Testing Exchange Rate Models Based on Rational Expectations versus Imperfect Knowledge Economics: A Scenario Analysis*

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Abstract

It is often argued that model based expectations are needed to ensure theoretical consistency of economic models. This paper argues that empirical consistency of basic theoretical assumptions is even more important. This entails carefully matching the basic assumptions underlying the theoretical model with the empirical regularities of the data as structured by a statistically adequate model. Since unit root nonstationarity is endemic in economic data, the paper argues that a correctly specified Cointegrated VAR model is likely to work well as a first statistical approximation. Within this model all basic assumptions on the model's shock structure and steady-state behavior can be formulated as testable hypotheses on common stochastic trends and cointegration in what is labelled 'a theory-consistent CVAR scenario'. As it allows testing of competing models, it is likely to enhance our ability to develop empirically relevant models. The scenario idea is applied to two different type of models for exchange rate determination, one based on so called rational expectations, the other on imperfect knowledge expectaions.

Keywords: Theory-Consistent CVAR, Imperfect Knowledge, Rational Expectations, International Puzzles, Long Swings, Persistence.

JEL Classification: F31, F41, G15, G17

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1 Introduction

International macroeconomics is known for its many empirical puzzles: the PPP puzzle, the long swings puzzle, the exchange rate disconnect puzzle, and the forward premium puzzle (Rogoff, 1996). Common for these puzzles is the fact that standard international parity conditions such as Purchasing Power Parity (PPP), Uncovered Interest rate Parity (UIP), and real interest rate differentials deviate from parity with a pronounced persistence.

While it is often argued that model based expectations are needed to ensure theoretical consistency, this paper argues that empirical consistency is even more important, requiring as a minimum that the model can adequately account for the persistency of data. The paper argues that (near) unit root persistence is likely to be informative about the underlying causes of the puzzling long swings behavior and, therefore, that statistical modelling using persistence as a structuring device is likely to uncover mechanisms that have generated such long swings in the data.

Figure 1 illustrates the ideas based on US-German exchange rate data. The upper panel shows relative prices with an upward sloping trend together with the nominal exchange rate with long persistent swings around a similar upward sloping trend. The lower panel shows the real exchange rate (deviation from the PPP) together with the real interest rate differential. It illustrates that, in an equilibrium context, a large imbalance in one economic variable needs to be compensated by a similar imbalance in another variable. Based on the graphical picture we can tentatively infer that there are two persistent trends in the data, the upward sloping trend in relative prices and the long persistent swings in the nominal exchange rate and that the long swings in the real exchange rates and the real interest rate differential are likely to be causally associated.

While a burgeoning literature has attempted to address theses long swings both theoretically and empirically, so far it does not seem to have reached a consensus about the main mechanisms. One major problem is that, while the expectation's formation in the foreign currency markets often represents a defining feature of an economic model, agents' expectations are generally not observed. To overcome this problem, this paper demonstrates that basic hypotheses about expectations formation can be translated into testable hypotheses on a theory consistent Cointegrated VAR (CVAR) scenario and that this can offer a scientifically credible way of choosing between competing models.

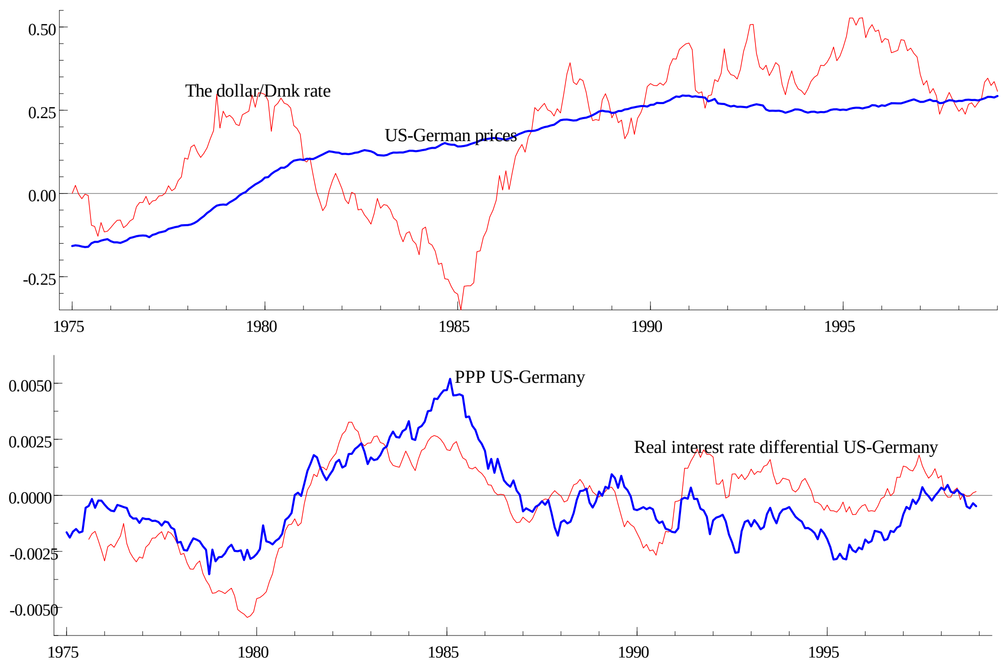

Figure 1: The graphs of the (mean and range adjusted) German-US price differential, pp, and the nominal exchange rate, s_{12} (upper panel), and the $ppp = pp - s_{12}$ and the real bond rate differential(lower panel).

As an illustration the paper tests whether models based on the Rational Expectations Hypothesis (REH) are able to satisfactorily explain the persistent movements of the data or whether less informationally demanding models based on imperfect knowledge economics (IKE) can improve our understanding of these puzzling features in the data.

Many REH-based models are taken to the data using calibration and Bayesian priors, restricting attention to a few specific features of the theoretical model which are then tested. But such tests can only make sense if the assumed structure of the economic model is correct. If the hypotheses are tested in the context of a fully specified statistical model the conclusions may change completely as demonstrated in Juselius and Franchi (2007)[1]. Spanos

[1] When testing REH-based models in the context of a statistically fully specified model, they have often been rejected. See for example the articles in the special issue "Using Econometric for Assessing Economic Models" (Juselius, 2009).

(2009) argues forcefully that the econometric procedures are valid *only* to the extent that the probabilistic assumptions constituting the underlying statistical model are satisfied vis-a-vis the data in question.

Therefore, a convincing test of the hypotheses underlying a theoretical model need to be carried out in the context of a well specified statistical model that can be considered an adequate description of the data generating process. A correctly specified VAR model can be seen as a broad description of the data generating process and, therefore, as an obvious candidate for such a model (Juselius, 2006, 2014). But because the statistical model and the theoretical model represent two different entities a bridging principle is needed. A theory-consistent CVAR scenario (Juselius, 2006, Juselius and Franchi, 2007, Møller, 2007) translate as many as possible of the assumptions underlying the theoretical model into testable hypotheses on the pulling and pushing forces of a CVAR model. Such a scenario describes a set of testable empirical regularities one should expect to see in the data if the basic assumptions of the theoretical model were empirically valid. Hoover and Juselius (2014) argue that it represents a designed experiment for data obtained by passive observations in the sense of Haavelmo (1944).

Since all prior hypotheses are tested in the unrestricted VAR this approach gives the data a rich context in which they are allowed to speak freely (Hoover et al., 2008). For example, most theoretical models are inherently consistent with a given number of exogenous shocks that cumulate to stochastic trends. This can be translated into testable hypotheses on the reduced rank of the CVAR. Most models also assume certain equilibrium relationships to be stationary, or (implicitly or explicitly) basic parity relationships to hold as stationary conditions. This can be formulated as testable hypotheses on the pulling forces of the CVAR model. A theoretical model that passes the first check of such basic properties is potentially an empirically relevant model.

The paper demonstrates the scenario approach by translating basic assumptions underlying an REH-based versus an IKE-based monetary model for exchange rate determination into a set of testable hypotheses on the CVAR model. The two scenarios are then tested on data for the German-US exchange rates, prices and interest rates. The results show that the REH-based scenario is empirically rejected on essentially all counts, whereas the IKE-based scenario obtain a remarkable support for essentially every single testable hypothesis.

The paper is organized as follows. Section 2 discusses principles underlying a theory-consistent CVAR scenario, Section 3 introduces an REH-based monetary model for exchange rate determination, Section 4 proposes a rule for associating expectations with observables and Section 5 formulates a theory-consistent CVAR scenario for the REH-based model. Section 6 introduces an imperfect knowledge based monetary model for exchange rate determination, Section 7 discusses how to anchor expectations to observable variables and Section 8 how to formulate a theory-consistent CVAR scenario for the IKE-based model. Section 9 introduces the empirical CVAR model, finds that the vector process is $I(2)$ and tests hypotheses on the order of integration of individual varaiables/relations. Section 10 estimates a structure of long-run relations in the foreign currency market and finds that the results are generally consistent with imperfect knowledge and self-reinforcing feed-back behavior. Section 11 concludes.

2 On the formulation of a theory-consistent CVAR scenario

The idea here is to classify variables and relations according to their persistency profiles measured by the order of integration. For example, a very stationary process would be classified as $I(0)$, a first order persistent process as $I(1)$ or near $I(1)$, a second order persistent process as $I(2)$ or near $I(2)$. See Juselius (2011) for detailed discussion. As it it would be implausible to expect economic variables to move away from their equilibrium values for infinite times, one may argue that most economic variables should be classified as *near* $I(1)$ or *near* $I(2)$. Hoowever, this does not exclude the possibility that over finite samples economic variables exhibit a persistence that is indistinguishable from a unit root or a double unit root process. In this sense a unit root should not necessarily be thought of as a structural parameter, but rather as a useful way of structuring the data variation. The subsequent sections will show that basic assumptions underlying the theory model can be translated into testable hypotheses on the order of integration within a theory-consistent CVAR scenario.

Theoretical relationships are often formulated in (unobservable) expectations of variables, whereas the empirical regularities that can be uncovered by a CVAR analysis are based on the observed data. To be able to derive the implications of the theoretical model and to formulate these in an internally consistent stochastic framework, we need some principles for how to associate expectations with observations.

A simple example illustrates: assume that (i) $x_t = x_{t-1} + \varepsilon_t$, i.e. $x_t \sim I(1)$ and (ii) $x_t = x_{t-1} + \Delta x_{t-1} + \varepsilon_t$, i.e. $x_t \sim I(2)$. A simple forecasting rule in the first case would be $x^e_{t|t+1} = x_t$ and in the second case $x^e_{t|t+1} = x_t + \Delta x^e_{t|t+1}$. Thus, in the former case we need to make an assumption about the persistency profile of the forecast error of the levels of x_t, i.e. $x^e_{t|t+1} - x_{t+1}$, in the second case about the forecast error of the differenced process, i.e. $\Delta x^e_{t|t+1} - \Delta x_{t+1}$. Assumption A exploits this simple idea:

Assumption A Assume that $x_t \sim I(1)$ or $I(2)$. When $x_t \sim I(1)$, the forecast error $(x^e_{t|t+1} - x_{t+1}) \sim I(0)$ where $x^e_{t|t+1}$ is the expected value of the variable x made at time t for $t+1$. When $x_t \sim I(2)$, the forecast error $(\Delta x^e_{t|t+1} - \Delta x_{t+1}) \sim I(0)$.

Note that Assumption A disregards $x_t \sim I(3)$, as it is considered empirically implausible, and $x_t \sim I(0)$, as it defines a non-persistent process for which cointegration and stochastic trends have no additional value[2].

Note also that $x_t \sim I(1)$ implies that $\Delta x_t \sim I(0)$, whereas $x_t \sim I(2)$ that $\Delta x_t \sim I(1)$ and $\Delta^2 x_t \sim I(0)$.

This leads to the following corollary:

Corollary When $x_t \sim I(1)$, x_t, x_{t+1} and $x^e_{t|t+1}$ have the same persistency property (order of integration). When $x_t \sim I(2)$, Δx_t, Δx_{t+1} and $\Delta x^e_{t|t+1}$ have the same persistency property, whereas x_{t+1} and $x^e_{t|t+1}$ do not necessarily share the same persistency properties.

As a consequence, when $x_t \sim I(1)$, a cointegration relation, $\beta' x_t$, has the same persistency property as $\beta' x^e_{t|t+1}$ or $\beta' x_{t+1}$ and when $x_t \sim I(2)$ a polynomially cointegrating relation, $\beta' x_t + \delta' \Delta x_t$, has the same persistency property as $\beta' x_t + \delta' \Delta x^e_{t|t+1}$ and $\tau' \Delta x_t$ will have the same persistency property as $\tau' \Delta x^e_{t|t+1}$ and $\tau' \Delta x_{t+1}$. See Appendix A for a definition of β, δ and τ.

Thus, Assumption A allows us to make valid inference about a long-run equilibrium relation in a theoretical model even though the postulated behavior is a function of expected rather than observed outcomes.

Based on the above, the steps behind a theory-consistent CVAR scenario can be formulated as follows:

[2] In growing economies the vast majority of economic variables would qualify as type I(1) or I(2) variables.

1. Translate the postulated behavioral relations of a theoretical model into a set of hypothetical conditions on their persistency properties. For example, most REH-based monetary models are consistent with the purchasing power parity and the uncovered interest rate parity holding as stationary (or at most as a near $I(1)$) conditions, whereas IKE-based models are consistent with the real exchange rate and the interest rate differential to be near $I(2)$ and cointegrate to $I(1)$ and further down to $I(0)$ by adding the inflation differential.

2. Express the expectations variable(s) as a function of observed variables. For example, according to Uncovered Interest Rate Parity (UIP), the expected change in the nominal exchange rate is equal to the interest rate differential. Hence, the persistency property of the latter is also a measure of the persistency property of the unobservable expected change in nominal exchange rate and can, therefore, be empirically tested.

3. For a given order of integration of the observed variable(s) associated with the unobserved expectations, derive the order of integration of all remaining variables.

4. Translate the stochastically formulated theoretical model into a theory-consistent CVAR scenario by formulating the basic assumptions underlying of the theoretical model as a set of testable hypotheses on cointegration relations and common trends.

5. Estimate a well-specified VAR model and check the empirical adequacy of the derived theory-consistent CVAR scenario.

When formulating a theory-consistent scenario one has to consider errors in the statistical model as well as postulated shocks in the theoretical model. The following notation will be used to discriminate between these different types of shocks:

a. $\varepsilon_{i,t} \sim Niid(0, \sigma_{\varepsilon}^2)$ is a white noise process.

b. $e_{i,t}$ is a stationary deviation from a long-run benchmark/equilibrium relation. It can generally be described as an ARMA process, $\rho(L)e_{i,t} = \theta(L)\varepsilon_t$ where $\rho(L)$ and $\theta(L)$ are lag polynomials.

c. $v_{i,t} = v_{i,t}^{(p)} + v_{i,t}^{(tr)}$ where $v_{i,t}$ is a stationary expectations error associated with either $x_{t|t+1}^e - x_{t+1}$ if $x_t \sim I(1)$ or $\Delta x_{t|t+1}^e - \Delta x_{t+1}$ if $x_t \sim I(2)$ and $v_{i,t}^{(p)}$ stands for a permanent expectations shock and $v_{i,t}^{(tr)}$ for a transitory. Note that if $v_{i,t}^{(p)} \neq 0$, then $(x_t - x_t^e)$ will exhibit stochastically trending behavior. Generally, $v_{i,t}^{(p)}$ is likely to be small compared to $v_{i,t}^{(tr)}$ and the stochastic trend in $(x_t - x_t^e)$ is likely to be small in absolute magnitude.

d. $u_{i,t} = f(\varepsilon_{i,t}), i = 1,..,p$ is a 'structural' shock. It is unobserved but assumed to be a linear function of the shocks, $\varepsilon_{i,t}$, to the system.

Note, that the VAR model is defined for $Niid$ errors, whereas the postulated steady-state errors of the theory model, $e_{i,t}$, and the expectations errors, $v_{i,t}$, while stationary, are not necessarily white noise.

3 REH-based monetary models for the real exchange rate

Among the class of REH-based monetary models, the overshooting model by Dornbush (1976) and Dornbush and Frankel (1988) belongs to the economists' standard toolbox. It attempts to explain persistent movements in the real exchange rate by price rigidities that cause the nominal exchange to overshoot its equilibrium value. The model also assumes that the rate of equilibrium adjustment to the PPP is identical for relative prices and nominal exchange rates (Frydman et al., 2007).[3] The rational bubble version of the monetary model (Blanchard and Watson, 1982) assumes that the nominal exchange rate is overshooting because at some point agents' forecasting behavior happens to become unrelated with fundamentals. This drives nominal exchange rate away from fundamental values in an explosive way until the market realizes its mistake, the bubble bursts, and the nominal exchange rapidly returns to its fundamental value.

While these models differ in various aspects, they share the assumptions that equilibrium in the goods market is characterized by Purchasing Power Parity and in the foreign currency market by UIP, and that the international Fisher parity holds as a stationary condition. Thus, these basic characteristic

[3] Cheung, Lai and Bergman (2004) find that this feature is not supported by empirical evidence.

features that will be exploited when formulating a theory-consistent CVAR scenario for this class of models.

PPP states that $S = P_d/P_f$, implying that the nominal exchange rate, S, should reflect relative prices, P_d/P_f. The real exchange rate is defined here as the log deviation from PPP:

$$q = s - p_d + p_f \tag{1}$$

where lower cases stand for logarithmic values and a subscript d stands for a domestic and f for a foreign economy. In equilibrium, the real exchange rate, q, is defined by relative prices being equal to the nominal exchange rate, i.e. $q_{ppp} = 0$. When prices are measured by a price index, the equilibrium value, q_{ppp}, is generally undefined and the observed average real exchange rate can be different from zero. The real exchange rate is assumed to deviate from its long-run equilibrium value, $\bar{q}$, by an equilibrium error e_t, which in the Dornbush/Frankel type of models is assumed to be an AR(1) process:

$$\Delta q_t = -\alpha(q_{t-1} - \bar{q}) + e_{q,t} \tag{2}$$

where $\bar{q}$ is the sample average, $0 < \alpha < 1$ measures the speed of adjustment, and $e_{q,t}$ is stationary. While q_t in (2) describes a stationary process some versions of the monetary model allow α to be very close to zero and, hence, the real exchange rate to be a near $I(1)$ process. For simplicity, the focus here is on the stationary case.

The Uncovered Interest Rate Parity (UIP):

$$i_{d,t} - i_{f,t} = (s^e_{t+1} - s_t) \tag{3}$$

where i stands for a nominal interest rate and a superscript e denotes an expected value.

The Fisher Parity states that nominal interest rate is equal to expected inflation rate plus an independent real interest rate. The real interest rate is assumed to reflect the ratio of average profit per capital in the economy. While the latter is less straightforward to measure on an aggregate level, it can roughly be associated with the real GDP growth rate. As the latter is generally found to be stationary with a non-zero mean, we assume here that the real interest rate is approximately stationary with a constant mean. The Fisher parity is defined as:

$$i_{j,t} = \Delta p^e_{j,t+1} + r_j, \qquad j = d, f \tag{4}$$

where r_j is an (unobserved) average real interest rate.

Finally, provided (2) and (3) hold, then the international Fisher Parity also holds:

$$(i_{d,t} - i_{f,t}) = (\Delta p^e_{d,t+1} - \Delta p^e_{f,t+1}). \tag{5}$$

4 Anchoring expectations to observables under REH

The purpose here is to derive theory consistent time-series properties for the relevant variables and relations using Assumption A to handle unobserved expectations. Let $v_{t+1} = (x_{t+1} - x^e_{t|t+1})$. When $x_t \sim I(1)$, $v_{t+1} \sim I(0)$. Furthermore, let $w_{t+1} = x_{t+1} - x_t$, then $x^e_{t|t+1} = x_t + (v_{t+1} - w_{t+1})$ where v_{t+1} is a forecast error and w_{t+1} is a shock to the process x_t. Both are stationary and likely to be of roughly the same magnitude. For simplicity, the notation $\nu_t = v_t - w_t$ will be used in the following.

In an REH world, the best predictor of the interest rate next period is

$$E_t(i_{t+1} \mid X_t) = i_t \tag{6}$$

which can be translated to interest rates being generated by random walk:

$$i_{j,t} = i_{j,t-1} + \varepsilon_{i,j,t}, \qquad j = d, f \qquad t = 1, ..., T \tag{7}$$

where $\varepsilon_{i,j,t}$ is a white noise process. Integrating over the sample period (7):

$$i_{j,t} = i_{j,0} + \sum_{s=1}^{t} \varepsilon_{j,s}, \tag{8}$$

where the cumulation of shocks is a measure of the stochastic trend in the interest rate. The condition that $(i_{d,t} - i_{f,t})$ is stationary implies that the stochastic trends in the two interest rates are identical, i.e. $\sum_{s=1}^{t}(\varepsilon_{d,s} - \varepsilon_{f,s}) = 0$.[4]

To derive the persistency properties for the remaining variables, $(s^e_{t+1} - s_t)$ is replaced with $\Delta s_t + v_{1,t}$ in (3):

$$(i_{d,t} - i_{f,t}) = \Delta s_{t+1} + v_{s,t+1}. \tag{9}$$

[4] This is of course a simplified description as in practice $\varepsilon_{j,t}$ contains both transitory and permanent shocks. For the interest rate differential to be stationary requires that the permanent shocks to the nominal interest rates are the same.

Under Assumption A, $\Delta s_t \sim I(0)$ and $v_{s,t}$ is a stationary error.

Eq. (4) can equivalently be expressed as:

$$\Delta p^e_{j,t+1} = i_{j,t} - r_{j,t}, \qquad j = d, f$$

where $r_{j,t} = r_j + e^r_t$. If $p_t \sim I(2)$,then $\Delta p_t \sim I(1)$, $(\Delta p_{t+1} - \Delta p_t) \sim I(0)$ and, under Assumption A, $(\Delta p^e_{t+1} - \Delta p_{t+1}) \sim I(0)$ as well as $(\Delta p^e_{t+1} - \Delta p_t) \sim I(0)$. Thus the inflation rate can be expressed as:

$$\Delta p_{j,t} = i_{j,t} - r_{j,t} + \nu_{j,t+1}, \qquad j = d, f \tag{10}$$

Given the assumption that $i_{j,t} \sim I(1)$ and $r_{j,t} \sim I(0)$, then $\Delta p_{j,t} \sim I(1)$ for the Fisher parity to hold. Inserting (8) in (10) gives an expression for the stochastic properties of the inflation rates:

$$\Delta p_{j,t} = i_{j,0} + \sum_{s=1}^{t} \varepsilon_{j,s} + r_{j,t} + \nu_{j,t+1}, \qquad j = d, f \tag{11}$$

Summing over (11) gives us an expression for domestic prices:

$$p_{j,t} = (i_{j,0} - r_j)t + \sum_{s=1}^{t}\sum_{i=1}^{s} \varepsilon_{j,s} + \sum_{s=1}^{t} e^r_{j,s} + \sum_{s=2}^{t+1} \nu_{j,s} + p_{j,0}, \qquad j = d, f \tag{12}$$

showing that prices being $I(2)$ is logically consistent with nominal interest being $I(1)$ and the Fisher parity holding as a stationary condition. The linear trend in prices derives from the initial value of the nominal interest rate corrected for the average real interest rate. This implies that the slope of the linear trend is approximately equal to the initial value of the expected inflation rate.

Replacing $(\Delta p^e_{d,t+1} - \Delta p^e_{f,t+1})$ with $(\Delta p_{d,t} - \Delta p_{f,t}) + \nu_{d,t+1} - \nu_{f,t+1}$ in (5) gives:

$$(i_{d,t} - i_{f,t}) = (\Delta p_{d,t} - \Delta p_{f,t}) + \nu_{d,t+1} - \nu_{f,t+1} \tag{13}$$

Thus, the REH-based monetary model predicts that $(i_{d,t} - i_{f,t}) \sim I(0)$ and $(\Delta p_{d,t} - \Delta p_{f,t}) \sim I(0)$. If $(\Delta p_{d,t} - \Delta p_{f,t}) \sim I(0)$ then $(p_{d,t} - p_{f,t}) \sim I(1)$, i.e. prices being individually $I(2)$ are cointegrated (1, -1) consistent with long-run price homogeneity.

Subtracting (9) from (13) gives:

$$\begin{aligned} \Delta p_{d,t} - \Delta p_{f,t} &= \Delta s_t + \nu_{d,t+1} - \nu_{f,t+1} - v_{s,t+1} \\ \Delta p_{d,t} - \Delta p_{f,t} - \Delta s_t &= \nu_{d,t+1} - \nu_{f,t+1} - v_{s,t+1} \\ \Delta q_t &= \nu_{q,t+1}, \end{aligned}$$

i.e. the change in real exchange rate can be expressed as the difference between an unanticipated shock and the forecast error. Integrating once gives an expression for q_t:

$$q_t = \sum_{j=1}^{t+1} \nu_{q,j} + q_0. \tag{14}$$

Thus, stationarity of the real exchange rate is consistent with the case where the permanent part of the unanticipated shocks to UIP and the international Fisher parity are identical and therefore cancel in (14).

5 A scenario for REH-based models

According to the stochastic properties derived above, prices are $I(2)$, the nominal exchange rate and interest rates are $I(1)$. Based on this, the behavioral equilibrium equations underlying the theoretical model can now be translated into a set of testable hypotheses on cointegration and common trends in the CVAR model.

The assumption that $(i_d - i_f) \sim I(0)$, implies that the two interest rates are cointegrated and, therefore, share one common stochastic trend. Because the stochastic properties of all the other variables have been derived from the stochastic properties of the interest rates, this is the only stochastic trend in the system. Hence, the theory-consistent CVAR is driven by one common stochastic trend and, the system is, therefore, equilibrium correcting to $p - 1 = 4$ cointegration relations. The common autonomous shock, $u_{1,t}$, cumulates once in the interest rates and the nominal exchange rate, but twice in the price variables[5] (see Juselius, 2006, Chapter 2.5) consistent with the

[5] Note that the common autonomous shock, $u_{1,t}$, can be associated with a exogenous shock outside the CVAR model. For example, most monetary models assume relative money supply shocks to be the pushing force.

assumptions in Section 4. Thus, the REH based CVAR scenario consistent with $\{r = 4, s_1 = 0, s_2 = 1\}$ is formulated as follows:

$$\begin{bmatrix} p_d \\ p_f \\ s \\ i_d \\ i_f \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \Sigma\Sigma u_1 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ c_1 \\ c_1 \end{bmatrix} [\Sigma u_1] + \begin{bmatrix} d_1 \\ d_2 \\ d_1 - d_2 \\ 0 \\ 0 \end{bmatrix} t + Z_t. \qquad (15)$$

where $\Sigma\Sigma u_1$ is a shorthand for $\sum_{s=1}^{t}\sum_{j=1}^{s} u_{1,s}$, Σu_1 a shorthand for $\sum_{j=1}^{t} u_{1,j}$ and Z_t is a catch-all for the short-term effects in the vector process. The common stochastic $I(2)$ trend affects both prices with identical coefficients, c_1, so that $(p_d - p_f) \sim I(0)$ implying long-run price homogeneity. The condition for PPP to be stationary is that $b_3 = b_1 - b_2$.

The coefficients c_i, b_i and d_i are not expressed as functions of the parameters of the theory model as this would require a specification also of the short-run dynamics. Thus, the CVAR scenario is only informative about the conditions under which the postulated long-run behavior of the model is empirically valid. M. Juselius (2010) argues that only such models that satisfactorily describe the long-run properties of the data need to be tested for their short-run implications.

Given the condition for long-run price homogeneity, one can apply the nominal-to-real transformation (Kongsted, 2005) without loss of information:

$$\begin{bmatrix} p_d - p_f \\ s \\ \Delta p_d \\ i_d \\ i_f \end{bmatrix} = \begin{bmatrix} b_1 - b_2 \\ b_3 \\ c_1 \\ c_1 \\ c_1 \end{bmatrix} [\Sigma u_1] + \begin{bmatrix} d_1 - d_2 \\ d_1 - d_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} t + Z_t. \qquad (16)$$

Because $(p_d - p_f) \sim I(1)$, it follows that $(\Delta p_d - \Delta p_f) \sim I(0)$. Thus under long-run price-homogeneity, inflation spread is stationary.

The scenario is consistent with $r = 4$ stationary cointegration relations. For example, the following relations are irreducible in the sense of Davidson (1998):

1. $(s - p_d + p_f) \sim I(0)$,
2. $(i_d - i_f) \sim I(0)$

3. $(i_d - \Delta p_d) \sim I(0)$

4. $(i_d - a_1(p_d - p_f) - a_2 t) \sim I(0)$ where $a_1 = c_1/(b_1 - b_2)$ and $a_2 = c_1(d_1 - d_2/(b_1 - b_2)$.

Linear combinations of these relations are also stationary and, hence, would also qualify as cointegration relations.

If relative prices and the nominal exchange rate are homogeneously related in all cointegration relations, then the scenario can, without loss of information, be formulated for the real exchange rate as follows:

$$\begin{bmatrix} s - p_d + p_f \\ \Delta p_d \\ \Delta p_f \\ i_d \\ i_f \end{bmatrix} = \begin{bmatrix} 0 \\ c_1 \\ c_1 \\ c_1 \\ c_1 \end{bmatrix} [\Sigma u_1] + Z_t. \tag{17}$$

showing as before that the the four irreducible cointegration relations correspond to the PPP, the UIP, and the Fisher parities:

1. $p_d - p_f - s \sim I(0)$

2. $i_d - i_f \sim I(0)$

3. $i_d - \Delta p_d \sim I(0)$,

4. $i_f - \Delta p_f \sim I(0)$,

Again, other irreducible relations can be found by linear combinations. For example, $(i_d - i_f) - (i_d - \Delta p_d) + (i_f - \Delta p_f) = (\Delta p_d - \Delta p_f) \sim I(0)$ is a linear combination of 2, 3, and 4.

Note that the $r = 4$ cointegrated relations in the transformed scenario (16) can be thought of as $r = 4$ polynomially cointegrated relations in the $I(2)$ scenario (15) where the relations 1 and 2 are formulated as directly stationary relations and the relations 3 and 4 as a polynomially cointegrated relations. See Johansen (1992) and Juselius (2006, Chapters 16-18).

6 Imperfect knowledge and the determination of nominal exchange rate

While essentially all asset price models assume that today's price depends on expected future prices, models based on rational expectations versus imperfect knowledge differ with respect to how agents are assumed to make forecasts and to react on forecasting errors. In RE-based models agents are adjusting back toward the equilibrium value of the theoretical model after they have made a forecast error, implying that expectations are endogenously adjusting to the "true" theoretical value. When "perfect knowledge" is replaced by imperfect knowledge, the role of expectations is likely to change. In IKE-based models individuals' forecasting strategies change in ways that cannot be fully pre-specified (Frydman and Goldberg, 2007). This means that expectational shocks can have a permanent effect on market outcomes and, hence act as an exogenous force in the model.

In heterogeneous agents models asset prices are similarly allowed to deviate persistently from their long-run fundamental values. For example, when the price is not too far away from its long-run value it may be too costly for agents to derive their forecasts based on rather complex, fundamentals-based models. In such periods, the chartists will be in the majority and their expectations will tend to drive prices away from fundamental long-term values (Hommes, 2005, 2013).

A common feature of imperfect knowledge models is the assumption that agents have diverse forecasting strategies - bulls hold long positions of foreign exchange and bet on appreciation while bears hold short positions and bet on depreciation. Imperfect knowledge speculators change their forecasting strategies depending on how far away the price is from the long-run benchmark value. As long as it is in the vicinity of its fundamental value, most agents are trend-followers but as soon as the price has moved sufficiently far away it will show an increasing tendency to revert back toward its fundamental value.

Thus, both the Frydman-Goldberg and the Hommes type of models predict that the real exchange rate will have a tendency to undergo swings around long-run benchmark values due to short-term speculative behavior which to a certain degree is unrelated with the state of long-run fundamentals.

In a world where agents change their forecasting rules in ways that cannot be prespecified model parameters cannot be assumed constant. Based on results in Frydman and Goldberg (2007) and Tabor (2014), Juselius and Assenmacher (2014) estimate a model where forecasting with nonconstant parameters seem to explain the long persistent swings typically found for the nominal exchange rate and the nominal interest rate (cf. Figure 1). Furthermore, as goods prices are usually not subject to speculative swings, the real exchange rate and the real interest rate inherit the long swings from the nominal magnitudes.

Therefore, in contrast to RE-based models, the real exchange rate cannot be assumed to be a stationary or a near I(1) process but, instead, is assumed to follow a random walk with a time-varying drift term (see Frydman and Goldberg, 2007):

$$\Delta(s_t - p_{d,t} - p_{f,t}) = \zeta_t + e_t^q \tag{18}$$

where e_t^q is stationary and the drift term, ζ_t, is assumed to follow an autoregressive process:

$$\zeta_t = \rho_t \zeta_{t-1} + \varepsilon_t^{\zeta}.$$

The parameter ρ_t may vary over different periods but generally so that its average value $\bar{\rho}$ is close to 1.0. Thus, while the REH-based model in (2) assumes that the differenced real exchange rate behaves like white noise, the IKE-based model in (18) assumes that it behaves like a very persistent near I(1) process. Hence, under imperfect knowledge the real exchange rate tend to exhibit long persistent swings typical of a near $I(2)$ process. The length of the swings are not predictable and the real exchange rate will entail swings of shorter and longer duration[6].

The long swings in the real exchange rate is likely to affect the speculative demand for foreign currency. To control for this feature Frydman and Goldberg (2007) add an uncertainty premium to the UIP condition:

$$(i_{d,t} - i_{f,t}) = s_{t+1}^e - s_t + up_t \tag{19}$$

where up_t stands for an uncertainty premium measuring agents' *loss* aver-

[6]This means that there can be sample periods when the evidence of I(2) is quite weak. See Juselius (2012, 2014) for an illustration of the difference between these two assumptions.

sion[7]. The condition (19), labelled the Uncertainty Adjusted UIP (UAUIP), is describing an economy where all speculators require a minimum return - an uncertainty premium - to speculate in the foreign exchange market. When the asset price is moving away from its long-run value, the uncertainty premium starts growing until the price reverses back towards equilibrium. In the foreign currency markets the uncertainty premium is closely associated with the PPP gap (Frydman and Goldberg, 2007)[8]. Using this, the UAUIP can be formulated as:

$$s_{t+1}^{e} - s_t = (i_{d,t} - i_{f,t}) - f(p_{d,t} - p_{f,t} - s_t) \tag{20}$$

Thus, the expected change in nominal exchange rate is not directly associated with the observed interest rate differential, but with the interest rate differential corrected for the PPP gap.

A further difference between RE-based and IK-based monetary models is that equilibrium in the goods market under the rational expectations hypothesis is consistent with a stationary real exchange rate and a stationary real interest rate differential, whereas under IKE it is defined as a stationary relation between the two:

$$\{(i_{d,t} - i_{f,t}) - (\Delta p_{d,t} - \Delta p_{f,t}) - \kappa_1(s_t - p_{d,t} + p_{f,t})\} \sim I(0) \tag{21}$$

7 Associating expectations with observables in an imperfect knowledge based model

Due to the uncertainty premium in foreign currency markets, the best predictor of the interest rate next period is not just the current interest rate, but the current interest rate together with its rate of change:

$$E_t(i_{t+1} \mid X_t) = i_t + \Delta i_t \tag{22}$$

where X_t stands for the information available at time t. In contrast, the RE-based model assume that the change has no predictive value and that best predictor is the present level of interest rate as shown in (6):

[7] The assumption that agents are loss averse, rather than risk averse, builds on the prospect theory by Kahneman and Tversky (1979).

[8] The assumption in Hommes (2005) that the proportion of chartists relative to fundamentalists decrease as the PPP gap grows is likely to capture a similar gap effect.

The first step in a theory-consistent CVAR scenario is to formulate a consistent description of the time series properties of the data given some basic assumption of the expectations formation. In the foreign currency markets expectations are primarily feeding into the model through the UAUIP condition which states that the expected change in the nominal exchange rate is given by the interest rate differential corrected for the uncertainty premium. As a consequence, the individual interest rate must also be affected by an uncertainty premium:

$$\Delta i_{j,t} = \omega_{j,t} + \varepsilon_{j,t} \text{ and } \varepsilon_{j,t} \sim Niid(0, \sigma^2_{\varepsilon,j}) \qquad j = d, f \tag{23}$$

where $\varepsilon_{j,t}$ is an unanticipated error and $\omega_{j,t}$ is a change in the domestic uncertainty premium assumed be a highly persistent AR(1) process:

$$\omega_{j,t} = \rho_{t,j}\omega_{j,t-1} + \varepsilon_{\omega j,t}, \text{ and } \varepsilon_{\omega j,t} \sim Niid(0, \sigma^2_{\varepsilon_\omega,j}) \qquad j = d, f.$$

The autoregressive coefficient $\rho_{t,j}$ is assumed to be approximately 1.0 in periods when the real exchange rate is in the neighborhood of its long-run benchmark value and $\ll 1.0$ when it is far away from this value. Since the periods when $\rho_{t,j} \ll 1.0$ are assumed to be short compared to the ones when $\rho_{t,j} \approx 1.0$, the average $\bar{\rho}_j$ is assumed to be close to 1.0. Thus, $\omega_{j,t}$ is assumed to be a near $I(1)$ process. Furthermore, $\sigma^2_{\varepsilon_\omega,j}$ is assumed small compared to $\sigma^2_{\varepsilon,j}$ (see Juselius, 2014) capturing the the stylized fact that the variance of the process is often much larger than the variance of the drift term.

Integrating (23) over t gives:

$$i_{j,t} = i_{j,0} + \sum_{s=1}^{t} \omega_{j,s} + \sum_{s=1}^{t} \varepsilon_{j,s}, \qquad j = d, f \tag{24}$$

Under the near $I(1)$ assumption of $\omega_{j,t}$, $\sum_{s=1}^{t} \omega_{j,s}$ is near $I(2)$, implying that nominal interest rates are near $I(2)$. Remember, however, that the shocks to the uncertainty premium, while persistent, are likely to be tiny compared to the interest rate shocks.[9] The process (24) is consistent with persistent swings of shorter and longer durations typical of observed interest rates.

Based on (24) the interest rate differential can be expressed as:

[9] Based on a simulation study, Juselius (2014) demonstrates for this case that univariate unit root tests almost always fail to detect the second near unit root whereas multivariate tests almost always find it.

$$(i_{d,t} - i_{f,t}) = (i_{d,0} - i_{f,0}) + \sum_{s=1}^{t}(\omega_{d,s} - \omega_{f,s}) + \sum_{s=1}^{t}(\varepsilon_{d,s} - \varepsilon_{f,s}). \qquad (25)$$

As the uncertainty premium $\omega_{j,t}$ is assumed to be near $I(1)$, the cumulation $\sum_{j=1}^{t}(\omega_{d,j} - \omega_{f,j})$ is near $I(2)$, unless $\omega_{d,j} - \omega_{f,j} = 0$. Because equality implies no uncertainty premium in the foreign currency market the latter would violate the imperfect knowledge assumption and the interest rate differential is, therefore, considered to be near $I(2)$.[10] Denoting $\sum_{j=1}^{t}(\omega_{d,j} - \omega_{f,j}) = \sum_{j=1}^{t}\omega_j$ leads to:

$$(i_{d,t} - i_{f,t}) = (i_{d,0} - i_{f,0}) + \sum_{j=1}^{t}\omega_j + \sum_{s=1}^{t}(\varepsilon_{d,s} - \varepsilon_{f,s}). \qquad (26)$$

Approximating $\sum_{j=1}^{t}\omega_j$ with a fraction, ϕ, of the PPP gap, $(p_d - p_f - s)_t$ gives:

$$(i_{d,t} - i_{f,t}) - \phi(p_d - p_f - s)_t = (i_{d,0} - i_{f,0}) + \sum_{s=1}^{t}(\varepsilon_{d,s} - \varepsilon_{f,s}), \qquad (27)$$

showing that the interest rate differential corrected for the uncertainty premium is $I(1)$, unless $\sum_{s=1}^{t}\varepsilon_{d,s} \approx \sum_{s=1}^{t}\varepsilon_{f,s}$ which is highly unlikely in an imperfect knowledge world. In the following they will be assumed different.

The real interest rate according to the Fisher parity is given by $r_{j,t} = i_{j,t} + \Delta p^e_{j,t+1}$. Under Assumption A, $\Delta p^e_{j,t+1} = \Delta p_{j,t+1} + v_{j,t+1}$ and $\Delta p_{j,t+1} = \Delta p_{j,t+1} + w_{j,t+1}$ with $v_{j,t} \sim I(0)$ and $w_{j,t} \sim I(0)$. Hence, the real interest rate can be formulated as:

$$r_{j,t} = i_{j,t} - \Delta p_{j,t} + \nu_{j,t+1}, \qquad j = d, f \qquad (28)$$

where $\nu_{j,t} = v_{j,t} + w_{j,t}$. Alternatively, (28) can be expressed for the inflation rate:

$$\Delta p_{j,t} = i_{j,t} - r_{j,t} + \nu_{j,t+1}, \qquad j = d, f \qquad (29)$$

Inserting (24) in (29) gives:

[10] Thus, a test of the hypothesis that $(i_{d,t} - i_{f,t})$ is near $I(2)$ or $I(1)$ is a test of whether the exchange rate determination in speculative currency markets is based on imperfect knowledge or rational expectations.

$$\Delta p_{j,t} = i_{j,0} + \sum_{s=1}^{t} \varepsilon_{j,s} + \sum_{s=1}^{t} \omega_{j,s} - r_{j,t} + \nu_{j,t+1}. \qquad j = d, f \tag{30}$$

If $\omega_{j,t}$ is near $I(1)$, then inflation would be near $I(2)$, unless $\sum_{s=1}^{t} \omega_{j,s}$ and $r_{j,t}$ contain the same near $I(2)$ trend. Assuming that the uncertainty premium, $\omega_{j,t}$, affects nominal interest rates, but not goods prices (which are likely to be determined by demand and supply in international goods market whereas not by speculation) then $r_{j,t}$ will contain $\sum_{s=1}^{t} \omega_{j,s}$, so that $\sum_{i=1}^{t} \omega_{j,i} - r_{j,s} = 0$. Under this assumption inflation is $I(1)$ and the real interest rate near $I(2)$ implying a delinking of inflation and the interest rate.

The inflation spread between the two countries becomes:

$$(\Delta p_{d,t} - \Delta p_{f,t}) = (i_{d,0} - i_{f,0}) + \sum_{s=1}^{t} (\varepsilon_{d,s} - \varepsilon_{f,s}) + (\nu_{d,t+1} - \nu_{f,t+1}), \tag{31}$$

showing that inflation spread is $I(1)$. An expression for relative prices is obtained by summing over (31):

$$p_{d,t} - p_{f,t} = p_{d,0} - p_{f,0} + (i_{d,0} - i_{f,0})t + \sum_{i=1}^{t} \sum_{s=1}^{i} (\varepsilon_{d,s} - \varepsilon_{f,s}) + \sum_{i=2}^{t+1} (\nu_{d,i} - \nu_{f,i}) \tag{32}$$

showing that relative prices are $I(2)$ around a linear trend. An expression for prices is obtained by summing over (30):

$$p_{j,t} = i_{j,0} \times t + \sum_{s=1}^{t} \sum_{i=1}^{s} \varepsilon_{j,i} + \sum_{s=2}^{t+1} \nu_{j,s} + p_{j,0}, \qquad j = d, f \tag{33}$$

showing that prices are $I(2)$ around a linear trend.

An expression for the change in nominal exchange rates can be derived from (19):

$$\Delta s_t = (i_d - i_f)_{t-1} - up_t - \nu_t \tag{34}$$

Inserting the expression for $(i_d - i_f)_{t-1}$ in (26) gives:

$$\begin{aligned} \Delta s_t &= (i_{d,0} - i_{f,0}) + \sum_{i=0}^{t-1} (\varepsilon_{d,i} - \varepsilon_{f,i}) + \sum_{i=0}^{t-1} \omega_i - \sum_{i=1}^{t} \omega_i + v_t \\ &= (i_{d,0} - i_{f,0}) + \omega_0 + \sum_{i=0}^{t-1} (\varepsilon_{d,s} - \varepsilon_{f,s}) - \omega_t + v_t \end{aligned}$$

where $\sum_{i=1}^{t} \omega_i = f(ppp)$ is the uncertainty premium measured by a fraction of the PPP gap. Summing over t gives an expression for the nominal exchange rate:

$$s_t = s_0 + (i_{d,0} - i_{f,0})t + \sum_{i=1}^{t-1} \sum_{s=1}^{i} (\varepsilon_{d,s} - \varepsilon_{f,s}) - \sum_{i=1}^{t} \omega_i + \sum_{i=1}^{t} v_i. \qquad (35)$$

Thus, the nominal exchange rate contains a local linear trend originating from the initial value of the interest rate differential, an I(2) trend describing the stochastic trend in the relative price, a near I(2) trend describing the long swings associated with the uncertainty premium, and I(1) trend originating from forecast errors.

An expression for the real exchange rate can now be obtained by subtracting (32) from (35):

$$s_t - p_{d,t} - p_{f,t} = s_0 - p_{d,0} - p_{f,0} - \sum_{i=1}^{t} \omega_i + \sum_{i=1}^{t} v_i - \sum_{i=2}^{t+1} (\nu_{d,i} - \nu_{f,i}), \quad (36)$$

showing that the real exchange rate is a near $I(2)$ process due to the uncertainty premium $\sum_{i=1}^{t} \omega_i$.

Finally, inserting the expression for (31) in (27) gives:

$$(i_{d,t} - i_{f,t}) - \phi(p_d - p_f - s)_t = \Delta p_{d,t} - \Delta p_{f,t} + (\nu_{d,t+1} - \nu_{f,t+1}), \qquad (37)$$

showing that the real interest rate differential cointegrates with the PPP gap to a stationary relation as assumed in (21).

In this case, prices are $I(2)$ and inflation rates $I(1)$, whereas both nominal and real interest rates are near $I(2)$. Thus, imperfect knowledge predicts that nominal and real interest rates are integrated of the same order and that the Fisher parity does not hold as a stationary condition.

To summarize: imperfect knowledge is consistent with the following testable hypotheses:

- $(p_{d,t} - p_{f,t}) \sim I(2)$,
- $s_t \sim I(2)$,
- $(i_{d,t} - i_{f,t}) \sim near\ I(2)$,
- $(s_t - p_d - p_f)_t \sim near\ I(2)$,

- $\{(i_{d,t} - i_{f,t}) - b_1(s_t - p_d - p_f)_t\} \sim I(1)$
- $\{(\Delta p_{d,t} - \Delta p_{f,t}) - b_2(i_{d,t} - i_{f,t}) + b_1(s_t - p_{d,t} - p_{f,t})\} \sim I(0)$
- $(i_{j,t} - \Delta p_{j,t}) \sim near\ I(2)$.

8 A theory consistent CVAR scenario for imperfect knowledge

The first step in a scenario describes how the underlying stochastic trends are assumed to load into the data provided the theory model is empirically correct. The results of the previous section showed that the data process $x_t = [p_{d,t}, p_{f,t}, s_t, b_{d,t}, b_{f,t}]$ should be integrated of order two and be affected by two stochastic trends, one originating from twice cumulated interest rate shocks, $\sum_{s=1}^{t}\sum_{i=1}^{s}\varepsilon_{j,i}$, and another from the once cumulated near $I(1)$ risk premium, $\sum_{i=1}^{t}\omega_i$. Two stochastic $I(2)$ trends that load into five variables implies three relations which are cointegrated $CI(2,1)$. These relations can be decomposed into r relations, $\beta' x_t$, that can become stationary by adding a linear combination of the growth rates, $\delta'\Delta x_t$, and s_1 linear combinations $\beta'_{\perp 1} x_t$ which can only become stationary by differencing. Thus, stationarity can be achieved by r polynomially cointegrated relations $(\beta' x_t + \delta'\Delta x_t) \sim I(0)$ and s_1 medium-run relations among the differences $\beta'_{\perp 1}\Delta x_t$. For a more detailed exposition see, for example, Juselius (2006, Chapter 17) and Appendix A for a definition of the VAR model for $I(2)$ data.

The three $CI(2,1)$ relations can be consistent with different choices of r and s_1 as long as $r + s_1 = p - s_2 = 3$ where s_2 is the number of $I(2)$ trends. Theoretically, the relation (27) predicts that $(i_{d,t} - i_{f,t})$ and $(s_t - p_{d,t} + p_{f,t})$ are cointegrated $CI(2,1)$ and (37) that $(i_{d,t} - i_{f,t})$, $(s_t - p_{d,t} + p_{f,t})$ and $(\Delta p_{d,t} + \Delta p_{f,t})$ are cointegrated $CI(2,2)$. Hence, $r \geq 1$ and the following three cases satisfy $r + s_1 = 3 : \{r = 1, s_1 = 2, s_2 = 2\}, \{r = 2, s_1 = 1, s_2 = 2\}$, and $\{r = 3, s_1 = 0, s_2 = 2\}$. The trace tests in the next section finds that $\{r = 2, s_1 = 1, s_2 = 2\}$ is tenable with the information in the data and the scenario below will be formulated for this case. The pushing forces are given by three autonomous shocks, $u_{1,t}$, $u_{2,t}$ and $u_{3,t}$, two of which cumulate twice to produce the two $I(2)$ trends and once to produce the corresponding $I(1)$ trends, whereas the third shock cumulates only once to produce an $I(1)$

trend. The pulling forces comprise two polynomially cointegrated relations and one medium-run relation between growth rates.

Based on the derivations of the previous section, it is possible to impose testable restrictions on some of the coefficients in the scenario. For example, relation (33) assumes that the uncertainty premium does not affect goods prices so that $(c_{12}, c_{22}) = 0$. Relation (36) assumes that the long-run stochastic trend in relative prices and nominal exchange rate cancel in $(p_d - p_f - s)$, so that $(c_{11} - c_{21}) = c_{31}$; Relation (24) assumes that the relative price trend does not load into the two interest rates, so that $(c_{14} = c_{15} = 0)$. Based on these restrictions, the imperfect knowledge scenario is formulated as:

$$\begin{bmatrix} p_d \\ p_f \\ s \\ i_d \\ i_f \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ c_{12} & 0 \\ c_{11} - c_{12} & c_{32} \\ 0 & c_{42} \\ 0 & c_{52} \end{bmatrix} \begin{bmatrix} \Sigma\Sigma u_1 \\ \Sigma\Sigma u_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \\ b_{51} & b_{52} & b_{53} \end{bmatrix} \begin{bmatrix} \Sigma u_1 \\ \Sigma u_2 \\ \Sigma u_3 \end{bmatrix} + Z_t, \tag{38}$$

where u_1 is considered to be a relative price shock and u_2 a shock to the uncertainty premium. The third shock, $u_{3,}$ can be interpreted as a medium-run trend originating from once cumulated forecast errors. All variables are $I(2)$ consistent with the derivations in the previous section. Since the two prices and the exchange rate share two stochastic $I(2)$ trends, there exists just one relation, $(p_d - w_1 p_f - w_2 s) \sim I(1)$ with $(w_1, w_2) \neq 1$.

Based on (38) the following three $CI(2,1)$ cointegration relations, can be found:

1. $\{(p_d - p_f - s) - a_1(i_d - i_f)\} \sim I(1)$ if $c_{32} - a_1(c_{42} - c_{52}) = 0$
2. $(i_d - a_2 p_d - a_3 s - \gamma t) \sim I(1)$, if $c_{42} - a_3 c_{32} = 0$ and $a_2 c_{11} + a_3(c_{11} - c_{21}) = 0$
3. $(i_f - a_4 p_f - a_5 s - \gamma t) \sim I(1)$ if $c_{52} - a_5 c_{32} = 0$ and $a_4 c_{21} + a_5(c_{11} - c_{21}) = 0$

The first relation corresponds to (27), whereas the two remaining relations, while not explicitely discussed above, are consistent with the theoretical model set-up. Note also that the inclusion of a linear trend in the relation means that trend-adjusted price/nominal exchange rate rather than the price itself is the relevant measure. Any linear combination of the three relations are of course also $CI(2,1)$.

To obtain stationarity, the above relations need to be combined with the growth rates in a way that cancels the $I(1)$ trends. The restrictions on the parameters that are consistent with stationarity are given below for the first polynomially cointegrated relation, $\beta_1' x_t + \delta_1' \Delta x_t$, corresponding to (37). The conditions for the other relations can be similarly derived.

1. $ppp - a_1(i_d - i_f) - a_6(\Delta p_d - \Delta p_f) \sim I(0)$, if $c_{32} - a_1(c_{42} - c_{52}) = 0$
   $\{(b_{11} - b_{21} - b_{31}) - a_1(b_{41} - b_{51}) - a_6 c_{11}\} = 0$,
   $\{(b_{12} - b_{22} - b_{32}) - a_1(b_{42} - b_{52}) + a_6 c_{12}\} = 0$, and

   $\{(b_{13} - b_{23} - b_{33}) + a_6(b_{43} - b_{53})\} = 0$
2. $(i_d - a_3 \Delta p_d - a_4 p_d - a_5 s - \gamma t) \sim I(0)$,

and one medium-run relation, $\beta_{\perp 1}' \Delta x_t$:

1. $(\Delta p_d + d_1 \Delta p_f + d_2 \Delta s + d_3 \Delta i_f) \sim I(0)$.

Linear combinations between the above stationary relations are, of course, also stationary.

9 The empirical *CVAR* model

The empirical analysis is based on German-US data for the post Bretton Woods, pre-EMU period. The sample starts in 1975:3 and ends in 1998:12. The VAR is specified with two lags and augmented with a number of dummy variables:

$$\begin{aligned} \Delta^2 x_t &= \Gamma \Delta x_{t-1} + \Pi x_{t-1} + \mu_0 + \mu_{01} Ds_{91.1,t} + \mu_1 t + \mu_1 t_{91.1} \\ &\quad + \phi_1 D_{tax,t} + \phi_2 Dp_{86.2} + \varepsilon_t, \end{aligned} \tag{39}$$

where $x_t = [p_{d,t}, p_{f,t}, s_t, b_{d,t}, b_{f,t}]$ and p_t stands for CPI prices, s_t for the Dmk/dollar exchange rate, b_t for long-term bond rates, a subscript d for Germany and a subscript f for USA, $t_{91:1,t}$ is a linear trend starting in 1991:1, $Ds_{91:1,t}$ is a step dummy starting in 1991:1, $D_{tax,t}$ is an impulse dummy accounting for three different excise taxes levied to pay for the German reunification, and $Dp_{86.2}$ is account for some large interest rate shocks as a result of the Plaza Accord.

From (48) in the Appendix it follows that an unrestricted constant will cumulate twice to a quadratic trend, and a trend to a cubic trend and similarly for the tax dummy and the broken trend. To avoid such effects, the coefficients of all quadratic and cubic trends are set to zero in the model. See Kongsted at al. (1999) or Juselius (2006, Chapter, 17).

Because the second rank condition is formulated as a reduced rank on the transformed $\mathbf{\Gamma}$ matrix, the latter is no longer unrestricted as in the $I(1)$ model. To circumvent this problem Johansen (1997) proposed a different parameterization which was modified by Paruolo and Rahbek (2007) as follows:

$$\Delta^2 x_t = \alpha \left[\rho' \begin{pmatrix} \tau \\ \tau_{01} \\ \tau_0 \end{pmatrix}' \begin{pmatrix} x_{t-1} \\ t_{91:1,t-1} \\ t-1 \end{pmatrix} + \begin{pmatrix} \delta \\ \delta_{01} \\ \delta_0 \end{pmatrix}' \begin{pmatrix} \Delta x_{t-1} \\ D_{s91:1,t-1} \\ 1 \end{pmatrix} \right]$$

$$+\alpha_{\perp\Omega}\kappa' \begin{pmatrix} \tau \\ \tau_{01} \\ \tau_0 \end{pmatrix}' \begin{pmatrix} \Delta x_{t-1} \\ D_{s91:1,t-1} \\ 1 \end{pmatrix} + \Phi_p D_{tax,t} + \varepsilon_t, \tag{40}$$

$$t = 1975.09 - 1998.12$$

where $\tau = [\beta, \beta_{\perp 1}]$ and $\alpha_{\perp\Omega} = \Omega\alpha_\perp(\alpha'_\perp \Omega \alpha_\perp)^{-1}$.

Finally, $\rho' = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ is a $(p - s_2) \times p_d$ matrix, where I_r is a $r \times r$ unit matrix and 0 stands for a zero matrix. Its main purpose is to pick up the β relations that belong to the r polynomially cointegrating relations.

9.1 Rank determination

The standard trace test procedure starts with the most restricted model ($r = 0, s_1 = 0, s_2 = 5$), continues to the end of the row, and proceeds similarly row-wise from left to right until the first non-rejection. Because the trace tests of $r = 0, 1$ were all strongly rejected, the first two rows have been omitted from Table 1. The first non-rejection is at ($r = 2, s_1 = 1, s_2 = 2$) with a p-value of 0.53.The characteristic roots of the model are also reported both for the unrestricted VAR model and for the relevant reduced rank cases. The unrestricted VAR contains five large roots, four of which are almost exactly on the unit circle (0.98) while the fifth is large (0.81) but not equally close to one. Thus, the choice of reduced rank indices should be consistent with

Table 1: Determination of the two rank indices

Rank Test Statistics								
$p-r$	r	$s_2=5$	$s_2=4$	$s_2=3$	$s_2=2$	$s_2=1$	$s_2=0$	
3	2			136.21 [0.00]	**52.97** [0.53]	38.62 [0.56]	39.85 [0.10]	
2	3				31.66 [0.72]	12.06 [0.99]	16.11 [0.49]	
Six largest characteristic roots:								
Unrestricted VAR			**0.99**	**0.99**	**0.98**	**0.98**	**0.81**	0.49
$r=2, p-r=3$			1.0	1.0	1.0	**0.96**	**0.96**	0.49
$r=2, s_1=2, s_2=1$			1.0	1.0	1.0	1.0	**0.95**	0.50
$r=2, s_1=1, s_2=2$			1.0	1.0	1.0	1.0	1.00	0.50

four or five unit roots. The case $\{r=2, s_1=1, s_2=2\}$ restricts five of the characteristic roots to be on the unit circle with the largest unrestricted root equal to 0.49 which confirms the adequacy of this choice.

Table 1 also reports the roots under the assumption that data are $I(1)$. The choice of $\{r=2, s_1=3, s_2=0\}$ would leave two large characteristic roots (0.96, 0.96) in the model. Such large roots would render any inference on stationarity completely unreliable. Altogether, the results seem to support the existence of two near $I(2)$ trends, one supposed to be associated with the long persistent movements in the relative price, the other with the long swings in the nominal exchange rate.

In contrast, the results provide little support for the RE-based scenario in Section 5 where the theory consistent case was found to be $\{r=4, s_1=0, s_2=1\}$ implying two (near) unit roots in the characteristic polynomial. Thus, the RE-based model seems unable to account for the second $I(2)$ trend capturing the long and persistent swings in real and nominal exchange rates around long-run PPP values.

To conclude, the reduced rank tests and the characteristic roots seem to favor the IK-based theoretical model rather than the RE-based model.

9.2 Testable hypotheses on integration and cointegration

All hypotheses in this section are and tested

The theory consistent RE- and IK-based CVAR scenarios imply different persistency properties that can be formulated and tested in the $I(2)$ VAR model. Two types of test procedures will be considered (see Johansen et al. (2010) for a detailed exposition).

The first type of tests imposes the same restriction on all τ relations and is formulated as $\tau = H\varphi$ (alternatively $R'\tau = 0$) where $\tau = [\beta, \beta_{\perp 1}]$:

1. Long-run price-homogeneity is assumed to hold in the RE-based model, but not in the IK-based model. It was rejected based on $\chi(3) = 11.14[0.00]$.

2. PPP is assumed to hold in the RE-based model, but not in the IK-based model. It was rejected based on $\chi(3) = 13.01[0.00]$.

3. The interest rate spread is assumed to hold in the RE-based model, but not in the IK-based model. It was rejected based on $\chi(3) = 34.19[0.00]$.

4. The broken linear trend is assumed excludable from the cointegration relations in the RE-based model, but not necessarily in the IK-based model. The results showed that the linear as well as the broken trend were individually excludable based on $\chi(3) = 6.02[0.11]$ and $\chi(3) = 3.61[0.31]$ respectively. The test of joint excludability was only borderline acceptable based on $\chi(6) = 11.17[0.08]$).

The small p-value in 4. suggests, however, that allowing for a broken trend in the cointegration relations is likely to improve the results. A sensitivity analysis showed that, while the main conclusions were reasonably robust to either including or leaving out the trend, including the trend seemed to improve model specification. It has, therefore, been included in the cointegration relations.

The second type of hypotheses is formulated as a known vector b_1 in τ, i.e. $\tau = (b_1, b_{1\perp}\varphi)$ where $b_{1\perp}\varphi$ defines the other vectors to be in the orthogonal space of b_1.[11] For example $b_1 = [1, 0, 0, 0, 0, 0, 0]$ tests whether the German price is a unit vector in τ. If not rejected, $p_{d,t}$ can be considered $I(1)$, if rejected $I(2)$. Note, however, that the above test does not include a linear (broken) trend in the relation and not allowing for a linear trend, if significant, is likely to bias the test towards rejection of $I(1)$. Therefore, the tests of the trend-adjusted price, the nominal and the real exchange rate being $I(1)$ are also reported in the table. It is formulated as $\beta = (H_1\varphi_1, H_2\varphi_2)$

[11] Note that in the $I(1)$ model this type of hypothesis is testing whether a variable/relation is $I(0)$, whereas in the $I(2)$ model whether it is $I(1)$.

where H_1 is a design matrix picking up the variable/relation in question and H_2 imposes just-identifying restrictions on the remaining $r-1$ cointegration relations. For example, $H_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is a test whether the trend-adjusted German price can be considered $I(1)$.

Table 2 reports the test results. Except for the German bond rate, all $I(1)$ hypotheses were rejected, most of them strongly so. This shows that that the processes have exhibited a pronounced persistence untenable with $I(1)$ behavior. The fact that the German bond rate could be rejected as $I(2)$ with a p-value of 0.24 suggests that the German bond rate has moved in a slightly less persistent manner than the other variables. In line with the discussion in Sections 6-7, the results support the relevance of allowing for an uncertainty premium in the currency market consistent with the IK-based model, whereas they are inconsistent with the RE-based model. Juselius and Assenmacher (2015) report strikingly similar results for Swiss-US data.

10 The long-run structure

The preferred model $\{r=2, s_1=1, s_2=2\}$ defines two stationary polynomially cointegrating relations, $\beta_i^{'}x_t + \delta_i'\Delta x_t,\ i = 1,2$ and one stationary medium-run relation in growth rates, $\beta_{\perp 1}'\Delta x_t$. The former can be interpreted as dynamic equilibrium relations in the following sense: When data are $I(2)$, $\beta^{'}x_t$ is generally $I(1)$ and describes a very persistent equilibrium error. In this case, $\delta'\Delta x_t$ also being $I(1)$ will react in a similar but compensating manner so that $\beta^{'}x_t + \delta'\Delta x_t$ is stationary. Thus, when discussing the adjustment dynamics in the $I(2)$ model, it is useful to interpret the coefficients α and δ as two levels of equilibrium correction: The δ adjustment describes how the growth rates, Δx_t, adjust to the long-run equilibrium errors, $\beta^{'}x_t$; the α adjustment describes how the acceleration rates, $\Delta^2 x_t$, adjust to the dynamic equilibrium relations, $\beta^{'}x_t + \delta'\Delta x_t$. This is illustrated below for the variable $x_{i,t}$:

$$\Delta^2 x_{i,t} = \cdots \sum_{i=1}^{r} \alpha_{ij}(\delta_i'\Delta x_{t-1} + \beta_i' x_{t-2}) + \cdots, j = 1, ..., p \tag{41}$$

where $\delta_i' = [\delta_{i1}, ..., \delta_{ij}, ..., \delta_{ip}]$ and β_i' is similarly defined.

The long and persistent swings away from fundamental PPP values implies self-reinforcing dynamic feedback mechanisms somewhere in the system

Table 2: Testing hypotheses of I(1) versus I(2)

		p_d	p_f	s	b_1	b_2	t_{91}	t	$\chi^2(v)$	$p-val$
Is German trend-adjusted price $I(1)$?										
$\mathcal{H}_1$	β_1'	1	-	-	-	-	*	*	43.8 (2)	0.00
Is US trend-adjusted price $I(1)$?										
$\mathcal{H}_2$	β_1'	-	1	-	-	-	*	*	39.4 (2)	0.00
Are relative prices $I(1)$?										
$\mathcal{H}_3$	τ_1'	1	-1	-	-	-	-	-	62.4 (4)	0.00
$\mathcal{H}_4$	β_1'	1	-1	-	-	-	-	*	41.8 (3)	0.00
$\mathcal{H}_5$	β_1'	1	-1	-	-	-	*	*	23.0 (2)	0.00
Is the nominal exchange rate $I(1)$?										
$\mathcal{H}_6$	τ_1'	-	-	1	-	-	-	-	22.2 (4)	0.00
$\mathcal{H}_7$	β_1'	-	-	1	-	-	-	*	22.2 (3)	0.00
$\mathcal{H}_8$	β_1'	-	-	1	-	-	*	*	12.2 (2)	0.00
Is real exchange rate $I(1)$?										
$\mathcal{H}_9$	τ_1'	1	-1	-1	-	-	-	-	10.6 (4)	0.03
Is German bond rate $I(1)$?										
$\mathcal{H}_{10}$	τ_1'	-	-	-	1	-	-	-	5.4 (4)	0.24
Is US bond rate $I(1)$?										
$\mathcal{H}_{11}$	β_1'	-	-	-	-	1	-	-	14.3 (4)	0.01
Is the bond rate differential $I(1)$?										
$\mathcal{H}_{12}$	τ_1'	-	-	-	1	-1	-	-	12.4 (4)	0.01

also called a positive and negative feedback mechanism (Soros, 2010). This can be empirically studied by checking the signs of β, δ, and α in the following way: If $\delta_{ij}\beta_{ij} > 0$, then the changes are equilibrium correcting to the levels; if $\alpha_{ij}\beta_{ij} < 0$ then the acceleration rates are equilibrium correcting to the long-run levels relation $\beta_i' x_{t-1}$; if $\alpha_{ij}\delta_{ij} < 0$ then the acceleration rates, $\Delta^2 x_{i,t}$, are equilibrium correcting to the differences, $\delta_i' \Delta x_{t-1.}$ In all other cases the system is equilibrium error increasing. Whether a variable is equilibrium error correcting or equilibrium error increasing is important for understanding the long swings puzzle.

Since all characteristic roots were inside the unit circle[12] the system is stable implying that any equilibrium error increasing behavior is compensated

[12] Roots inside the unit disk imply non-explosive behavior as they are calculated as eigenvalues of the characteristic polynomials (Juselius, 2006).

Table 3: An identified long-run structure in β

$$\tilde{\beta} = (h_1 + H_1\varphi_1, \ldots, h_r + H_r\varphi_r)\,,\ \chi^2(6) = 4.60[0.60]$$

	$p_{d,t}$	$p_{f,t}$	s_t	$b_{d,t}$	$b_{f,t}$	$t_{91.1}$	t
$\tilde{\beta}_1^{'}$	**−0.01** [−32.2]	**0.01** [32.2]	**0.01** [32.2]	**1.00**	**−1.00**	−	−
$\tilde{\delta}_1^{'}$	*0.17*	**1.10**	*−0.47*	*−0.00*	*0.00*	−0.006	0.013
α_1'	**0.47** [15.7]	**−0.10** [−3.5]	*1.62* [3.2]	**−0.01** [−4.5]	**0.02** [4.1]		
$\tilde{\beta}_2^{'}$	−	**−0.01** [74.3]	**−0.01** [74.3]	−	**1.00**	**0.00** [3.1]	**0.00** [2.0]
$\tilde{\delta}_2^{'}$	**−0.22**	*0.89*	*0.04*	0.00	−0.01	−0.001	0.040
α_2'	**0.65** [9.8]	**0.40** [6.2]	**3.42** [3.1]	**−0.03** [−5.0]	0.01 [1.4]	−	−
$\tilde{\beta}_{\perp,1}^{'}$	**1.00**	**−0.262**	**−0.250**	0.015	−0.005	0.001	-0.002[1)]

1) The trend estimate has been multiplied by 10000. significant values in bold face.

by error correcting behavior somewhere else in the system. Thus, while variables can move away from their long-run stable equilibrium path for extended periods of time, sooner or later the equilibrating forces will set in, for example due to an increasing uncertainty premium, and pull the variable back toward equilibrium.

Johansen et al. (2009) discusses a test of over-identifying restrictions on $\beta^{'}x_t$. A test of over-identifying restrictions on both $\beta^{'}x_t$ and $\delta'\Delta x_t$ has recently been worked out in Mosconi and Paruolo (2015) but has not yet been implemented. Table 3 reports an overidentified structure on β, together with the corresponding unrestricted estimates of δ and $\beta_{\perp 1}'$.

The first polynomially cointegrated relation, $\tilde{\beta}_1^{'}\tilde{x}_t + \tilde{\delta}_1^{'}\Delta\tilde{x}_t$, is given by:

$$(b_d - b_f)_t - 0.01(p_{d,t} - p_{f,t} - s_t) + 0.17\Delta p_{d,t} + 1.1\Delta p_{f,t} - 0.47\Delta s_t + ...$$

It corresponds closely to the UAUIP relation (20) in which the PPP gap is a proxy for the uncertainty premium and the expected change in the exchange rate is proxied by the short-term changes in the process $(0.17\Delta p_{d,t}+1.1\Delta p_{f,t}-0.47\Delta s_t)$. One implication of the estimated relation is that the real exchange rate can deviate persistently from its long-run value as long as the interest rate differential moves in a compensating manner.

In the medium-run the δ coefficients show that both inflation rates equilibrium correct to the long-run PPP ($\alpha_{ij}\beta_{ij} < 0$) but very slowly so. The speed of adjustment for US inflation rate to the PPP gap is approximately 0.01 implying that it would take on average 6-8 years for the real exchange rate to return to its long-run value if the inflation rate alone were to adjust. German inflation is equilibrium increasing in the medium run ($\delta_{ij}\beta_{ij} < 0$) while error-correcting in the long-run. The coefficient $\alpha_{11} = 0.47$ shows that German price has adjusted quickly (in two months on average) when the interest rate differential exceeds the 0.01 fraction of the real exchange rate. For the nominal exchange rate the δ adjustment is equilibrium error increasing ($\delta_{ij}\beta_{ij} < 0$) while the α adjustment in Δx_t is equilibrium error correcting ($\alpha_{ij}\delta_{ij} < 0$). Altogether it supports the IKE hypothesis that over the medium run the nominal exchange rate will have a tendency to move away from long-run benchmark values while over the longer run it will move back towards equilibrium. For both bond rates, the δ adjustment is equilibrium error increasing ($\delta_{ij}\beta_{ij} < 0$), suggesting that the behavior of long-term interest rates is key to understanding the long swings movements in the currency market over the chosen period.

The second polynomially cointegrated relation, $\tilde{\beta}_1'\tilde{x}_t + \tilde{\delta}_1'\Delta\tilde{x}_t$, is given by:

$$b_{f,t} - 0.01(p_{f,t} + s_t) + 0.00000trend + 0.89\Delta p_{f,t} - 0.22\Delta p_{d,t} + ... \qquad (42)$$

corresponding approximately to the third $CI(2,1)$ relation in the scenario analysis of Section 8. It shows that an increase/decrease in theUS bond rate has been associated with an increase/decrease in the trend-adjusted US price denominated in Dmk. Both the US prices and the Dmk/dollar rate are equilibrium error increasing in the medium run ($\delta_{ij}\beta_{ij} < 0$) but error correcting over the long run ($\alpha_{ij}\beta_{ij} < 0$). The medium-run stationary relation between growth rates is formulated as:

$$\Delta p_{d,t} \simeq 0.25(\Delta p_{f,t} + \Delta s_t) \qquad (43)$$

$$\Delta(s - p_d - p_f)_t \simeq 0.75(\Delta p_{f,t} + \Delta s_t), \qquad (44)$$

where (43) suggests that the drift in German inflation rate has been associated with the drift in US inflation rate measured in Dmk, pointing to the importance of imported inflation for German prices.[13] Equivalently, (44) can

[13] That the coefficient differs from 1.0 is consistent with the finding that $(p_d - p_f - s) \sim I(2)$ and, hence, $(\Delta p_d - \Delta p_f - \Delta s) \sim I(1)$. It shows that German inflation rate has generally been lower than the US inflation rate when measured in the same currency.

be expressed to show that the drift in the real exchange rate has been associated with the drift in the US inflation rate and the nominal exchange rate.

11 Conclusions

This paper argues that accounting for near unit roots in the model provides a powerful way of classifying data into persistent and less persistent directions and that this can be used for testing and comparing competing models. The idea is formalized as theory consistent CVAR scenario which describes the empirical regularities we should expect to see if the theory model is empirically relevant. To solve the problem of unobservable expectations, the paper proposes to associate the latter with theory-consistent observable variables. As an illustration, the paper demonstrates how to translate basic assumptions underlying a monetary model for real exchange rate determination based on "rational" contra imperfect knowledge expectations into testable hypotheses in the CVAR model.

The empirical findings provided overwhelmingly strong support for the relevance of the imperfect knowledge type of models as opposed to the rational expectations based models. Between the two models, only the former could explain the near $I(2)$ properties of the data and, thus, the long and persistent swings in the real exchange rate. These swings were shown to be consistent with the existence of an uncertainty premium in foreign currency markets measured by the PPP gap. The latter was found to be a near $I(2)$ process and caused the real interest rate differential and the real interest to be near $I(2)$.

Such persistent movements in important determinants for th real economy as the real exchange rate and the real interest rate shows that speculative behavior in the foreign currency market is likely to have serious implications for the macroeconomy and points to the importance of understanding the pronounced persistence typical of economic data. The failure of extant models to foresee the recent financial and economic crisis and to propose adequate policy measures in its aftermath is strong evidence for this conclusion.

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13 Appendix A: The CVAR

13.1 The Basic VAR model for I(2) data

For convenience of interpretation, an unrestricted VAR(2) is formulated in acceleration rates, changes and levels:

$$\Delta^2 x_t = \Gamma \Delta x_{t-1} + \Pi x_{t-1} + \mu_0 + \mu_1 t + \varepsilon_t, \tag{45}$$

where $x_t = [p_{d,t}, p_{f,t}, s_t, b_{d,t}, b_{f,t}]$ with $p_{d,t}, p_{f,t}$ measuring German/US prices, s_t the Dmk-dollar rate, and $b_{d,t}, b_{f,t}$ German/US interest rates, and $\varepsilon_t \sim$

$Niid(0, \Omega)$.

The hypothesis that x_t is $I(1)$ is formulated as a reduced rank hypothesis on Π:

$$\Pi = \alpha\beta' \text{ , where } \alpha, \beta \text{ are } p \times r \tag{46}$$

and that x_t is $I(2)$ is formulated as an additional reduced rank hypothesis

$$\alpha'_{\perp}\Gamma\beta_{\perp} = \xi\eta', \text{ where } \xi, \eta \text{ are } (p-r) \times s_1. \tag{47}$$

where $\alpha_{\perp}, \beta_{\perp}$ are the orthogonal complements of α, β respectively. The first reduced rank condition is associated with the levels of the variables and the second with the differenced variables. The intuition is that the differenced process also contains unit roots when data are $I(2)$.

The moving average representation of subject to (46) and (47) expresses the variables x_t as a function of once and twice cumulated errors and deterministic components:

$$\begin{aligned} x_t \;=\; & C_2 \sum_{j=1}^{t} \sum_{i=1}^{j} (\varepsilon_i + \mu_0 + \mu_1 i) + C_1 \sum_{j=1}^{t} (\varepsilon_j + \mu_0 + \mu_1 j) \\ & + C^*(L)(\varepsilon_t + \mu_0 + \mu_1 t) + A + Bt, \end{aligned} \tag{48}$$

where $C_2 = \beta_{\perp 2}(\alpha'_{\perp 2}\Psi\beta_{\perp 2})^{-1}\alpha'_{\perp 2}$. The remaining matrices, C_1 and $C^*(L)$, are also functions of the parameters of the VAR model but will not be discussed here. See Johansen, 1992 for further detail.

To facilitate the interpretation of the $I(2)$ trends and how they load into the variables, it is useful to denote $\breve{\beta}_{\perp 2} = \beta_{\perp 2}(\alpha'_{\perp 2}\Psi\beta_{\perp 2})^{-1}$ or $\breve{\alpha}'_{\perp 2} = (\alpha'_{\perp 2}\Psi\beta_{\perp 2})^{-1}\alpha'_{\perp 2}$ so that

$$C_2 = \breve{\beta}_{\perp 2}\alpha'_{\perp 2} \text{ or } C_2 = \beta_{\perp 2}\breve{\alpha}'_{\perp 2} \tag{49}$$

Using (49) it is straightforward to interpret $\alpha'_{\perp 2}\sum_{j=1}^{t}\sum_{i=1}^{j}\varepsilon_i$ (alternatively $\breve{\alpha}'_{\perp 2}\sum_{j=1}^{t}\sum_{i=1}^{j}\varepsilon_i$) as an estimate of the s_2 second order stochastic trends which load into the variables x_t with the weights $\breve{\beta}_{\perp 2}$ (alternatively $\beta_{\perp 2}$).

13.2 A Simple Illustration

To introduce notation and the basic idea of structuring the data into pulling and pushing forces, I shall use a simple 3-dimensional VAR model for $x'_t =$

$[x_1, x_2, x_3]$, where the variables for example could be the nominal exchange rate and domestic and foreign prices. The model distinguishes between $p-r$ pushing and r pulling forces. I assume that $(r = 1, p - r = 2)$

The $I(2)$ model has a more complicated structure. The vector x_t is now integrated of order 2 and the $p-r$ stochastic trends are divided into s_1 first order and s_2 second order stochastic trends, i.e. $p - r = s_1 + s_2$. The r cointegration relations, $\beta' x_t$, are generally integrated of order 1, i.e. they cointegrate from $I(2)$ to $I(1)$, often denoted $CI(2,1)$, and becomes stationary by adding a linear combination of the growth rates, $\delta' \Delta x_t$. In addition there are s_1 linear combinations, $\beta'_{\perp 1} x_t \sim I(1)$, which can become stationary exclusively by differencing, i.e. $\beta'_{\perp 1} \Delta x_t \sim I(0)$. Thus, the $I(2)$ model contains $p - s_2$ $CI(2,1)$ relations, $\tau' x_t$, where $\tau = (\beta, \beta_{\perp 1})$. Furthermore, when $r - s_2 > 0$, it is possible to find $r - s_2$ relations $\beta' x$ which are stationary without adding the growth rates.

Under the assumption $(r = 1, s_1 = 1, s_2 = 1)$, the pulling forces are described by the equilibrium $\Sigma\Sigma$error correcting model $\Delta^2 x_t = \alpha(\beta' x_{t-1} + \delta' \Delta x_{t-1}) + \zeta \tau' \Delta x_{t-1} + \varepsilon_t$, where $\tau = [\beta, \beta_{\perp 1}]$, i.e. as:

$$\begin{bmatrix} \Delta^2 x_{1,t} \\ \Delta^2 x_{2,t} \\ \Delta^2 x_{3,t} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} (\beta' x_{t-1} + \delta' \Delta x_{t-1}) + \begin{bmatrix} \zeta_{11} & \zeta_{21} \\ \zeta_{12} & \zeta_{22} \\ \zeta_{13} & \zeta_{23} \end{bmatrix} \begin{bmatrix} \beta' \Delta x_{t-1} \\ \beta'_{\perp 1} \Delta x_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}$$

where $\beta' x_{t-1} + \delta' \Delta x_{t-1}$ describes a deviation from a dynamic long-run equilibrium relation, and $\beta' \Delta x_{t-1}$ and $\beta'_{\perp 1} \Delta x_{t-1}$ describe deviations from two medium-run equilibrium relations among growth rates.

The pushing forces are given by the common stochastic trends form $x_t = \beta_{\perp 2} \Sigma\Sigma u_s + B \Sigma u_i + ... \varepsilon_t$, i.e. as:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix} = \begin{bmatrix} \beta_{\perp 2,1} \\ \beta_{\perp 2,2} \\ \beta_{\perp 2,3} \end{bmatrix} \left[\sum_{i=1}^{t} \sum_{s=1}^{i} u_{1,s} \right] + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{t} u_{1,i} \\ \sum_{i=1}^{t} u_{2,i} \end{bmatrix} + ...$$

where $u_{1,t} = \alpha'_{\perp 2} \varepsilon_t$ is an autonomous shock that double cumulates over time and $u_{2,t} = \alpha'_{\perp 1} \varepsilon_t$ is an autonomous shocks that cumulates once over time. $\alpha_\perp = [\alpha_{\perp,1}, \alpha_{\perp,2}]$, is a 3×2 matrix orthogonal to α, defining the two shocks as linear combination of the VAR residuals $\hat{\varepsilon}_t$. $\beta_{\perp 2}$ is a 3×1 vector orthogonal to $\{\beta, \beta_{\perp 1}\}$ measuring how the $I(2)$ stochastic trend loads into the variables.

Table 4: Misspecification tests

Multivariate tests:					
Autocorrelation:	Lag 1: $\chi^2(25) = 35.0$ [0.09]				
	Lag 2: $\chi^2(25) = 27.9$ [0.31]				
ARCH:	Lag 1: $\chi^2(225) = \mathbf{372.9}$ [0.00]				
	Lag 1: $\chi^2(450) = \mathbf{640.8}$ [0.00]				
Normality:	$\chi^2(10) = \mathbf{47.8}$ [0.00]				
Univariate tests:					
	$\Delta^2 pp_t$	$\Delta^2 s_t$	$\Delta^2(b_1 - b_2)$	$\Delta^2 p_d$	$\Delta^2 b_1$
ARCH	0.07 [0.96]	3.81 [0.15]	**17.23** [0.00]	3.04 [0.22]	**29.62** [0.00]
Skew.	-0.12	0.02	0.05	-0.11	0.11
Kurt.	3.00	3.68	4.67	4.72	5.15
Norm.	0.70 [0.70]	6.45 [0.04]	**26.80** [0.00]	**27.83** [0.00]	**39.15** [0.00]
R^2	0.60	0.17	0.24	0.71	0.20

14 Appendix: Mispecification tests

Table 4 reports various model specification tests which show that the interest differential and the US bond rate model do not pass the ARCH and the residual normality test. Non-normality and ARCH are typical features of financial variables, but adding more dummies is not necessarily a good solution. As the non-normality is primarily due to excess kurtosis but not skewness and the VAR results are reasonably robust to moderate ARCH and excess kurtosis I continue with this model.