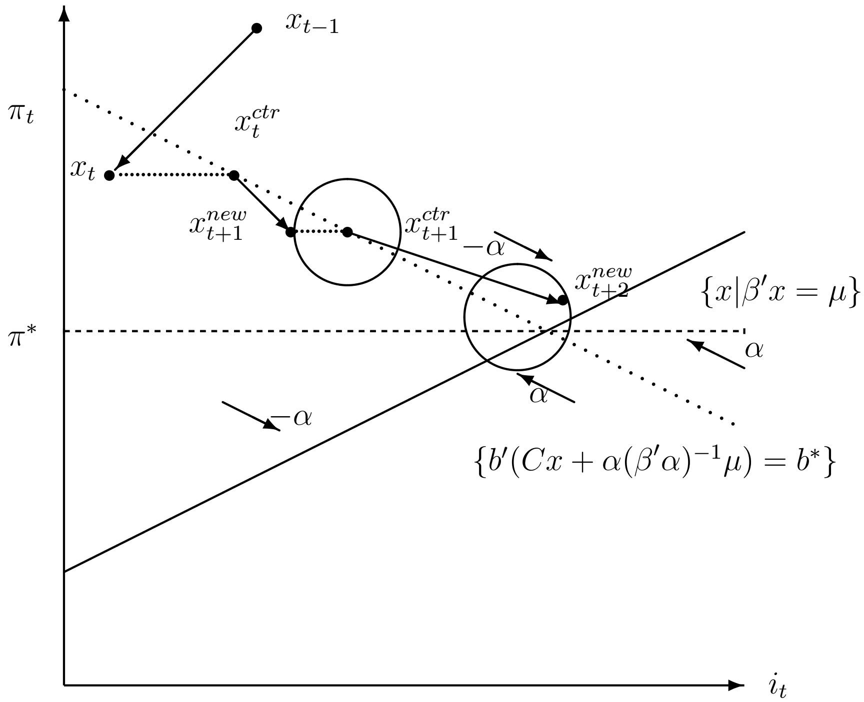

Figure 3: The point x_t is moved by the central bank to x_t^{ctr}, and the equations generate the point x_{t+1}^{new}, which in a VAR(2) model is based upon x_t^{ctr} and x_t. This point is moved to x_{t+1}^{ctr}, and the equations generate x_{t+2}^{new} from x_{t+1}^{ctr} and x_t^{ctr}. The dashed line is $\pi_t = \pi^*$, the dotted line is the set of points for which the long-run value is π^*, and finally the full line is the attractor set $\{x|\beta'x = \mu\}$.

and Theorem 4 proves that $b'x_t^{new}$ is stationary around b^*.

Figure 2 shows how the point x_t is moved by the central bank onto the dotted line. Once there, the dynamics of the process will take it towards the point on the attractor set defined by the desired target value π^*. This is so provided the central bank moves it back to the dotted line at all time points, i.e. uses the central bank interest rate in an error correcting manner. Therefore, if the bank introduces a feedback rule involving the target variable, for example $\pi_t - \pi_t^*$, π_t will be rendered stationary by the dynamics of the model.

This simple example illustrates that it is possible to control a nonstationary variable to become stationary around a desired value provided there is no long-run neutrality between the instrument and the target. Section 3 provides a general formulation of the control problem and gives the conditions for controllability summarized by Theorems 2, 5, and 6. Theorem 4 shows that even in the more general situation with more lags, it is possible to control a target variable using the idea (10) of the simple example.

2.4 Structural VAR

The results have so far been formulated for the reduced form VAR in equation (6), but it seems more plausible that both the central bank and the market take into account current as well as lagged values of the process when deciding on their actions. Therefore, we assume now that the equations are given by a structural VAR and that the new value of the process x_t^{new} is generated by the equations:

$$A_0 \Delta x_{t+1} = \alpha_0(\beta' x_t - \mu) + \varepsilon_{t+1}^0,$$

where the variance of ε_{t+1}^0 is diagonal. The control rule, which takes into account current values, is of a similar form as before:

$$x_t^{ctr} = x_t + a(\kappa' x_t - \kappa^*),$$

but the equation generating the new value now becomes:

$$A_0 x_{t+1}^{new} = A_0 x_t^{ctr} + \alpha_0(\beta' x_t^{ctr} - \mu) + \varepsilon_{t+1}^0.$$

Dividing through by A_0^{-1} the reduced form becomes:

$$x_{t+1}^{new} = x_t^{ctr} + A_0^{-1}\alpha_0(\beta' x_t^{ctr} - \mu) + A_0^{-1}\varepsilon_{t+1}^0,$$

which has the form (6) with $\alpha = A_0^{-1}\alpha_0$ and $\varepsilon_t = A_0^{-1}\varepsilon_{t+1}^0$ and the previous results can be applied directly. Thus, although the general results of Section 3 are formulated for the reduced form VAR they can be applied to the structural VAR as well.

2.5 Example

We conclude this section by a simple example which illustrates the implications of the results in Theorem 2. We consider as a stylized example the model for inflation, π_t, output gap, y_t, and interest rate, i_t, analyzed by Rudebusch and Svensson (1999). They assume that the (de-meaned) variables satisfy the equations:

$$\begin{aligned}
\pi_{t+1} &= \sum_{i=1}^{4} \alpha_{\pi i}\pi_{t-i+1} + \alpha_y y_t + \varepsilon_{t+1}, \\
y_{t+1} &= \beta_{y1} y_t + \beta_{y2} y_{t-1} - \beta_r(\bar{\imath}_t - \bar{\pi}_t) + \eta_{t+1},
\end{aligned}$$

where $\sum_{i=1}^{4}\alpha_{\pi i}=1$ (i.e. $\pi_t\sim I(1)$), $\bar{\pi}_t=\frac{1}{4}\sum_{i=0}^{3}\pi_{t-i}$, and similarly for $\bar{\imath}_t$. By setting $\alpha_{\pi 1}=1, \alpha_{\pi i}=0, i=2,3,4,$ $\beta_{y2}=0$, and replacing $(\bar{\pi}_t,\bar{\imath}_t)$ by (π_t,i_t) the example can be simplified to fit into the VAR(1) framework. To close the system an equation for the interest rate is added and a constant is included in the equations. This gives us the following system:

$$\begin{aligned}\pi_{t+1} &= \pi_t+\alpha_y(y_t-\mu_y)+\varepsilon_{t+1},\\ y_{t+1} &= \beta_y(y_t-\mu_y)-\beta_r(i_t-\pi_t+\mu_\pi)+\eta_{t+1},\\ i_{t+1} &= i_t+\delta_{t+1},\end{aligned}$$

which, for $x_t=(\pi_t,y_t,i_t)$, can be expressed as:

$$\Delta x_{t+1}=\begin{pmatrix}\alpha_y & 0\\ \beta_y-1 & -\beta_r\\ 0 & 0\end{pmatrix}\left[\begin{pmatrix}0 & -1\\ 1 & 0\\ 0 & 1\end{pmatrix}' x_t-\begin{pmatrix}\mu_y\\ -\mu_\pi\end{pmatrix}\right]+\begin{pmatrix}\varepsilon_{t+1}\\ \eta_{t+1}\\ \delta_{t+1}\end{pmatrix}.$$

The system of equations defines a cointegrated $I(1)$ process if

$$I_2+\beta'\alpha=\begin{pmatrix}\beta_{y1} & -\beta_r\\ -\alpha_y & 1\end{pmatrix}$$

has eigenvalues bounded by one. Under this assumption y_t and $i_t-\pi_t$ are stationary, but i_t and π_t are nonstationary and the long-run impact matrix is

$$C=\begin{pmatrix}0&0&1\\0&0&0\\0&0&1\end{pmatrix}.$$

According to Theorem 2 a central bank could achieve a desired inflation rate, $\pi^*=2\%$, say, by using the interest rate as an instrument and adopting the following control rule:

$$i_t^{ctr}=i_t+\lambda(\pi_t-\pi^*),$$

where λ should be chosen so that the new process remains $I(1)$ rather than becomes explosive, see condition (16). Note that the long-run impact of a shock to the interest rate on the inflation rate $b'Ca=1$ is positive in this case. In this particular example the process actually becomes stationary, as the cointegrating rank is extended from 2 to 3. Another possibility would be to use rule (10) which in this example becomes:

$$\begin{aligned}i_t^{ctr} &= i_t+(\pi^*-\pi_t)+(1,0,0)\alpha(\beta'\alpha)^{-1}(y_t-\mu_y,i_t-\pi_t+\mu_\pi)'\\ &= i_t+(\pi^*-\pi_t)-(i_t-\pi_t+\mu_\pi)\\ &= \pi^*-\mu_\pi.\end{aligned}$$

Thus, π_t can be made stationary around π^* by making i_t stationary around $\pi^* - \mu_\pi$.

3 A linear control rule in the VAR(k) model

This section formulates the control problem within the general VAR(k) model with a trend and discusses a control rule which can depend on current and lagged variables. This is done by expressing the VAR model in companion form, thereby reducing the control problem to a VAR(1) model with no trend. In this model we discuss how to control m target variables, $b'x_t$, given m instruments, $a'x_t$, and define two basic processes, the controlled process x_t^{ctr} and the new process x_t^{new}. The main results are formulated in Theorem 2, 5, and 6.

Note that we define the purpose of the control of a nonstationary variable as partly to remove the nonstationarity and partly to give it the desired mean. Another problem is how to choose the control rule so that the variance of the target variable is minimized, provided the first two goals have been achieved. It has been discussed for example in Preston and Pagan (1982) and will not be treated here.

3.1 Defining the general control problem

We assume that the $I(1)$ process x_t, for $t = 1, \dots, T$, is given by:

$$\Delta x_t - \gamma = \alpha(\beta'(x_{t-1} - \gamma(t-1)) - \mu) + \sum_{i=1}^{k-1} \Gamma_i(\Delta x_{t-i} - \gamma) + \varepsilon_t, \qquad (11)$$

where Δx_t and $\beta' x_t$ are stationary and $\alpha'_\perp(I_p - \sum_{i=1}^{k-1} \Gamma_i)\beta_\perp$ has full rank (see Johansen 1996, Theorem 4.2). We define $\bar{\alpha} = a(a'a)^{-1}$ so that $a'\bar{a} = I_m$. The process x_t^{ctr} defines the actions by the central bank:

$$x_t^{ctr} = x_t^{new} + \bar{a}\kappa'(x_t^{new} - \gamma t) + \bar{a}(\sum_{i=1}^{k-1} \kappa'_i(x_{t-i}^{ctr} - x_t^{new} + \gamma i) - \kappa^*), \qquad (12)$$

and the process x_t^{new} defines the actions by the market:

$$x_{t+1}^{new} = (I_p + \alpha\beta')x_t^{ctr} - \alpha\beta'\gamma t - \alpha\mu + \sum_{i=1}^{k-1} \Gamma_i(\Delta x_{t+1-i}^{ctr} - \gamma) + \gamma + \varepsilon_{t+1}, \quad (13)$$

where $\kappa, \kappa_1, \ldots, \kappa_{k-1}$ are $p \times m$ matrices, and κ^* is $m \times 1$. We assume that the central bank makes the policy decision based on the present value of x_t^{new} and past values of x_t^{ctr} and that the market generates the next new value x_{t+1}^{new} based on current and past values of x_t^{ctr}. The basic concepts of the control problem can now be formally defined.

Definition 1

(i) Instrument variables $a'x_t, (a\,(p \times m))$ have the property that one can change their value by an intervention, so that the value $a'x_t$ can be replaced by $a'x_t + v$, for any $v \in R^m$.

(ii) Final target variables, or just target variables, $b'x_t$, $(b\,(p \times m))$ are the variables one would like to influence so that $b'x_t^{new}$ (possibly trend adjusted) becomes stationary with mean b^*, the desired target value, using the control rule (12) and the instruments $a'x_t$.

(iii) Intermediate target variables $c'x_t$, $(c\,(p \times m))$ are variables that are cointegrated with the final target $b'x_t$, so that there exists a stationary relation $c'x_t + \phi b'x_t$ where the matrix $\phi\,(m \times m)$ has full rank.

(iv) A set of target variables $b'x_t$ with target value b^* is controllable by the instrument variable $a'x_t$ and the control rule $(\kappa, \kappa_1, \ldots, \kappa_{k-1}, \kappa^*)$ if, by intervening at all time points using the control rule (12), the new process defined by (13) is $I(1)$ and has the property that the target, $b'x_t^{new}$ (possibly trend adjusted) becomes stationary with mean b^*.

It is convenient for the formulation of the main results to introduce the stacked process $\tilde{x}_t$ and the companion form of model (11). We define

$$\tilde{x}_t = \begin{pmatrix} x_t - \gamma t \\ x_{t-1} - \gamma(t-1) \\ \vdots \\ x_{t-k+1} - \gamma(t-k+1) \end{pmatrix}, \tilde{\varepsilon}_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; \tilde{\mu} = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

so that (11) becomes

$$\Delta\tilde{x}_t = \tilde{\alpha}(\tilde{\beta}'\tilde{x}_{t-1} - \tilde{\mu}) + \tilde{\varepsilon}_t,$$

where

$$\tilde{\alpha} = \begin{pmatrix} \alpha & \Gamma_1 & \dots & \Gamma_{k-1} \\ 0 & I_p & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & I_p \end{pmatrix}, \ \tilde{\alpha}_\perp = \begin{pmatrix} \alpha_\perp \\ -\Gamma_1' \alpha_\perp \\ \vdots \\ -\Gamma_{k-1}' \alpha_\perp \end{pmatrix},$$

$$\tilde{\beta} = \begin{pmatrix} \beta & I_p & \dots & 0 \\ 0 & -I_p & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & -I_p \end{pmatrix}, \ \tilde{\beta}_\perp = \begin{pmatrix} \beta_\perp \\ \beta_\perp \\ \vdots \\ \beta_\perp \end{pmatrix}.$$

Thus, the cointegration rank of $\tilde{x}_t$ is $r + (k-1)p$, and we find

$$\tilde{\alpha}_\perp' \tilde{\beta}_\perp = \alpha_\perp' (I_p - \sum_{i=1}^{k-1} \Gamma_i) \beta_\perp = \alpha_\perp' \Gamma \beta_\perp,$$

which for an $I(1)$ process has full rank. Finally, we find the long-run impact matrix

$$\tilde{C} = \tilde{\beta}_\perp (\tilde{\alpha}_\perp' \tilde{\beta}_\perp)^{-1} \tilde{\alpha}_\perp' = \begin{pmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{pmatrix} C \begin{pmatrix} I_p \\ -\Gamma_1' \\ \vdots \\ -\Gamma_{k-1}' \end{pmatrix}', \tag{14}$$

where

$$C = \beta_\perp (\alpha_\perp' \Gamma \beta_\perp)^{-1} \alpha_\perp'.$$

The process x_t can be recovered from the first p components of $\tilde{x}_t$. We now define the extended instrument, target and control matrices of dimension $pk \times m$

$$\tilde{a} = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tilde{b} = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \tilde{\kappa} = \begin{pmatrix} \kappa - \sum_{i=1}^{k-1} \kappa_i \\ \kappa_1 \\ \vdots \\ \kappa_{k-1} \end{pmatrix}.$$

The equations (12) and (13) can be written in the form (5) and (6) using $(\tilde{\alpha}, \tilde{\beta}, \tilde{a}, \tilde{\kappa}, \tilde{\mu}, \kappa^*)$ and

$$\tilde{x}_t^{new} = \begin{pmatrix} x_t^{new} - \gamma t \\ x_{t-1}^{ctr} - \gamma(t-1) \\ \vdots \\ x_{t-k+1}^{ctr} - \gamma(t-k+1) \end{pmatrix}, \ \tilde{x}_t^{ctr} = \begin{pmatrix} x_t^{ctr} - \gamma t \\ x_{t-1}^{ctr} - \gamma(t-1) \\ \vdots \\ x_{t-k+1}^{ctr} - \gamma(t-k+1) \end{pmatrix}.$$

3.2 Controllability of a nonstationary variable when $|b'Ca| \neq 0$

We first consider the situation where the target variable $b'x_t$ is nonstationary and the condition $|b'Ca| \neq 0$ is satisfied.

Theorem 2 *If x_t is a VAR(k) process given by (11), and we apply the recursively defined control rule (12), then the stacked process $\tilde{x}_t^{new}$ defined by (13) is given by the VAR(1) model*

$$\tilde{x}_{t+1}^{new} = (I_{kp} + \tilde{\alpha}\tilde{\beta}')(I_{kp} + \bar{\tilde{a}}\tilde{\kappa}')\tilde{x}_t^{new} - \tilde{\alpha}\tilde{\mu} - (I_{kp} + \tilde{\alpha}\tilde{\beta}')\bar{\tilde{a}}\kappa^* + \tilde{\varepsilon}_{t+1}. \qquad (15)$$

The stacked process $\tilde{x}_t^{new}$ is an $I(1)$ process with $r+m+(k-1)p$ cointegrating relations if and only if

$$\rho\begin{pmatrix} I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha} & (I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha})\tilde{\beta}'\bar{\tilde{a}} \\ \tilde{\kappa}'\tilde{\alpha} & I_m + \tilde{\kappa}'(I_{pk} + \tilde{\alpha}\tilde{\beta}')\bar{\tilde{a}} \end{pmatrix} < 1. \qquad (16)$$

In this case

$$|\kappa'Ca| \neq 0, \qquad (17)$$

and $\beta'(x_t^{new} - \gamma t) - \mu$ and $\kappa'(x_t^{new} - \gamma t) - \kappa^*$ are stationary with mean zero. Furthermore, the variables

$$x_{t-i}^{ctr} - x_t^{new} + \gamma i, \;\; i = 1, \ldots, k-1,$$

are stationary with mean zero. The stacked process $\tilde{x}_t^{new}$ has a long-run impact matrix given by

$$\tilde{C}^{new} = \begin{pmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{pmatrix} (C - Ca(\kappa'Ca)^{-1}\kappa'C) \begin{pmatrix} I_p \\ -\Gamma_1' \\ \vdots \\ -\Gamma_{k-1}' \end{pmatrix}'. \qquad (18)$$

The m nonstationary target variables $b'(x_t - \gamma t)$ can be made stationary with mean b^* if $(\tilde{\kappa}, \kappa^*)$ can be chosen so that, $\kappa = b\lambda, \kappa^* = \lambda' b^*$, for some $\lambda \in R^{m\times m}$ provided (16) is not violated. Thus, a necessary condition is that $|b'Ca| \neq 0$.

The proof is given in the Appendix. The implication of this result is that if $(\tilde{\kappa}, \kappa^*)$ is introduced as a control rule $\kappa'(x_t^{new} - \gamma t)$ becomes stationary provided condition (16) is satisfied. For example, a non-stationary inflation rate can, under certain conditions, be made stationary around a given value by changing the interest rate proportional to the deviation of the actual inflation rate from the desired value. The necessary condition (17), i.e. $|b'Ca| \neq 0$, is needed to rule out explosive behavior.

In the simple case of one target and one instrument, the condition requires that the long-run impact of a shock (an intervention) to the instrument variable on the target variable must be non-zero. Therefore, controllability as defined here is inconsistent with long-run neutrality of target to instrument. The role of the coefficient λ in the control of $b'x_t$ is to secure that the policy rule does not generate explosive roots, and at the same time to secure the stationarity of $b'(x_t^{new} - \gamma t)$.

The role of the intermediate target is given in

Proposition 3 *Under the assumptions of Theorem 2, let $c'x_t$ be an intermediate target and $b'x_t$ the final target. If $c'(x_t - \gamma t)$ is controllable by the instrument $a'x_t$, then the final target $b'(x_t - \gamma t)$ is controllable by $a'x_t$.*

Proof. Let $\gamma = 0$ and let b^* be the desired target for $b'x_t$. Because $\phi b'x_t + c'x_t$ is stationary with some mean τ, say, then it follows from Theorem 2 that $\phi b'x_t^{new} + c'x_t^{new}$ is also stationary with mean τ. Because $c'x_t$ is controllable we choose to control it to become stationary with mean $\tau - \phi b^*$. It follows that $b'x_t$ becomes stationary around b^*. ∎

Theorem 2 shows that under given conditions, the control rule $(\tilde{\kappa}, \kappa^*)$ makes $\kappa'(x_t - \gamma t)$ stationary around κ^*. Therefore, to make $b'x_t$ stationary we can simply choose $\tilde{\kappa}$ so that $\kappa = b\lambda$, but the results simplify considerably if $\tilde{\kappa}'\tilde{\alpha} = 0$, see condition (16). Since $\tilde{\beta}'\tilde{x}_t^{new}$ remains stationary, the choice of $\tilde{\kappa} = \tilde{b}\lambda + \tilde{\beta}\xi$, for some ξ, would also make $\tilde{b}'\tilde{x}_t^{new} = b'(x_t^{new} - \gamma t)$ stationary. Therefore, by choosing ξ so that $\tilde{\kappa}'\tilde{\alpha} = \lambda'\tilde{b}'\tilde{\alpha} + \xi'\tilde{\beta}'\tilde{\alpha} = 0$, we get $\xi' = -\lambda'\tilde{b}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}$ and $\tilde{\kappa}' = \lambda'\tilde{b}' - \lambda'\tilde{b}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\beta}' = \lambda'\tilde{b}'\tilde{C}$. The parameter λ is chosen as $\lambda = -(b'Ca)^{-1}$ in the next result, which extends the result in (10) to the case with more lags.

Theorem 4 *Let x_t be an $I(1)$ process given by (11), and assume that a and* *b satisfy* $|b'Ca| \neq 0$. *If we apply the recursively defined control rule*

$$\begin{aligned} x_t^{ctr} &= x_t^{new} - a(b'Ca)^{-1}b'C[\Gamma(x_t^{new} - \gamma t) \\ &\quad - \textstyle\sum_{i=1}^{k-1} \Gamma_i(x_{t-i}^{ctr} - x_t^{new} + \gamma i) - (b^* - b'(I_p - C\Gamma)\bar{\beta}\mu)] \\ &= x_t^{new} - a(b'Ca)^{-1}[b'(x_t^{new} - \gamma t) - b^* \\ &\quad + b'(I_p - C\Gamma)\bar{\beta}(\beta'(x_t^{new} - \gamma t) - \mu) - b'C \textstyle\sum_{i=1}^{k-1} \Gamma_i(x_{t-i}^{ctr} - x_t^{new} + \gamma i)] \end{aligned} \tag{19}$$

then the stacked process $\tilde{x}_t^{new}$ *is a cointegrated VAR(1) process and* $\beta'(x_t^{new} - \gamma t)$ *and* $b'(x_t^{new} - \gamma t)$ *are stationary around* μ *and* b^*. *Moreover* $b'C(x_t^{new} - \sum_{i=1}^{k-1} \Gamma_i x_{t-i}^{ctr})$ *is white noise around its mean.*

The proof is given in the Appendix. The next subsection discusses two other situations where controllability can be achieved, but at the price of introducing a trend in the first case and of loosing cointegration in the second case.

3.3 Controllability when $|b'Ca| = 0$

If $|b'Ca| = 0$, we define two particular cases: $b'\beta_\perp = 0$ and $\alpha'_\perp a = 0$, in which we can achieve controllability under suitable conditions. We define the matrix

$$\tilde{\eta} = (I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha})(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\beta}'\bar{\tilde{a}},$$

which is used below to formulate some conditions. The proofs are given in the Appendix.

Theorem 5 *Let* x_t *be a VAR(k) process given by (11), and let us apply the recursively defined control rule (12). If* $\beta'_\perp \kappa = 0$, *so that* $\kappa' Ca = 0$, *then* $\kappa'(x_t - \gamma t)$ *is stationary. In this case the new process,* $\tilde{x}_t^{new}$, *is a cointegrated* $I(1)$ *process with cointegration rank* $r + (k-1)p$ *if and only if*

$$\rho(I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha} + \tilde{\eta}\tilde{\kappa}'\tilde{\alpha}) < 1. \tag{20}$$

The cointegrating relations are $\tilde{\beta}^{new} = \tilde{\beta}$ *and the adjustment vectors are*

$$\tilde{\alpha}^{new} = \tilde{\alpha}(I_r + \tilde{\beta}'\bar{\tilde{a}}\tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}) + \bar{\tilde{a}}\tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}. \tag{21}$$

In general $\tilde{x}_t^{new}$ has a trend, but $\kappa'(x_t^{new} - \gamma t)$ is stationary with mean given by

$$E(\tilde{\kappa}'\tilde{x}_t) + (I_m + \tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\eta})^{-1}\tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\eta}(\kappa^* - E(\tilde{\kappa}'\tilde{x}_t)).$$

The m stationary target variables $b'x_t$, with $b'\beta_\perp = 0$, so that $b'Ca = 0$, can be controlled to have mean b^* if the $m \times m$ matrix

$$\Theta = \tilde{b}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\eta} \tag{22}$$

has full rank and the control rule $(\tilde{\kappa}, \kappa^*)$ is chosen so that $\tilde{\kappa} = \tilde{b}\lambda$, that is, $\kappa = \lambda b, \kappa_i = 0$, (20) is satisfied, and finally

$$\kappa^* = \lambda' b^* - \Theta^{-1}E(b'(x_t - \gamma t) - b^*). \tag{23}$$

The result of this theorem allows us to control a stationary target variable by changing the dynamics of the process to produce the desired mean without changing the stationarity of the variable. The value of κ^* shows how the deviation of $E(b'(x_t - \gamma t))$ from the desired target b^* influences κ^*. Note that the control is only possible when Θ has full rank which need not be the case. For example, it is not possible to control $b'x_t$ if $k = 1$ and $\beta' x_t$ is white noise, because then $I_r + \beta'\alpha = 0$ and $\Theta = 0$. Note also that the price paid for changing the mean of a stationary target variable is to introduce a new trend in the process.

Theorem 6 *If x_t is a VAR(k) process given by (11), and we apply the recursively defined control rule (12), and if $\alpha'_\perp a = 0$, so that $\kappa' Ca = 0$, then the instrument is already part of α ($sp(a) \subset sp(\alpha)$). In this case the new process $\tilde{x}_t^{new}$ is an $I(1)$ process with cointegration rank $r + (k-1)p$, if and only if (20) holds. In this case $\tilde{\alpha}^{new} = \tilde{\alpha}$, and $\tilde{\beta}^{new} = \tilde{\beta} + \tilde{\kappa}\tilde{\eta}'$. Thus $\tilde{\beta}\overline{\tilde{\eta}}$ is replaced by $\tilde{\beta}\overline{\tilde{\eta}} + \tilde{\kappa}$. The constant term is $-\tilde{\alpha}(\tilde{\mu} + \tilde{\eta}\kappa^*)$, so that $\tilde{x}_t^{new}$ has no trend, and $E(\beta^{new\prime}(x_t^{new} - \gamma t))$ is given by the first r components of $\tilde{\mu} + \tilde{\eta}\kappa^*$. In particular*

$$E((\tilde{\beta}\overline{\tilde{\eta}} + \tilde{\kappa})'\tilde{x}_t^{new}) = \overline{\tilde{\eta}}'E(\tilde{\beta}'\tilde{x}_t) + \kappa^*.$$

In this case the m target variables $b'(x_t - \gamma t)$ can be made stationary with mean b^* if $\tilde{\eta}$ has rank m, and the control rule $(\tilde{\kappa}, \kappa^*)$ is chosen so that (20) is satisfied and $\tilde{\kappa} = \tilde{b}\lambda - \tilde{\beta}\overline{\tilde{\eta}}$, and $\kappa^* = \lambda' b^* - \overline{\tilde{\eta}}'E(\tilde{\beta}'\tilde{x}_t)$ for some $\lambda \in R^{m \times m}$.

The result here shows that if we want to control a non-stationary variable $b'(x_t - \gamma t)$, but the instrument is part of α, then this is possible, but we introduce the new cointegrating relation $(\tilde{\beta}\bar{\bar{\eta}} + \tilde{\kappa})'\tilde{x}_t^{new}$ at the price of eliminating one of the existing relations, $\tilde{\beta}\bar{\bar{\eta}}$, in order to achieve the goal. Notice that it does not in general hold that $\tilde{\kappa}$ is a cointegrating relation for $\tilde{x}_t^{new}$. Thus when applying the result to control a given target $b'x_t$ we do not choose $\tilde{\kappa}$ equal to $\tilde{b}\lambda$, but rather find a $\tilde{\kappa}$ so that $\tilde{\beta}\bar{\bar{\eta}} + \tilde{\kappa} = \tilde{b}\lambda$, as this will make $\tilde{b}'\tilde{x}_t^{new} = b'(x_t^{new} - \gamma t)$ stationary.

4 A monthly VAR model for US monetary data

We apply the VAR model (11) to the following monthly monetary US variables

$$x_t' = [m - p, y, \Delta p, R3, R6, B10, Ff, Trg]_t,$$

observed for the period 1985:8 to 1999:2, a total of 163 observations[1]. All data series are from the database EcoWin (1999). The variable m denotes the log of the average money stock (M3), p is the log of the monthly CPI, y is the log of real GDP[2], Δp denotes the monthly inflation rate, $R3_t$, $R6_t$ are the three and six month tbill rates, $B10$ is the 10 year bond rate, Ff_t denotes the federal funds rate, and Trg_t is the federal funds target rate. The last four variables are monthly averages of daily observations. The nominal interest rates have been transformed to monthly rates and divided by 100 to achieve comparability with log price changes. Graphs of the monthly data are given in Appendix B, Figures 8 and 8.

The domestic monetary transmission mechanism describes how changes in the intermediate targets - trend-adjusted money $(m - m^*)$ and the short-term interest rate (R_s) - dynamically affect the domestic economy through the subsequent adjustment of the long-term interest rates (R_l), the output gap $(y-y^*)$, and finally the inflation rate (Δp). The following simple diagram serves as an illustration:

[1] The availability of the federal target rate is the motivation for beginning the sample at 1985:8.

[2] The monthly GDP values have been interpolated from quarterly data using the procedure interpolate.scr in RATS, Estima, assuming a random walk + drift model.

$$
\begin{array}{ccccccc}
 & Central\ bank\, interv. & & & The\, market & & \\
 & \swarrow \quad \searrow & & & \swarrow \quad \searrow & & \\
\overset{(-)}{m - m^*} & \leftrightarrows & \overset{(+)}{R_s} & \longrightarrow \overset{(+)}{R_l} & \longrightarrow \overset{(-)}{y - y^*} & \longrightarrow & \overset{(-)}{\Delta p}
\end{array}
\tag{24}
$$

The theoretical results in Section 3 demonstrated that the central bank instrument, here the federal funds rate, can be used either directly to control the final goal variable, here the inflation rate, or indirectly to control an intermediate target, here the short-term market rates. In the first case we need to establish controllability of the inflation rate by the federal funds rate as a condition on the C matrix. In the second case we need to establish cointegration between the intermediate target and the inflation rate as well as controllability of the intermediate target by the federal funds rate. In either case, given that the conditions for controllability are satisfied, we can find a natural control rule for the inflation rate as a function of the discrepancy between the actual and desired (intermediate) target and the equilibrium errors from the economy-wide steady-state relations, see (19).

4.1 Model specification

The federal funds target rate cannot be considered a true stochastic process generated from Gaussian errors, as it has often been kept fixed for weeks at a time. Nonetheless, based on the graph in Appendix B, Figure 8, it seems reasonably safe to treat the monthly average as a stochastic process. Because the federal funds target rate is likely to contain information on how the Fed has responded to changing market conditions, i.e. on an implicit policy rule, the federal funds target rate has been included as a system variable in the VAR model[3].

Three dummies were needed to account for the following outliers:

$$D_t' = [D86.03, D86.12, D87.11],$$

where Dxx, yy describes a permanent impulse dummy at time 19xx.yy. All empirical results are based on a VAR(2) model which was tested for misspecification based on a number of multivariate tests. Calculations were

[3] As a sensitivity check the analysis was done setting the target rate as an exogenous variable. Since the main conclusions remained unaltered, we report only the results of the full system analysis.

Table 1: Misspecification tests, characteristic roots, and weak exogeneity for the monthly data

Univ. tests:	Δm_t	Δy_t	$\Delta^2 p_t$	$\Delta R3_t$	$\Delta R6_t$	$\Delta B10_t$	ΔFf_t	ΔTrg_t
ARCH(2)	1.6	0.2	0.1	**15.3**	1.8	0.1	2.1	**13.7**
J.-B.(2)	0.6	4.8	6.9	3.6	0.0	0.8	1.0	**30.0**
$\hat{\sigma}_\varepsilon \times 100$	0.26	0.15	0.13	0.01	0.01	0.02	0.01	0.01
R^2	0.75	0.66	0.70	0.49	0.42	0.37	0.67	0.52
$Tracetest:$	335	234	156	98	**52**	24	11	3
(90% quantile)	(182)	(146)	(115)	(87)	(63)	(42)	(25)	(12)
The roots:	1.0	1.0	1.0	1.0	0.91	0.80	0.80	0.43
Test of weak exogeneity for $r = 4$:								
$\chi^2_{0.95}(4) = 9.5$	25.5	27.6	40.2	**8.6**	**2.7**	**4.0**	30.9	23.1

performed using CATS in RATS (Hansen and Juselius, 1994) and GiveWin (Doornik and Hendry, 1998).

The main specification results are presented in Table 2. The multivariate LM test for first order residual autocorrelation, distributed as $\chi^2(64)$, gave a test statistic of 81.0 and was accepted with a $p-$value of 0.07. The multivariate normality test $\chi^2(16)$ was rejected based on a test statistic of 87.7. The univariate Jarque-Bera tests for normality indicate that the non-normality is exclusively in the equation for the federal funds target rate, which as argued above is not truly a stochastic process. Univariate residual ARCH tests similarly accept for all equations except the federal funds target rate and the 3 month tbill rate. However, the asymptotic analysis of cointegration remains valid for non normal errors and even ARCH effects, see Dennis, Hansen and Rahbek (2002), so we shall proceed with the analysis. Additionally, Table 2 reports the trace tests for cointegration rank, the roots of the VAR(2) model and weak exogeneity tests. The trace test suggests four cointegrating relations and, hence, four common trends. With the choice of $r = 4$, the modulus of the largest stationary root is 0.91 in the model.

It appears that the 6 month tbill rate and the 10 year bond rate individually can be assumed weakly exogenous for the long-run parameters β, whereas the 3 month tbill rate is a borderline case. The test of joint exogeneity of the first two gave a test statistic of $9.0 < \chi^2_{0.95}(8) = 15.5$ and a p-value of 0.34, whereas the test that all three are jointly exogenous was strongly rejected based on a test statistic of $53.0 > \chi^2_{0.95}(12)$ and a p-value of 0.00. The finding that there are two common stochastic trends between the interest rates suggests the following trend representation:

$$\begin{bmatrix} Ff_t \\ R3 \\ R6 \\ B10 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix} \begin{matrix} \sum \varepsilon_{R6,i} \\ \sum u_{10,i} \end{matrix} + ...$$

Thus, consistent with the analysis below it will not, in general, be possible to find cointegration between pairs of interest rates. Since most theory models assume that there is just one autonomous shock, the monetary policy shock, influencing the whole term structure from the short end to the long, this result may seem surprising. However, under the assumption that shocks to the 6 month tbill rate reflect expectations in the financial market to monetary policy changes, that shocks to the 10 year bond rate reflect long-term inflationary expectations, and that these expectations have been exogenous to the system, the exogeneity finding makes perfectly sense. For example, financial market behavior based on imperfect knowledge expectations would be consistent with this outcome. See for example Frydman and Goldberg (2002).

4.2 Cointegration results

According to Definition 1, a necessary condition for controllability of a final target through the intermediate target is cointegration between an intermediate target and the final target. More generally, the cointegration properties between instruments, intermediate and final targets contain important information on the monetary transmission mechanism as illustrated by the simple diagram at the beginning of this section. Therefore, we first test cointegration between the inflation rate and each of the interest rates and then between the interest rates relative to each other.

These results are reported in Table 2 where the hypotheses $\mathcal{H}_1 - \mathcal{H}_{15}$ are of the form $\beta = \{H\phi, \psi\}$, i.e. they test restrictions on a single vector and leave the other vectors unrestricted (Johansen and Juselius, 1992). Except for $\mathcal{H}_{15}$, $H\phi$ tests whether two variables are cointegrated $(1, a)$, where a is freely estimated. If there exists cointegration between pairs of variables, this procedure should find it.

The hypotheses $\mathcal{H}_1 - \mathcal{H}_5$ are tests of cointegration between the inflation rate and each of the interest rates. There is weak evidence that the inflation rate is cointegrated (1, -0.4) with the federal funds rate (or the target rate).

Table 2: Testing pairwise cointegration properties. Monthly data

	m-p	*y*	*Δp*	*R3*	*R6*	*B10*	*Ff*	*Trg*	$\chi^2(v)$	*p.val.*
$\mathcal{H}_1$	0	0	1	-0.43	0	0	0	0	11.4(4)	0.02
$\mathcal{H}_2$	0	0	1	0	-0.45	0	0	0	11.5(4)	0.02
$\mathcal{H}_3$	0	0	1	0	0	-0.56	0	0	18.8(4)	0.00
$\mathcal{H}_4$	0	0	1	0	0	0	-0.37	0	8.9(4)	0.06
$\mathcal{H}_5$	0	0	1	0	0	0	0	-0.37	8.5(4)	0.07
$\mathcal{H}_6$	0	0	0	1	-1.04	0	0	0	19.4(4)	0.00
$\mathcal{H}_7$	0	0	0	1	0	-1.56	0	0	26.5(4)	0.00
$\mathcal{H}_8$	0	0	0	1	0	0	-0.81	0	20.2(4)	0.00
$\mathcal{H}_9$	0	0	0	1	0	0	0	-0.81	20.1(4)	0.00
$\mathcal{H}_{10}$	0	0	0	0	1	-0.66	0	0	26.2(4)	0.00
$\mathcal{H}_{11}$	0	0	0	0	1	0	-0.78	0	15.5(4)	0.00
$\mathcal{H}_{12}$	0	0	0	0	1	0	0	-0.79	15.8(4)	0.00
$\mathcal{H}_{13}$	0	0	0	0	0	1	-0.53		23.9(4)	0.00
$\mathcal{H}_{14}$	0	0	0	0	0	1	0	-0.53	23.8(4)	0.00
$\mathcal{H}_{15}$	0	0	0	0	0	0	1	-1	0.81(5)	**0.98**

Among the remaining hypotheses $\mathcal{H}_6 - \mathcal{H}_{15}$, cointegration is found only for $\mathcal{H}_{15}$, which describes the spread between the federal funds rate and the target rate. Stationarity is accepted with a p-value of 0.98. The estimated coefficient between the 3 and 6 month tbill rates in $\mathcal{H}_6$ is close to unity, but the spreads are, nevertheless, found to be nonstationary. See also the graph in panel (c) of Figure 4.

Although stationarity of the interest rate spread is of particular interest, it is not necessary to test this hypothesis since the less restrictive hypothesis of cointegration $(1, a)$ was already rejected in $\mathcal{H}_6 - \mathcal{H}_{14}$. In Figure 4 the graphs of the spread between the federal funds rate and each of the tbill rates are given in panel (a) and (b), between the 3 and 6 month tbill rates in panel (c) and in panel (d) between the 10 year bond rate and the 6 month tbill rate. In spite of the close comovements between the interest rates, as illustrated by the graphs in Figure 8 in Appendix B, the spread is found to be nonstationary. This suggests that the shocks to the interest rate spread have been small but, nevertheless, permanent in this period consistent with the finding of two common trends in the term structure.

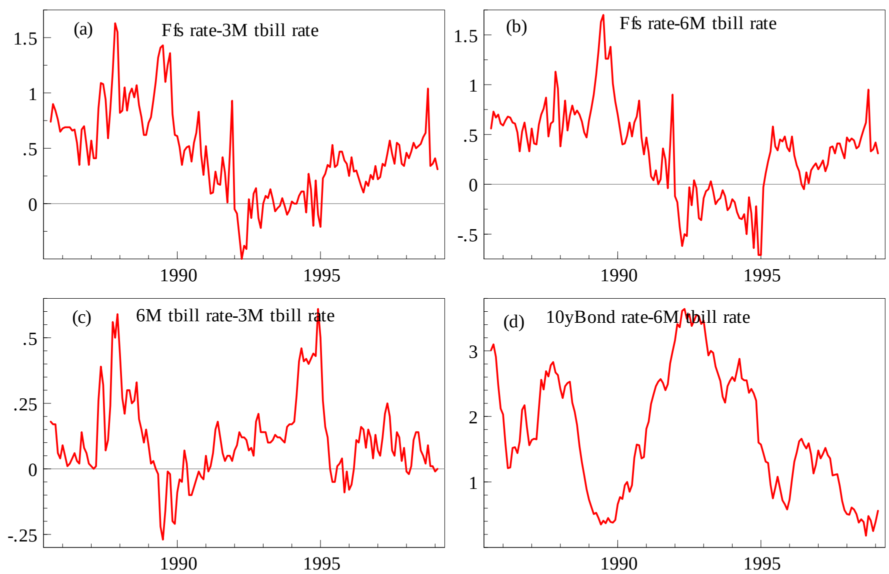

Figure 4: The graphs of the spread between (a) the federal funds rate and the 3 months tbill rate, (b) the federal funds rate and the 6 months tbill rate, (c) the 6 months tbill rate and the 3 months tbill rate, and (d) the 10 years bond rate and the 6 months tbill rate.

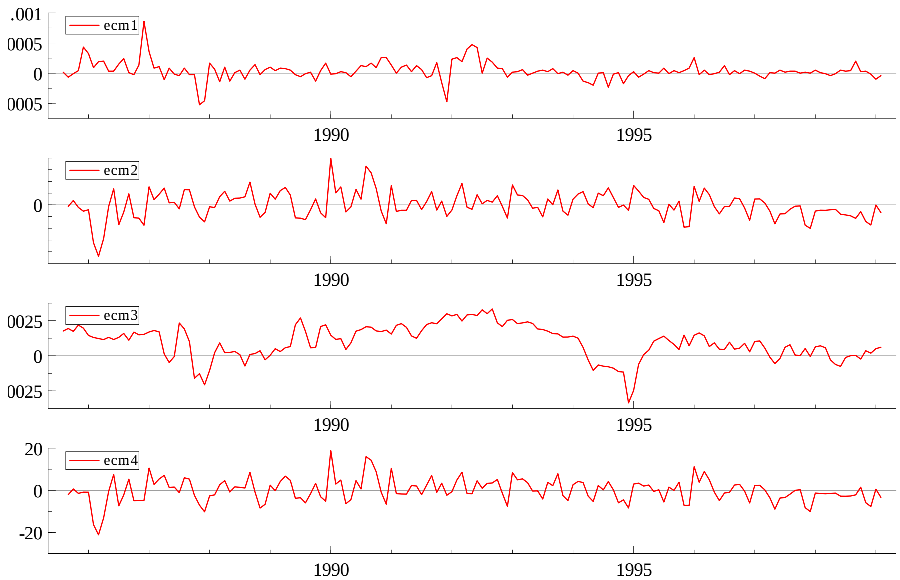

Figure 5: The graphs of the four identified cointegration relations in Table 4 for the monthly data.

4.3 An identified cointegration structure

Table 3 reports an identified structure of the cointegrating vectors $\hat{\beta}$ and their corresponding short-run adjustment parameters $\hat{\alpha}$. Significant coefficients are in bold face. The 14 overidentifying restrictions produced a test statistic of 9.02, which compared to $\chi^2_{0.95}(12) = 21.0$ can be accepted with a $p-$value of 0.83. Graphs of the four cointegrating relations are given in Figure 5. The stability of the model parameters have been checked by the recursive test procedures discussed in (Hansen and Johansen, 1999). The recursively calculated 95% confidence sets around $\hat{\beta}_t$ contained β_{T_1} and β_T for all $t = T_1 =$1991:6$, \ldots, T =$1999:2. The recursively calculated coefficients of β_{ij} and α_{ij} showed remarkable constancy.

The first cointegration relation corresponding to $\mathcal{H}_{15}$ in Table 2 says that the Fed has followed its announced target rate. Note that the inflation rate is not significantly affected by this relation, nor is any of the market determined interest rates.

Table 3: A fully identified representation of the cointegrating relations for the monthly model.

Cointegration vectors β t-values in brackets				
Var.	$\hat{\beta}_{1.}$	$\hat{\beta}_{2.}$	$\hat{\beta}_{3.}$	$\hat{\beta}_{4.}$
$(m-p)_t$	0.0	0.0	0.0	**0.004** (2.0)
y_t	0.0	0.0	0.0	$-$**0.038** ($-$4.0)
Δp_t	0.0	**0.55** (18.3)	0.0	**1.0**
$R3_t$	0.0	0.0	**1.0**	0.0
$R6_t$	0.0	**1.0**	$-$**1.13** ($-$89.6)	0.0
$B10_t$	0.0	0.0	0.**13** (9.7)	$-$**0.48** ($-$5.3)
Ff_t	**1.0**	**-1.0**	0.0	0.0
Trg_t	**-1.0**	0.0	0.0	0.0
Trend	0.0	0.0	0.0	0.0

Adjustment coefficients α t-values in brackets				
Eq.	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$
$\Delta(m-p)_t$	$-$**5.21** (-2.6)	$-$**4.28** (-4.0)	$-$**1.47** ($-$3.6)	**2.61** (4.5)
Δy_t	$-$**2.32** (-2.1)	$-$**1.34** (-2.3)	$-$0.15 ($-$0.7)	**1.50** (4.6)
$\Delta^2 p_t$	1.62 (1.7)	0.35 (0.7)	0.24 (1.3)	$-$**1.16** ($-$4.3)
$\Delta R3_t$	$-$0.11 (-1.2)	0.05 (0.9)	$-$0.01 ($-$0.8)	$-$0.00 ($-$0.1)
$\Delta R6_t$	$-$0.16 ($-$1.5)	0.05 (0.8)	0.02 (1.0)	$-$0.00 ($-$0.2)
$\Delta B10_t$	$-$0.05 ($-$0.4)	$-$0.04 ($-$0.6)	$-$0.01 ($-$0.5)	0.06 (1.5)
ΔFf_t	**-0.28** (-3.4)	**0.19** (4.3)	0.03 (1.6)	$-$**0.09** ($-$3.7)
ΔTrg_t	**0.48** (4.5)	**0.18** (3.3)	$-$0.01 ($-$0.6)	$-$**0.09** ($-$2.8)

The second relation,

$$Ff - R6 = 0.55\Delta p + stat.comp,$$

shows that the spread between the federal funds rate and R6 contains the same stochastic trend as the inflation rate. The federal funds rate is significantly error correcting to this relation, suggesting that it may contain an element of a policy rule; the cumulated shocks to the federal funds rate relative to the 6 month tbill rate are the same as those to the inflation rate. Thus, under the assumption that shocks to the 6 months tbill rate reflect expectations in the financial market to monetary policy changes, the second relation suggests that the difference between the actual value of the federal funds rate and its expected change depends on the inflation rate.

The third relation is a homogeneous interest rate relation between the 3 and 6 month tbill rates and the 10 year bond rate,

$$R3 - R6 = 0.13(R6 - B10) + stat.comp.$$

It suggests that the tbill spread and the spread between the bond rate and the 6 month tbill rate share a common stochastic trend, which according to the interpretation in Section 4.1 would be the difference between expectations to monetary policy changes and long-term inflationary expectations. The inflation rate and the money stock are significantly affected, suggesting that deviations from this relation are associated with movements in the speculative demand for money. The negative adjustment coefficient in the money equation would also be consistent with the Fed causing a contraction in money stock in order to increase the Ff rate as a response to a change in the yield curve. It is, however, more surprising that the market interest rates are not directly error correcting to this relation.

The fourth relation contains elements of an IS curve relation and a Phillips curve relation,

$$(y - trend) = 0.10(m - p - trend) - 12.7(B10 - \Delta p) + 13.9\Delta p + stat.comp.$$

Trend-adjusted real GDP is positively related to real trend-adjusted $m3$, negatively to real long-term interest rate (the IS curve effect) and positively to the inflation rate (the Phillips curve effect). The federal funds rate is significantly adjusting to this relation which can be interpreted as the Fed's policy reaction to the output gap relative to its determinants. Both the real

Table 4: The long-run impact matrix C for the monthly data. (t-values in brackets)

	ε_m	ε_y	$\varepsilon_{\Delta p}$	ε_{R3}	ε_{R6}	ε_{B10}	ε_{Ff}	ε_{Trg}
$m_t - p_t$	**3.89**	-1.27	-1.15	**-153.55**	**146.70**	**-49.67**	20.17	**58.24**
	(3.2)	(-0.5)	(-0.4)	(2.0)	(1.9)	(-2.3)	(1.0)	(2.1)
y_t	**0.82**	**2.22**	1.69	-23.78	24.95	**-17.90**	8.35	**19.61**
	(2.2)	(2.8)	(1.8)	(-1.0)	(1.1)	(-2.8)	(1.3)	(2.4)
Δp_t	0.00	**0.15**	**0.13**	-0.19	-1.18	0.45	**0.88**	0.54
	(0.2)	(4.0)	(3.1)	(-0.2)	(-1.1)	(1.5)	(2.9)	(1.4)
$R3_t$	0.04	**0.14**	**0.14**	-1.09	2.46	-0.22	0.11	0.90
	(1.3)	(2.3)	(2.0)	(-0.6)	(1.4)	(-0.5)	(0.2)	(1.4)
$R6_t$	0.03	**0.12**	**0.13**	-1.07	2.32	-0.09	0.10	0.80
	(1.2)	(2.1)	(1.9)	(-0.6)	(1.4)	(-0.2)	(1.1)	(1.3)
$B10_t$	-0.00	0.01	0.04	-0.51	0.03	**1.07**	0.31	0.02
	(0.0)	(0.2)	(1.0)	(-0.5)	(0.0)	(3.6)	(1.1)	(0.1)
Ff_t	0.04	**0.19**	**0.19**	-1.22	2.00	0.04	0.46	1.08
	(1.1)	(2.7)	(2.3)	(-0.6)	(1.0)	(0.1)	(0.8)	(1.5)
Trg_t	0.04	**0.19**	**0.19**	-1.18	1.94	0.05	0.47	1.08
	(1.1)	(2.8)	(2.4)	(-0.6)	(1.0)	(0.1)	(0.9)	(1.5)

Note: The large coefficients to interest rates in the first two rows are due to the data transformation described at the beginning of Section 4.

GDP and the inflation rate are error correcting to this relation, whereas the real money stock is not.

Whether the variables are controllable or not is crucially dependent on the long-run impact matrix, C. Table 4 reports the estimated $\hat{C}$. It can be seen that for different market rates R_i, the entries $C_{Ri,Ff}$ and $C_{Ri,Trg}$ are not significant from zero indicating that the market rates cannot be controlled by either Ff or Trg[4]. On the other hand the inflation rate can be controlled by the Fed (using the federal funds rate) because $C_{\Delta p,Ff} \neq 0$, but not by Trg, since $C_{\Delta p,Trg}$ is not significantly different from zero. The coefficient $C_{\Delta p,Ff} = 0.9$ is positive (against the prior belief of a negative impact on the inflation rate in the long run).

Furthermore, the two negative elements in the inflation row, $c_{\Delta p,R3}$ and $c_{\Delta p,R6}$, corresponding to the tbill rates, are not significantly different from zero. The most significant long-run impact on the inflation rate is from shocks to the output gap, signifying a 'demand-pull' effect on the inflation rate. Since the output gap was cointegrating with the real long-term interest

[4] This result was shown to be roboust to changes in the model specification, such as weak exogeneity restrictions and/or overidentifying restrictions on β.

rate a policy rule that could influence the level of this interest rate could potentially reduce the inflation rate. But long-term bond rate, being weakly exogenous, exhibited no long-run impact from shocks to the federal funds rate, nor the federal funds target rate.

Hence, the widely held belief that the Fed is able to reduce the US CPI inflation rate by increasing the federal funds rate does not obtain empirical support based on this information set.

4.4 Simulating the new process

As an example of the effect of rule (19) on the variables in the economy, we have used the estimated parameters and residuals from the VAR model of this section to generate the process x_t^{new} based on (11):

$$x_{t+1}^{new} = x_t^{ctr} + \hat{\alpha}(\hat{\beta}' x_t^{ctr} - \hat{\mu}) + \hat{\Gamma}_1 \Delta x_t^{ctr} + \hat{\Phi} D_{t+1} + \hat{\varepsilon}_{t+1}.$$

Without interventions the process x_t would be generated by:

$$x_{t+1} = x_t + \hat{\alpha}(\hat{\beta}' x_t - \hat{\mu}) + \hat{\Gamma}_1 \Delta x_t + \hat{\Phi} D_{t+1} + \hat{\varepsilon}_{t+1}.$$

By subtracting x_{t+1} from x_{t+1}^{new} we find an expression for x_{t+1}^{new} in terms of $x_{t+1}, x_t, \Delta x_t, x_t^{ctr}$ and Δx_t^{ctr} given by:

$$x_{t+1}^{new} = x_{t+1} + (I_p + \hat{\alpha}\hat{\beta}')(x_t^{ctr} - x_t) + \hat{\Gamma}_1 \Delta(x_t^{ctr} - x_t),$$

which together with (19)

$$x_t^{ctr} = x_t^{new} - a(b'\hat{C}a)^{-1}[b' x_t^{new} - b^* + b'(\hat{C}\hat{\Gamma} - I_p)\hat{\beta}(\hat{\beta}'\hat{\beta})^{-1}(\hat{\beta}' x_t^{new} - \hat{\mu}) \\ + b'\hat{C}\hat{\Gamma}_1(x_t^{new} - x_{t-1}^{ctr})]$$

can be used to derive the new process. The intervention, given by the difference $a'(x_t^{ctr} - x_t^{new})$, is graphed in Figure 6 together with the Federal Funds rate. The additional interventions needed to make the inflation rate stationary are very small indeed.

Because the long-run impact of permanent shocks to the federal funds rate was significant (though positive), the inflation rate is controllable. By applying the control rule (19), the inflation rate would have become stationary around a target mean of 2% as illustrated in Figure 7. However, the difference between the 'new' inflation rate and the observed inflation is not very large in this period.

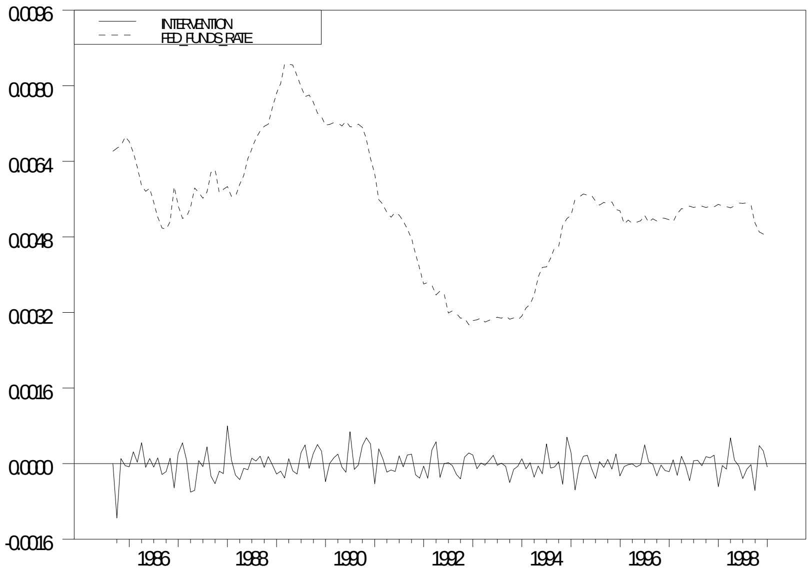

Figure 6: Graphs of the federal funds rate (dotted line) and the derived interventions, $Ff_t^{ctr} - Ff_t^{new}$ (solid line) using control rule (19).

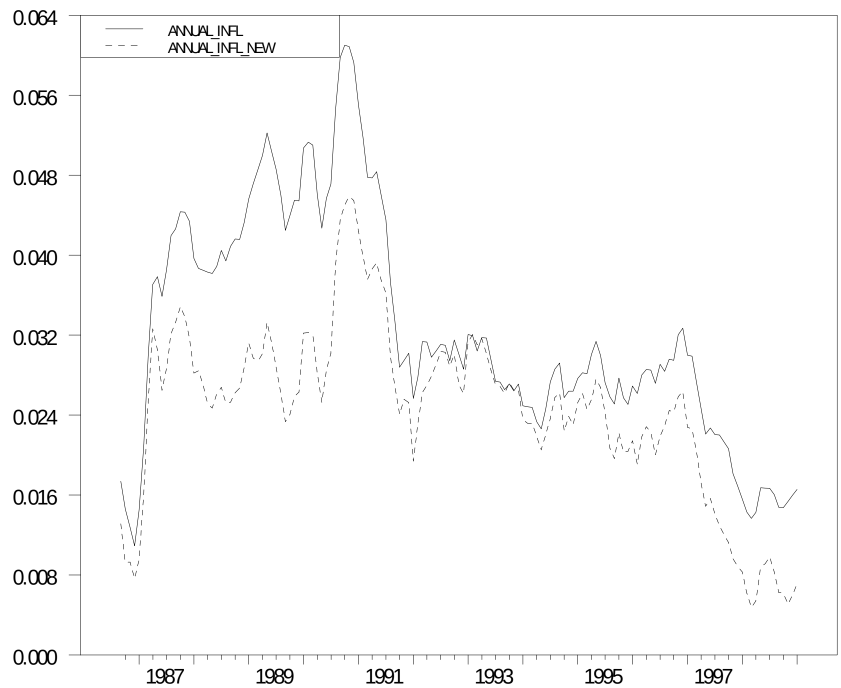

Figure 7: The annual inflation rate (solid line) and the new annual inflation rate (dotted line) using control rule (19) at all time points

5 Conclusion

We have investigated the cointegrated VAR with the purpose of formulating a control problem as that of making a nonstationary target variable stationary around a given target value using a given instrument variable. The basic assumption is that the market takes into account the value set by the bank x_t^{ctr} when producing the new values of the variables x_{t+1}^{new}. The properties of the new process, x_t^{new}, when the linear control rule $(\tilde{\kappa}, \kappa^*)$ is applied at all time points depends on the relation between the control rule κ and the behavior of the economy β and between the instruments a and the dynamics of the error correction α. Under the assumption that the new process is still $I(1)$ and not explosive, say, implying that $|\kappa' C a| \neq 0$, Theorem 2 shows that the m vectors κ become cointegration vectors and, hence, stationary. This gives the conditions for when m given targets $b'x_t$ can be controlled to (trend) stationarity around b^*. We show by an example in Theorem 4 how one could choose the control rule provided $|a'Cb| \neq 0$. In Theorem 5 we extend the results to the case where $\kappa' x_t$ is stationary, so that $\kappa' C a = 0$, and in Theorem 6, we extend the results to the case where $\alpha_\perp' \kappa = 0$, so that again $\kappa' C a = 0$.

As an illustration we apply the theoretical results to a model for US monthly data comprising the federal funds rate, the federal funds target rate, the inflation rate, three market interest rates, money stock and output gap. Based on a VAR(2) with four cointegrating relations we estimate the long-run impact matrix C and find that here is a significant (but positive) long-run impact from the federal funds rate to the inflation rate. Thus, under the four basic assumptions reported in the Introduction, the Federal Reserve Bank would have been able to make the inflation rate stationary using the federal funds rate. We show by an example how this could have been done.

However, the estimated long-run impacts of shocks to the federal funds rate (or the target rate) on the three market interest rates were all insignificant. This implies that the latter cannot be directly controlled and, therefore, would not qualify as intermediate targets. Thus, if inflation targeting is effective, as is widely believed, it does not seem to be so through the interest rate channel. To conclude, based on this information set we do not find empirical evidence that the Federal Reserve Bank can reduce the CPI inflation rate in the long run by raising the federal funds rate.

6 Appendix A

Proof of Lemma 1:

The solution of (1) with initial value x_t is

$$x_{t+h} = (I_p + \alpha\beta')^h x_t + \sum_{i=0}^{h-1} (I_p + \alpha\beta')^i (\varepsilon_{t+h-i} - \alpha\mu). \tag{25}$$

From the relations

$$\alpha'_\perp (I_p + \alpha\beta')^h = \alpha'_\perp \text{ and } \beta'(I_p + \alpha\beta')^h = (I_r + \beta'\alpha)^h \beta'$$

it follows from (25) that

$$\begin{aligned} E(\alpha'_\perp x_{t+h}|x_t) = & \quad \alpha'_\perp x_t, \\ E(\beta' x_{t+h}|x_t) = & \quad (I_r + \beta'\alpha)^h \beta' x_t - \textstyle\sum_{i=0}^{h-1} (I_r + \beta'\alpha)^i \beta'\alpha\mu \\ \rightarrow & \quad - \textstyle\sum_{i=0}^{\infty} (I_r + \beta'\alpha)^i \beta'\alpha\mu = \mu. \end{aligned}$$

Hence, using (3) we find

$$E(x_{t+h}|x_t) = \beta_\perp (\alpha'_\perp \beta_\perp)^{-1} E(\alpha'_\perp x_{t+h}|x_t) + \alpha(\beta'\alpha)^{-1} E(\beta' x_{t+h}|x_t) \rightarrow C x_t + \alpha(\beta'\alpha)^{-1}\mu.$$

Because $\beta' x_{t,\infty} = \beta'(C x_t + \alpha(\beta'\alpha)^{-1}\mu) = \mu$, it follows that $x_{t,\infty}$ is in the attractor set (4), i.e. is an equilibrium, or steady state point. See also Proietti (1997), Bruneau and Jondeau (1999), and Bedini and Mosconi (2000).

Proof of Theorem 2:

We let $\gamma = 0$. Substituting (12) into (13) we derive the equation for the new process

$$\tilde{x}_{t+1}^{new} = (I_{kp} + \tilde{\alpha}\tilde{\beta}')(I_{kp} + \bar{\tilde{a}}\tilde{\kappa}')\tilde{x}_t^{new} - \tilde{\alpha}\tilde{\mu} - (I_{kp} + \tilde{\alpha}\tilde{\beta}')\bar{\tilde{a}}\kappa^* + \tilde{\varepsilon}_{t+1},$$

which proves (15). The error correction form of the equation is

$$\Delta\tilde{x}_{t+1}^{new} = (\tilde{\alpha}, (I_{kp} + \tilde{\alpha}\tilde{\beta}')\bar{\tilde{a}}) \begin{pmatrix} \tilde{\beta}' \tilde{x}_t^{new} - \tilde{\mu} \\ \tilde{\kappa}' \tilde{x}_t^{new} - \kappa^* \end{pmatrix} + \tilde{\varepsilon}_{t+1}. \tag{26}$$

The cointegrating space is spanned by the $r + (k-1)p + m$ vectors $\beta^{new} = (\tilde{\beta}, \tilde{\kappa})$, and the adjustment coefficients have changed to

$$\tilde{\alpha}^{new} = (\tilde{\alpha}, (I_{kp} + \tilde{\alpha}\tilde{\beta}')\bar{\tilde{a}}),$$

so that the space of adjustment coefficients is spanned by the $r+(k-1)p+m$ vectors $(\tilde{\alpha}, \tilde{a})$. The process is an $I(1)$ process if and only if $\rho(I_{r+m+(k-1)p} + \tilde{\beta}^{new\prime}\tilde{\alpha}^{new}) < 1$, which is equivalent to condition (16). In this case the processes

$$\tilde{\beta}'\tilde{x}_t^{new} = (x_t^{new\prime}\beta, x_t^{new\prime} - x_{t-1}^{ctr\prime}, \ldots, x_{t-k+1}^{ctr\prime} - x_{t-k}^{ctr\prime})'$$

$$\tilde{\kappa}'\tilde{x}_t^{new} = \kappa' x_t^{new} + \textstyle\sum_{i=1}^{k-1} \kappa_i'(x_{t-i}^{ctr} - x_t^{new})$$

are stationary around $\tilde{\mu}$ and κ^* respectively. This shows that $\beta' x_t^{new} - \mu$, and $x_{t-i}^{ctr} - x_t^{new}$, and hence also $\kappa' x_t^{new} - \kappa^*$ are stationary with mean zero. If condition (16) is satisfied then

$$\left| \begin{pmatrix} \tilde{\beta}'\tilde{\alpha} & (I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha})\tilde{\beta}'\tilde{a} \\ \tilde{\kappa}'\tilde{\alpha} & \tilde{\kappa}'(I_{pk} + \tilde{\alpha}\tilde{\beta}')\tilde{a} \end{pmatrix} \right| = |\tilde{\beta}'\tilde{\alpha}||\tilde{\kappa}'\tilde{a} - \tilde{\kappa}'\tilde{a}(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\beta}'\tilde{a}| = |\tilde{\beta}'\tilde{\alpha}||\tilde{\kappa}'\tilde{C}\tilde{a}|$$

is non-zero, and from (14) we find

$$\tilde{\kappa}'\tilde{C}\tilde{a} = \kappa' C a$$

which proves (17). Thus there are $r + m + (k-1)p$ cointegrating vectors $(\tilde{\beta}, \tilde{\kappa})$ and hence $p - r - m$ common trends.

To derive the new matrix $\tilde{C}^{new}$ we first calculate

$$\begin{pmatrix} \tilde{\beta}'\tilde{\alpha} & (I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha})\tilde{\beta}'\tilde{a} \\ \tilde{\kappa}'\tilde{\alpha} & \tilde{\kappa}'(I_{pk} + \tilde{\alpha}\tilde{\beta}')\tilde{a} \end{pmatrix}^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$\begin{aligned}
A_{11} &= (\tilde{\beta}'\tilde{\alpha})^{-1} + (\tilde{\beta}'\tilde{\alpha})^{-1}(I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha})\tilde{\beta}'\tilde{a}(\tilde{\kappa}'\tilde{C}\tilde{a})^{-1}\tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}, \\
A_{21} &= -(\tilde{\kappa}'\tilde{C}\tilde{a})^{-1}\tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}, \\
A_{12} &= -(\tilde{\beta}'\tilde{\alpha})^{-1}(I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha})\tilde{\beta}'\tilde{a}(\tilde{\kappa}'\tilde{C}\tilde{a})^{-1}, \\
A_{22} &= (\tilde{\kappa}'\tilde{C}\tilde{a})^{-1},
\end{aligned}$$

and then find

$$\begin{aligned}
\tilde{C}^{new} &= I_{pk} - (\tilde{\alpha}, (I_{pk} + \tilde{\alpha}\tilde{\beta}')\tilde{a}) \begin{pmatrix} \tilde{\beta}'\tilde{\alpha} & (I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha})\tilde{\beta}'\tilde{a} \\ \tilde{\kappa}'\tilde{\alpha} & \tilde{\kappa}'(I_{pk} + \tilde{\alpha}\tilde{\beta}')\tilde{a} \end{pmatrix}^{-1} (\tilde{\beta}, \tilde{\kappa})' \\
&= I_{pk} - \tilde{\alpha}A_{11}\tilde{\beta}' - \tilde{\alpha}A_{12}\tilde{\kappa}' - (I_{pk} + \tilde{\alpha}\tilde{\beta}')\tilde{a}A_{21}\tilde{\beta}' - (I_{pk} + \tilde{\alpha}\tilde{\beta}')\tilde{a}A_{22}\tilde{\kappa}',
\end{aligned}$$

which reduces to the expression given in (18). The last statement follows by choosing $\tilde{\kappa}, \kappa^*$ as indicated.

Proof of Theorem 4:

We define

$$\begin{aligned}
\kappa &= -(b'C\bar{a})^{-1}b'C\Gamma,\ \kappa_i = (b'C\bar{a})^{-1}b'C\Gamma_i,\ i = 1, \ldots, k-1 \\
\tilde{a} &= (a', 0, \ldots, 0)', \\
\kappa^* &= -(b'C\bar{a})^{-1}(b^* - b'(I_p - C\Gamma)\bar{\beta}\mu),
\end{aligned}$$

so that $\kappa - \sum_{i=1}^{k-1} \kappa_i = -(b'C\bar{a})^{-1}b'C$ and

$$\tilde{\kappa}' = -(b'C\bar{a})^{-1}b'C(I_p, -\Gamma_1, \ldots, -\Gamma_{k-1}).$$

It is seen that $\tilde{\kappa}'\tilde{\alpha} = 0$ and $\tilde{\kappa}'\bar{\tilde{a}} = -I_m$, so that condition (16) simplifies to

$$\rho \begin{pmatrix} I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha} & (I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha})\tilde{\beta}'\bar{a} \\ 0 & 0 \end{pmatrix} < 1,$$

which is satisfied by the assumption about the process x_t. Multiplying (15) by $\tilde{\kappa}'$, and using $\tilde{\kappa}'\tilde{\alpha} = 0, \tilde{\kappa}'\bar{\tilde{a}} = -I_m$, we find an equation for $\tilde{\kappa}'\tilde{x}_t^{new}$:

$$\tilde{\kappa}'\tilde{x}_{t+1}^{new} = \kappa^* + \tilde{\kappa}'\tilde{\varepsilon}_{t+1},$$

which shows that

$$\tilde{\kappa}'\tilde{x}_t^{new} = -(b'C\bar{a})^{-1}b'C(x_t^{new} - \gamma t - \sum_{i=1}^{k-1} \Gamma_i(x_{t-i}^{ctr} - \gamma(t-i))$$

is a white noise process around its mean κ^*, and that

$$\kappa'(x_t^{new} - \gamma t) = -(b'C\bar{a})^{-1}b'C\Gamma(x_t^{new} - \gamma t)$$

is stationary with mean $\kappa^* = -(b'C\bar{a})^{-1}[b^* - b'(I_p - C\Gamma)\bar{\beta}\mu]$.

From the relation

$$C\Gamma = (C\Gamma - I_p) + I_p = (C\Gamma - I_p)\bar{\beta}\beta' + I_p,$$

we find, noting that $E(\beta' x_t^{new}) = E(\beta' x_t) = \mu$:

$$E(b'C\Gamma(x_t^{new} - \gamma t)) = E(b'(x_t^{new} - \gamma t)) + b'(C\Gamma - I_p)\bar{\beta}\mu = b^* - b'(I_p - C\Gamma)\bar{\beta}\mu,$$

which shows that

$$E(b'(x_t^{new} - \gamma t)) = b^*.$$

Proof of Theorem 5:

If $\beta'_\perp \kappa = 0$, then $\tilde{\beta}'_\perp \tilde{\kappa} = \beta'_\perp \kappa = 0$, and $\tilde{\kappa} = \tilde{\beta}(\tilde{\alpha}'\tilde{\beta})^{-1}\tilde{\alpha}'\tilde{\kappa}$. Equation (26) takes the form

$$\Delta \tilde{x}_{t+1}^{new} = (\tilde{\alpha} + (I_{pk} + \tilde{\alpha}\tilde{\beta}')\bar{\tilde{a}}\tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1})\tilde{\beta}'\tilde{x}_t^{new} - \tilde{\alpha}\tilde{\mu} - (I_{pk} + \tilde{\alpha}\tilde{\beta}')\bar{\tilde{a}}\kappa^* + \tilde{\varepsilon}_{t+1}.$$

This shows that the new process is an $I(1)$ process with cointegrating relations $\tilde{\beta}$ and adjustment coefficients $\tilde{\alpha}^{new} = \tilde{\alpha} + (I_{pk} + \tilde{\alpha}\tilde{\beta}')\bar{\tilde{a}}\tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}$, if and only $\rho(I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha}^{new}) < 1$, see (2), where the matrix is

$$I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha}^{new} = \tilde{\beta}'\tilde{\alpha}(I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha} + \tilde{\eta}\tilde{\kappa}'\tilde{\alpha})(\tilde{\beta}'\tilde{\alpha})^{-1},$$

which has eigenvalues equal to those of is $(I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha} + \tilde{\eta}\tilde{\kappa}'\tilde{\alpha})$, see (20). The constant term is not in general contained in the space spanned by $\tilde{\alpha}^{new}$, and the new process will therefore in general have a linear trend. From

$$\begin{aligned}\tilde{\beta}'\Delta \tilde{x}_{t+1}^{new} &= (\tilde{\beta}'\tilde{\alpha})(I_{r+(k-1)p} + \tilde{\eta}\tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1})\tilde{\beta}'x_t^{new} \\ &\quad -\tilde{\beta}'\tilde{\alpha}(\tilde{\mu} + \tilde{\eta}\kappa^*) + \tilde{\beta}'\varepsilon_{t+1},\end{aligned}$$

we find that

$$E(\tilde{\beta}'x_t^{new}) = (I_{r+(k-1)p} + \tilde{\eta}\tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1})^{-1}(\tilde{\mu} + \tilde{\eta}\kappa^*)$$

and

$$\begin{aligned}&E(\tilde{\kappa}'x_t^{new}) = \tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}(I_{r+(k-1)p} + \tilde{\eta}\tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1})^{-1}(\tilde{\mu} + \tilde{\eta}\kappa^*) \\ &= (I_m + \tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\eta})^{-1}(E(\tilde{\kappa}'\tilde{x}_t) + \tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\eta}\kappa^*) \\ &= E(\tilde{\kappa}'\tilde{x}_t) + (I_m + \tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\eta})^{-1}\tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\eta}(\kappa^* - E(\tilde{\kappa}'\tilde{x}_t)),\end{aligned}$$

using the formula

$$\xi'(I_{r+(k-1)p} + \tilde{\eta}\xi')^{-1} = (I_m + \xi'\tilde{\eta})^{-1}\xi',$$

for $\xi' = \tilde{\kappa}'\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}$. If we take $\tilde{\kappa}, \kappa^*$, so that $\kappa = b\lambda$, $\kappa_i = 0$, then $\tilde{\kappa} = \tilde{b}\kappa$, and $E(b'x_t^{new}) = E(\tilde{b}'\tilde{x}_t^{new}) = b^*$ if κ^* is chosen so that

$$\lambda' b^* = \lambda' E(\tilde{b}'\tilde{x}_t) + (I_m + \lambda'\Theta)^{-1}\lambda'\Theta(\kappa^* - \lambda' E(\tilde{b}'\tilde{x}_t)),$$

with solution

$$\kappa^* = \Theta^{-1}(b^* - E(\tilde{b}'\tilde{x}_t)) + \lambda' b^*,$$

which proves (23).

Proof of Theorem 6:

If $\alpha_\perp' a = 0$, then $\tilde{\alpha}_\perp' \tilde{a} = 0$, and $\overline{\tilde{a}} = \tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\beta}'\overline{\tilde{a}}$, and the new process satisfies

$$\Delta\tilde{x}_{t+1}^{new} = \tilde{\alpha}(\tilde{\beta}' + \tilde{\eta}\tilde{\kappa}')\tilde{x}_t^{new} - \tilde{\alpha}(\tilde{\mu} + \tilde{\eta}\kappa^*) + \tilde{\varepsilon}_{t+1}.$$

The cointegrating space is spanned by $\tilde{\beta}^{new} = \tilde{\beta} + \tilde{\kappa}\tilde{\eta}'$, and the adjustment coefficients are $\tilde{\alpha}^{new} = \tilde{\alpha}$. This determines an $I(1)$ process if and only if $\rho(I_{r+(k-1)p} + \tilde{\alpha}'\tilde{\beta} + \tilde{\alpha}'\tilde{\kappa}\tilde{\eta}') < 1$, which is equivalent to (20). Note that the constant term is proportional to $\tilde{\alpha}$, so that no trend is generated in the process, and that $E(\tilde{\beta}^{new\prime}\tilde{x}_t^{new}) = \tilde{\mu} + \tilde{\eta}\kappa^*$. In particular the new cointegrating relation $\tilde{\beta}^{new}\overline{\tilde{\eta}} = \tilde{\beta}\overline{\tilde{\eta}} + \tilde{\kappa}$ has mean

$$E((\tilde{\beta}\overline{\tilde{\eta}} + \tilde{\kappa})'\tilde{x}_t^{new}) = \overline{\tilde{\eta}}'(\tilde{\mu} + \tilde{\eta}\kappa^*) = E(\overline{\tilde{\eta}}'\tilde{\beta}'\tilde{x}_t) + \kappa^*.$$

If b is a set of m target variables we choose the control rule $(\tilde{\kappa}, \kappa^*)$ so that (20) is satisfied, and so that $\tilde{\beta}\overline{\tilde{\eta}} + \tilde{\kappa} = \tilde{b}\lambda$, that is $\tilde{\kappa} = \tilde{b}\lambda - \tilde{\beta}\overline{\tilde{\eta}}$. Then

$$\lambda' E(\tilde{b}'\tilde{x}_t^{new}) = \overline{\tilde{\eta}}'\tilde{\mu} + \kappa^* = \lambda' b^*,$$

for $\kappa^* = \lambda' b^* - \overline{\tilde{\eta}}'\tilde{\mu}$.

7 References

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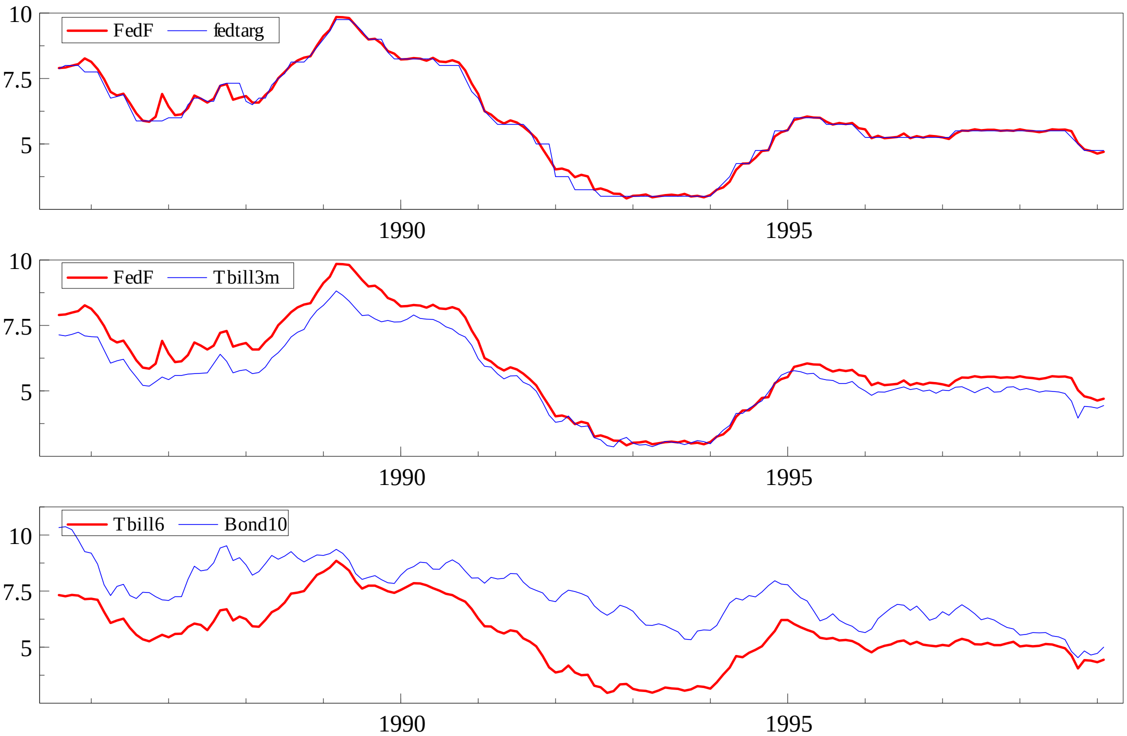
8 Appendix B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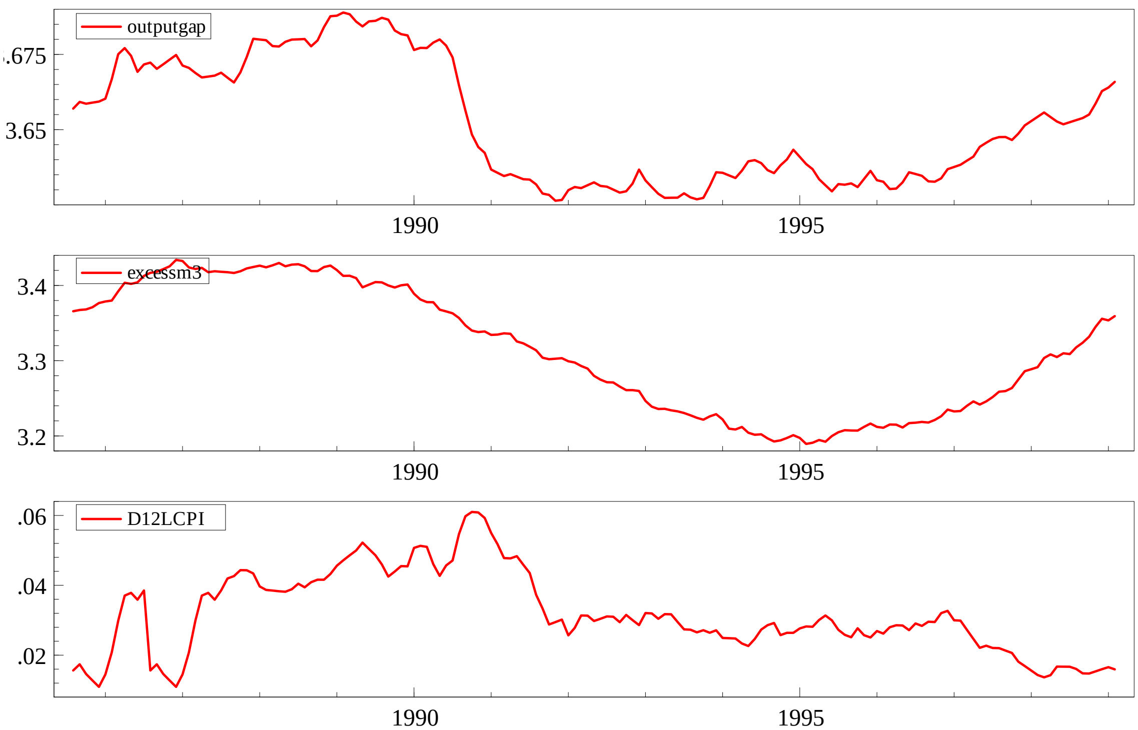

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