Complements versus Substitutes and Trends in Fertility Choice in Dynastic Models*

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Abstract

Demographers have long emphasized decreased mortality and 'economic development' as the main contributors generating the demographic transition (DT). In economics, the Barro-Becker (BB) model of fertility choice, though simple and intuitive, has not been successful at reproducing changes in fertility in line with the demography literature—at least in its original formulation. We show that this is due to an implicit assumption that number and welfare of children are complements, a byproduct of high intertemporal elasticity of substitution (IES) typically assumed in the fertility literature. Not only is this assumption not necessary, but qualitative and quantitative properties of the model in terms of fertility choice change dramatically when substitutability and low IES are assumed. These results do not require non-homotheticities or any other major changes to the basic BB model but emphasize productivity *growth rates* as opposed to income *levels* to interpret 'economic development.' We find that with an IES of one-third, model predictions of changes in fertility amount to two-thirds of those observed in U.S. data since 1800. The increase in productivity growth accounts for 90 percent of the predicted fall in fertility before 1880; and changes in mortality account for 90 percent of the predicted fall from 1880 to 1990.

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1 Introduction

Between 1800 and 2000, fertility choice in the U.S. underwent a dramatic transition. Depending on estimates, it fell from about 6 children per woman to around 2 children per woman. This demographic transition (DT)—from a state of high fertility and high mortality to low fertility and low mortality—has been documented by demographers and historians in the U.S., and in every other developed country in the world. Over this same period, natural rates of population growth (net of immigration) also fell, but less substantially. Demographers have not reached a consensus on the root causes of this transformation, but list several changes as likely main contributors. These include reductions in Infant and Child Mortality Rates (IMR's and CMR's) and the process of industrialization and economic development. The work in demography and history on this topic has largely been limited to empirical studies with almost no choice based modeling of the type seen in economics. Even there, issues of the timing of changes has lead to controversy about which of these causes is most important.[1]

Much of the economic literature on the topic has focused on estimating the effects of these causal variables on fertility within optimizing models. There is an added difficulty that must be faced to do this however. While it is straightforward to include changes in IMR's and CMR's in the analysis, it is unclear how to study the effects of the 'process of industrialization and development.' For example, does this phrase mean changes in the levels of income, or changes in the growth rate of income? Both of these occurred over this period.

Early work on the DT in economics was done by Barro and Becker (Becker and Barro (1988) and Barro and Becker (1989)) who developed an intuitive and analytically tractable model based on dynastic altruism—parents have children because they themselves receive utility benefits from the happiness of their descendants. Because of the construction, the model has a natural recursive structure, with time consistent preferences across generations. This allows the model to be studied using standard techniques in economics.

Despite this intuitive appeal, the Barro-Becker (BB hereafter) model, in its original version, has shortcomings as a model of the DT. These can be summarized by three properties of the model: 1) certainty equivalent versions of the model have the property that surviving fertility always increases in response to reductions in IMR and CMR, births can go either up or down under usual parameterizations, but when it does go down, this change is usually quite modest (see Doepke (2005) for a useful summary); 2) in its usual incarnations (with constant elasticities), the model has a balanced growth path (BGP) and hence, changes in income *levels* per se, have no effect on fertility; 3) while increases in income growth rates imply reductions in fertility in partial equilibrium (i.e., fixed interest rates) they imply increased

[1] See Chesnais (1992) for a useful discussion of a literature that dates back to at least Notestein (1945).

fertility under usual parameterizations when general equilibrium effects are included (i.e., interest rates are endogenous). In this sense, the DT is a challenge in the BB framework. Because of these weaknesses authors attempting to build quantitative models of the DT have been forced to make serious changes in the model (e.g., non-homothetic preferences) and/or look for other forcing variables (e.g., changes in child-labor laws, etc.) (see below).

In this paper, we revisit the ability of versions of the BB model to match the basic facts of the DT. We show that previous failures are in large part due to important, but unnecessary, restrictions in parameterizations. In brief, most of the authors who have examined the quantitative properties of the BB model (including BB themselves) have restricted attention to flow utility functions with very low curvature (i.e., high intertemporal elasticity of substitution, IES), typically less than log utility. This has two important implications. First, it implies that family size and descendant welfare are complements in the utility of the parent. (More generally, it implies that family size and child 'quality' are complements—see below). Since the changes suggested by demographers above typically increase descendant welfare, it follows that the demand for children increases in response to these changes. In contrast, we focus on versions of the model built on high curvature in utility (i.e., low IES)—more in line with the growth and real business cycle literatures. In this case, family size and descendant welfare are substitutes in the utility of the parent (or family size and child quality are substitutes). Because of this, increases in productivity growth rates cause population growth rates and births to fall (substantially) in keeping with what is seen in the data.

Second, high IES has another direct implication which is primarily important for quantitative reasons. The properties of the BB model are much like those of any growth model (indeed this analogy is exact in some cases). In particular, when the IES is high, reductions in the costs of investment goods are met with large increases in growth rates. In the BB model, the analogues are the costs of raising surviving children (which decrease when IMR's and CMR's decrease) and population growth. Thus, if the IES is high, when IMR's and CMR's fall, the number of surviving children increases substantially and the reduction in births is modest. In contrast, when IES is low, reductions in IMR's and CMR's are met with small increases in surviving fertility and population growth rates and large reductions in overall birth rates.

To build intuition for these results, we derive simple comparative statics at the household level in Section 2. These highlight the effect of the complementarity versus substitutability assumption for changes that exogenously increase children's utility. For simplicity, we abstract from bequests and assume (exogenously growing) labor income throughout Sections 2 to 5, but, as is shown in Section 6, the same effects are present in versions of the model with a child quality choice (either in the form of direct bequests or education). Indeed, when the IES is low, fertility and child quality itself are substitutes in the value function of the family head and hence similar comparative statics go through.

In Section 3, we introduce longevity through random survival of adults and child mortality to derive the dynasty's utility and law of motion for population. The simplicity of the model then allows us to derive analytical comparative statics across balanced growth paths (BGP) in Section 4. In keeping with the discussion above, we analyze the fertility response to permanent changes in productivity growth rates, longevity and youth mortality and distinguish between population growth rates and total births. We contrast the results for an IES above or below unity in this case. Our findings are summarized in Proposition 1. First, when the IES is below (above) one, we find that all measures of fertility fall (increase) in response to increases in productivity growth. Second, the size of fertility responses to increases in survival probabilities is decreasing in the IES (i.e., smaller increases in the population growth rate and larger reductions in birth rates as the IES decreases).

In Section 5, we study quantitative predictions of the simple model. We calibrate child costs in the model to match the recent fertility experience in the U.S. given its economic circumstances. We find that with standard parameterizations (an IES of two), calibrated costs as a fraction of income are unrealistically high—the maximum number of children a two-parent household could feasibly have is less than three. The corresponding maximum for an IES of one third is twelve children—thus, a lower IES resolves a commonly encountered puzzle in the fertility literature. We then simulate the U.S. experience since 1800, taking the timing of events and all three changes in economic environment into account. We find that with an IES of one third, the model accounts for about two-thirds of the observed decrease in the Crude Birth Rate. In terms of population growth rates, the model accounts for about one half of observed changes in the U.S.. From these two findings, we conclude that the changes emphasized by demographers were definitely important forces behind the DT. Moreover, the model also has implications that speak to the debate about which of these causes was most important and when. Interestingly, about 90 percent of model predicted changes in fertility before 1880 are accounted for by changes in productivity growth rates, while changes in mortality account for about 90 percent of the predicted fertility decrease thereafter. Thus, we find: it was a combination of both mortality and development with the latter being the most important factor early in the transition and mortality being the most important factor later on. We find that changes in longevity alone have only a small effect on fertility choice. Finally, these changes do not account for all of the reduction in fertility seen in the data. Thus, there is ample room for changes in child costs and education policies as emphasized by other authors (see below).

In Section 6, we generalize the model and study the inclusion of various types of investments into the model. First, we show that dynasty size and quality per surviving descendant are substitutes versus complements depending on whether the IES is above or below unity as before. We then interpret quality as physical capital (or bequests). The findings shed light on the original result in the BB papers, namely that holding the interest rate fixed, an increase in productivity growth decreases fertility, independently of the IES.[2] What we find is that, when the interest rate is determined in equilibrium, the effect of productivity growth depends on the size of the IES again. That is, for low values of IES, population growth rates fall when productivity growth rates increase. However, the IES threshold for which the sign of this effect changes is now less than one. As before, the size of effects of mortality is decreasing in the IES. Quantitatively, with a Cobb-Douglas production function, labor augmenting technological change and an IES of one-third, overall effects are slightly larger than without capital. This result suggests that even this model bypasses important channels such as investments in human capital or complementarities in the production function. We briefly address such channels and show that similar results go through.

Many other economists have also studied trends in fertility but have focused on different channels and setups.[3] We view these channels as complementary to those studied here. First, Becker, Murphy, and Tamura (1990) and Tamura (1994, 1996) use standard BB preferences but the crucial feature for the fertility decrease is an increase in the rate of return to human capital across multiple equilibria in models with increasing returns. Low IES brings about a utility version of Becker's quantity-quality trade-off because number and utility of children are now substitutes. Hence, under this assumption, the requirements on technology to generate the DT may well be weaker. Second, our findings also help clarify the results of several authors who have recently studied quantitative versions of the BB model to examine its ability to track the basic trends over the last 200 years in fertility choices in response to increases in productivity growth and decreases in mortality. These include Mateos-Planas (2002), Doepke (2005), and Bar and Leukhina (2007). We show that the differences in their assumptions, regarding the IES, and hence, implicitly, complementarity versus substitutability between number and utility of children, is key to reconciling their seemingly contradictory results. Further, in a BB model with IES above one and non-convexities in education choice, Doepke (2004) generates the change in fertility by increases in the importance of skill intensive technologies. The speed of the transition is analyzed resulting

[2] See Becker and Barro (1988), p. 19, and Barro and Becker (1989), p.494.

[3] See Galor (2005) for a recent survey.

from the timing of changes in child labor laws and education subsidies (see also Doepke and Zilibotti (2005)). In a BB type model with unitary elasticity, Fernandez-Villaverde (2001) uses falling capital prices when capital and skilled labor are complements as a key feature. The quantitative results in these papers may well change if lower values of IES were allowed. Finally, as noted, our model, like many based on dynastic altruism, has a balanced growth path (BGP) property. Therefore changes in the *level* of productivity has no permanent effect on fertility choice. Rather it is the *acceleration* in income growth that causes permanent changes with low IES. This distinguishes our theory from many others in the field, including Galor and Weil (2000) and Greenwood and Seshadri (2002). Both ultimately generate the decrease in fertility by increased *levels* of income with non-homotheticities in utility. Our theory is based on growth rates in conjunction with family size and descendants welfare being substitutes in utility. The distinction between levels versus growth rates provides a new focus of study to the demography literature. This list is necessarily incomplete of course, but we think that the key role played by the IES highlighted in our results suggests that further study of the traditional explanations in demography with these other channels is warranted.

Our finding that the model predictions are so much more in line with the data when family size and descendant welfare are substitutes is suggestive, but, does not mean that this is the correct specification of preferences. There is a small literature that uses micro data to try to determine whether family size and child quality are complements or substitutes (see Schultz (2005) for a recent review). This research is ongoing, but does not rule out the possibility that family size and child quality are substitutes in the value function—the key property that we exploit here.

2 Barro-Becker altruism revisited: complements versus substitutes

In this section, we discuss Barro-Becker type preferences and show how one implicit restriction, namely positive utility, has lead to the assumption of complementarity between number and utility of children. We also show why this assumption is crucial in determining the fertility response to permanent changes in productivity growth, youth mortality and longevity.

The standard presentation of the Barro-Becker model usually begins with a description of the preferences of a period-t adult. It is assumed that parents care about three separate objects:

i) their own consumption in the period, c_t;

ii) the number of children they have, n_t;

iii) the average utility of their children, U_{t+1}.

This is usually specialized further. It is assumed that utility of the typical time-t household is of the form:

$$U_t = u(c_t) + \beta g(n_t) \sum_{i=1}^{n_t} \frac{1}{n_t} U_{it+1}$$

where U_{it+1} is the utility of the ith child of the parent. Assuming equal treatment and no heterogeneity $U_{it+1} = U_{i't+1} = U_{t+1}$ for all i, i', this simplifies to:

$$U_t = u(c_t) + \beta g(n_t) U_{t+1}.$$

Intuitively, the following are desirable properties of utility:

1.) Parents like the consumption good:
utility is increasing and concave in own consumption, c_t;

2.) Parents are altruistic (with respect to born children):
holding n_t fixed and increasing U_{t+1} increases (strictly) the utility of the parent;

3.) Parents like having children:
holding U_{t+1} fixed and increasing n_t increases (strictly) the utility of the parent.

4.) The increase described in 3.) is subject to diminishing returns.

The first property is satisfied as long as u is increasing and concave. The second has implications for what g can be. Since $\partial(u(c) + \beta g(n)U)/\partial U = \beta g(n)$, it follows that (2.) implies that $g(n) > 0$ for all n. The third requirement is less straightforward. Although this requirement makes intuitive sense, some issues arise because of the special restrictions implicit on functional forms. For example, suppose that $U_{t+1} > 0$. Then (3.) implies that $g(n)$ must be increasing in n. On the other hand, if $U_{t+1} < 0$, (3.) implies that $g(n)$ should be decreasing in n. It follows that if it is possible for U_{t+1} to be either positive OR negative, it is impossible to satisfy all of these requirements simultaneously. In sum, (1.)-(3.) are mutually inconsistent without some sort of restrictions on the possible values for U_{t+1}. Similar issues arise with respect to (4.). If U_{t+1} is restricted to be positive, (4.) requires g to be concave while if U_{t+1} is restricted to be negative, (4.) requires that g is convex.

This is not to say that these conditions cannot be satisfied. We must simply assume that either $U_{t+1} > 0$ always or $U_{t+1} < 0$ always and then make the appropriate assumptions on g.[4] Without an assumption like this, the natural monotonicity properties of utility cannot be guaranteed. Thus, we are left with two options:

[4] One issue that arises here is whether or not, with negative utility of born children, parents are necessarily making potential children worse off by having them. This will not be true if the utility of unborn potential children is a large enough negative number. Further, the same representation for the choice problem of the individual parent will hold (approximately) if the fraction of potential children that can feasibly be born is small. See the Appendix in Jones and Schoonbroodt (2008) for more details on this point.

I. Assume $g(n)$ is non-negative, strictly increasing and concave and $U > 0$.

II. Assume $g(n)$ is non-negative, strictly decreasing and convex and $U < 0$.

As it turns out, the choice between these two alternatives has important implications for the properties of the model. This can be illustrated in a simple example. First, consider how the solution of the problem of a time zero parent changes when the growth rate in wages is changed. In the simplest case, they face a problem of the form:

$$\max_{\{c_0,n_0\}} \quad u(c_0) + \beta g(n_0) U_1$$
$$s.t. \quad c_0 + \theta_0 n_0 \leq w_0,$$

where θ_0 is the cost of raising a child to survive to adulthood and w_0 is the wage rate in time 0. Increased wage growth only enters this problem through the indirect effect of changing U_1. That is, if wages grow faster, future generations will have larger choice sets and hence, U_1 will be larger. The first order condition for this problem is:

$$LHS(n_0) \equiv \theta_0 u'(w_0 - \theta_0 n_0) = \beta U_1 g'(n_0) = RHS(n_0). \tag{1}$$

The left hand side of this equation is the marginal cost in terms of period 0 consumption of having an extra child and is increasing in n_0, while the right hand side is the marginal benefit and is decreasing.

A change in U_1 has different effects depending on which case we are in. In particular, whether an increase in U_1 increases or decreases the right hand side depends on whether U_1 is positive or negative. That is:

$$\frac{\partial RHS(n_0, U_1)}{\partial U_1} = \beta g'(n_0) = \frac{\partial^2 U_0}{\partial n_0 \partial U_1}. \tag{2}$$

When U_1 is positive, g is increasing and hence, the cross partial in (2) is positive—children and the utility of children are complements in the utility of the parent.[5] In this case, it follows that a change in wage growth shifts the right hand side up causing n_0 to increase. Fertility is increasing in the rate of growth of wages. When U_1 is negative, g is decreasing and hence, this is negative—children and the utility of children are substitutes in utility. In this case, it follows that the right hand side shifts down and n_0 falls. Fertility is

[5] Note that in this simple version U_1 is endogenous to the child or period-1 adult, but exogenous to the period-0 parent. Hence, "complements" and "substitutes" are a slight abuse of language. In Section 6, we introduce bequests which gives parents some control over children's utility alleviating the abuse of language.

decreasing in the rate of growth of wages. Thus, whether an increase in wage growth (i.e., an acceleration of industrialization) increases or decreases fertility is completely determined by this assumption. Also clear is that this effect is quite general—it is not restricted to changes in the rate of growth of wages.

I. If $U_1 > 0$, increasing U_1 increases the marginal utility of children. Because of this, anything that increases U_1 will lead to a greater desire for children (unless something else changes to offset that).

II. If $U_1 < 0$, increasing U_1 (smaller number in absolute value) lowers the marginal utility of children, making larger family sizes LESS desirable. Again, this is holding everything else equal.

From this discussion, we can also get a sense about how changes in the survival rates of children depend on this choice. It is common in the literature (e.g., Barro and Becker (1989), Doepke (2005), Bar and Leukhina (2007)) to model decreases in youth mortality as a reduction in the cost of producing a surviving child—a reduction in θ_0. Notice that this will also typically increase U_1 if the reduction in mortality is permanent (i.e., future costs of producing surviving children also decrease). A decrease in θ_0 causes the left hand side of the equation to shift down. Hence, when children and their utility are complements ($U_1 > 0$), it follows that fertility will increase. In the opposite case ($U_1 < 0$), there are off-setting effects and the sign of the change cannot be predicted without more detailed analysis. This discussion is complicated by the fact that U_1 is not exogenous. Below we derive a planner's problem with explicit mortality formulations to generalize these intuitions. Finally, note that in this simple case, the parent has no direct method of affecting U_1, and hence, although one part of a quality-quantity trade-off is present—the direct preference part—other aspects of it are missing (e.g., increasing U_1 through leaving a larger bequest or spending more on the education of the child). This is a weakness in some ways, but allows us to focus our attention on the importance of the role of preferences much more transparently. In Section 6, we address extensions in which the parent has some control over children's utility and show that the main results derived so far go through.

We conclude that the assumption one makes about whether children and their utility are substitutes or complements in the utility of the parent—and with it the implicit assumptions about both the sign of U and the monotonicity of g—has important qualitative implications about the properties of such models. Almost all work based on the Barro-Becker model to date has focused on the first case and this has had important implications about quantitative results using these extensions.

One question that comes to mind is: Are children and their well-being really substitutes in parent's utility? To provide some intuition, suppose the utility of the child is positively related to the purchase, by the parent of an investment good, say private education. And suppose the price of private education falls. When family size and utility are complements in utility, it follows that both education and family size increase, while when they are substitutes, education increases substantially, but family size falls. In Section 6, we include such investments as choice variables into the model and review empirical evidence. To the best of our knowledge, there is currently no good evidence for assuming that number and well-being of children are complements—at least not in the context of the nineteenth and early-twentieth century U.S.. Instead, we find indirect evidence to the contrary since with low elasticity and substitutability, the model is able to generate trends in fertility similar to those observed over the past 200 years in response to observed changes in productivity growth rates, mortality and longevity. These are also the drivers most emphasized by demographers.

3 A simple dynastic model of fertility choice

In this section we lay out the basic model we will analyze for the remainder of the paper and link choices for the IES in dynastic planner's problems to the discussion in Section 2. The model we study is a version of the Barro-Becker model with two changes. The first is to include adult survival. The second is to restrict attention to wage income (for simplicity).

3.1 Youth mortality and longevity

To address the effect on fertility choice of changes in mortality rates, we introduce child mortality and adult longevity by adopting a certainty equivalence formulation.[6]

We follow the original Barro-Becker work and assume that parents care only about surviving children, $n_{s,t} = \pi_s n_{b,t}$, where π_s is survival rate of a birth to adulthood and $n_{b,t}$ is the number of births. The total cost to the household of all births is then given by $\theta_{b,t} n_{b,t}$, where $\theta_{b,t}$ is the cost, in terms of goods, of the birth of each child.[7] Thus, even children that do not survive to adulthood can be costly.

We assume that the survival probability for adults is age independent and given by $\pi < 1$ (see Blanchard (1985)). This formulation allows us to perform comparative statics

[6] See Ben-Porath (1976), Sah (1991), Wolpin (1997), Eckstein, Mira, and Wolpin (1999), Kalemli-Ozcan (2002, 2003), and Doepke (2005) for models where family level uncertainty is included.

[7] Although we write the costs of children here in terms of goods, it is straightforward to reinterpret it as requiring the time of the parent. In this case, $\theta_{b,t}$ is directly proportional to the wage.

with respect to longevity (or, expected working life) and calibrate to reasonable life lengths without adding additional state variables.[8]

To simplify preference aggregation below, and keep the number of state variables in the model to a minimum, it is useful to impose some structure to the preferences of parents over their own future utility (if they survive) and that of their surviving children. We assume that the parent's continuation utility is a function only of expected family size and future per capita utility:

$$U_t = u(c_t) + \beta g(\pi + \pi_s n_{b,t})U_{t+1}.$$

For example, when $\pi = 0$ these preferences revert to those used in the original Barro-Becker model. At the other extreme, $\pi = 1$, people live forever and care about themselves exactly as much as they care about each surviving child.[9]

Finally, the household's budget constraint is:

$$c_t + \theta_{b,t} n_{b,t} \leq w_t.$$

Notice that all the intuitive results relating to complementarity and substitutability derived in the previous subsection go through with these changes. An additional effect is also present. This is that increased longevity (i.e., π) decreases the marginal utility of having an extra child, regardless of the sign of U_{t+1}.

3.2 Dynasty utility and intertemporal elasticity

To fully specify the model we need an explicit mechanism for the determination of the utility of subsequent generations. Note that by construction, any time-t preferences have a natural time consistency property: there is no inherent conflict in preferences between the agents in period t and period $t+1$. Therefore, the sequence of decisions made by the individual time t agents are exactly the same as what would be decided for them by the time 0 agent. To study this version of the problem, we can sequentially substitute $U_1, U_2, ..., U_{t+1}, ...$ into the utility function of the time 0 agent, the dynasty head, to get:

$$U_0 = \sum_{t=0}^{\infty} \beta^t \left[\Pi_{k=0}^{t-1} g(\pi + \pi_s n_{b,k}) \right] u(c_t).$$

[8] Barro and Sala-i-Martin (1995), Chapter 9, introduce longevity in a similar manner in a continuous time version of the Barro-Becker model.

[9] We treat parental longevity in this reduced form way for simplicity. An interesting extension would be to the case in which there are two outcomes, survival and not, with expected utility over the two as the parents utility function.

Because of the term $\Pi_{k=1}^{t-1} g(\pi + \pi_s n_{b,k})$, this utility function is typically not concave as written. However, as discussed in Alvarez (1999), under certain conditions, this can be rewritten as a concave problem in dynasty aggregate variables. Assume that $g(x) = x^{\eta}$, $\eta < 1$, and let $N_0 = 1$ and $N_t = \Pi_{k=0}^{t-1}(\pi + \pi_s n_{b,k})$, the expected total number of adults (parent and descendants) alive during period t evaluated in $t-1$. Assuming certainty equivalence/law of large numbers, we get the following law of motion for population:

$$N_{t+1} = \pi N_t + N_{s,t} = \Pi_{k=0}^{t}(\pi + \pi_s n_{b,k})$$

where $N_{s,t} = n_{s,t} N_t$ is the total number of surviving children born in period t. Then $\Pi_{k=1}^{t-1} g(\pi + \pi_s n_{b,k}) = g(\Pi_{k=1}^{t-1}(\pi + \pi_s n_{b,k})) = g(N_t)$, and so preferences for the dynasty head can be rewritten as:

$$U_0 = \sum_{t=0}^{\infty} \beta^t g(N_t) u\left[\frac{C_t}{N_t}\right]$$

where $C_t = N_t c_t$ is total (or aggregate) consumption in period t. Note that this assumes that consumption is the same for all adults in a period. U_t for $t > 0$ is defined similarly.

Following the discussion above, since $g(N) = N^{\eta}$ is always positive, there are two possible ways to satisfy conditions (1.)-(4.) above:

I. Assume that $u(c) \geq 0$ for all $c \geq 0$, that u is strictly increasing and strictly concave and that $0 < \eta < 1$;

II. Assume that $u(c) \leq 0$ for all $c \geq 0$, that u is strictly increasing and strictly concave and that $\eta < 0$.

Either of these are consistent with the entire set of intuitive requirements laid out in the original Barro-Becker papers. Typically we want more, however. For existence and uniqueness of a solution to the planner's problem below, the extra desirable properties are that U_0 as written here is increasing and concave in (C, N). We therefore specialize further and assume that and $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $\sigma > 0$. Let $V(C_t, N_t) = \frac{N_t^{\eta+\sigma-1} C_t^{1-\sigma}}{1-\sigma}$ denote the period-t flow utility in aggregates. Given this functional form, there are two sets of parameter restrictions that satisfy the natural monotonicity and concavity restrictions of V:[10]

AI. The first completes the standard assumption in the fertility literature:

$$0 < 1 - \sigma \leq \eta < 1$$

In this case, $U_t > 0$ for all $(C, N) \in R_+^2$.

[10] The low elasticity case, $\eta < 0$, has also been considered in Alvarez (1999), Proposition 2, and mentioned in Barro and Sala-i-Martin (1995), Chapter 9, footnote 18. Mateos-Planas (2002) used $\sigma = 3$ and $\eta = -2.78$ in his quantitative analysis. None of these papers pointed out the relationship between IES and complements versus substitutes, nor did they derive the comparative statics across these two parameter configurations.

AII. The second allows for intertemporal elasticities of substitution in line with the growth and business cycle literatures:

$$0 > 1 - \sigma \geq \eta$$

In this case, $U_t < 0$ for all $(C, N) \in R_+^2$.[11]

The conditions in assumption AI can be read as follows: if V is positive (first inequality) then for $V_N > 0$, we need $\eta + \sigma - 1 \geq 0$ (second inequality). To ensure overall concavity, we need $V_{CC} < 0$, $V_{NN} < 0$, and $V_{CC}V_{NN} - V_{CN}V_{NC} > 0$. This requires $\eta < 1$ (third inequality). In case AI, $V_{CN} = V_{NC} > 0$ and hence C and N are complements in utility. The conditions in configuration AII can be read as follows: if V is negative (first inequality) then for $V_N > 0$, we need $\eta + \sigma - 1 \leq 0$ (second inequality). This immediately ensures overall concavity. In case AII, $V_{CN} = V_{NC} < 0$ and hence C and N are substitutes in utility.[12] In the case where, $\eta = 1 - \sigma$ (allowed under both configurations), utility becomes a function of aggregate consumption only. Hence, conditions for monotonicity and concavity of V involve V_C and V_{CC} only.[13]

3.3 The planner's problem

Using the dynasty's preferences derived above, we follow the approach from Alvarez (1999) in which a time zero dynastic head chooses the time paths of aggregate, dynasty level variables. The problem solved is:

[11] One concern is that, for $\pi = 0$ and $\sigma > 1$, $U_0 = -\infty$ if for any i and t, $n_{it} = 0$. From the dynastic point of view, if at any point in time, any descendant has zero children that branch of the dynasty has $-\infty$ continuation utility implying that the time 0 decision maker also has $-\infty$ utility. This is particularly relevant for questions such as those in Doepke (2005) and Kalemli-Ozcan (2003) where the probability of all born children dying is strictly positive. One can get around this problem by adding small constants to consumption and children in the utility and thereby preserve the properties of the model to a large extent while bounding utility away from $-\infty$. Using $\pi > 0$ also prevents this problem.

[12] Note that, with this utility function, high elasticity is always associated with complementarity and vice versa. An alternative formulation that would allow us to disentangle these two effects at the household level is given by:

$$u(c) = \bar{u} + \frac{c^{1-\sigma}}{1-\sigma}.$$

By adjusting $\bar{u}$ appropriately, one can consider $\sigma > 1$ but non-negative utility (i.e. complements) or $\sigma < 1$ and negative utility (i.e. the substitutes case) with the assumptions on $g(.)$ adjusted appropriately. One disadvantage of this formulation is that there is no BGP in the sense that fertility and population growth will have a trend as long as $\bar{u} \neq 0$. Given this, we chose to derive analytical comparative statics across BGPs with $\bar{u} = 0$. Hall and Jones (2007) use this formulation to generate a trend to the health spending share of GDP.

[13] Note that the intertemporal elasticity of substitution in consumption is only partially expressed in σ. The actual elasticity also involves N_t. In the case where $\eta = 1 - \sigma$, the analogy is exact. We will nevertheless stick to this abuse of language for the remainder of the paper.

$$\max_{\{C_t,N_{b,t},N_{s,t},N_t\}} \quad \sum_{t=0}^{\infty} \quad \beta^t \frac{N_t^{\eta+\sigma-1}C_t^{1-\sigma}}{1-\sigma} \tag{3}$$

$$\begin{aligned} s.t. \quad & C_t + \theta_{b,t}N_{b,t} \leq w_t N_t \\ & N_{t+1} \leq \pi N_t + N_{s,t} \\ & N_{s,t} \leq \pi_s N_{b,t}, \\ & N_0 \quad given \end{aligned}$$

This problem can be reformulated by eliminating $N_{b,t}$ and defining $\theta_{s,t} \equiv \frac{\theta_{b,t}}{\pi_s}$, the cost of producing a surviving child. Thus, the cost of raising a child to working age depends on the survival probability—an increase in π_s decreases $\theta_{s,t}$. Further one can eliminate $N_{s,t}$ and C_t and solve for N_{t+1}, the period-$t+1$ stock of population given N_t.

Under either of the sets of parameter restrictions given above (AI and AII), this (time zero) maximization problem has a concave objective function and a convex constraint set. Thus, the problem has a unique solution, concave value functions, etc. (see Alvarez and Stokey (1998)).

4 Equilibrium and comparative statics

In this section, we describe the solution to the model and present analytical comparative statics results across Balanced Growth Paths (BGPs). We discuss how fertility changes when productivity growth rates and survival rates are changed exogenously. We find that both the sign and size of these effects depend critically on which parameter configuration holds.

4.1 Equilibrium populations

The first order condition for the stock of population in period-$t+1$, N_{t+1}, is given by:

$$\begin{aligned} &\theta_{s,t}N_t^{\eta+\sigma-1}C_t^{-\sigma} \\ &\quad = \underbrace{\beta\left[w_{t+1} + \theta_{s,t+1}\pi\right] N_{t+1}^{\eta+\sigma-1}C_{t+1}^{-\sigma}}_{A} + \underbrace{\beta\frac{(\eta+\sigma-1)}{(1-\sigma)}N_{t+1}^{\eta+\sigma-2}C_{t+1}^{1-\sigma}}_{B}. \end{aligned} \tag{4}$$

The intuition for this is as follows. On the left is the marginal cost in terms of changed current utility of increasing N_{t+1} (i.e., of producing an extra child). This cost is just the direct utility cost of reduced consumption today (rescaled by the fact that it only takes $\theta_{s,t}$ units of C to make one extra unit of N). On the right hand side are the two pieces of the marginal benefits next period from increasing N_{t+1}. These are: (A.) the value of the extra output the dynasty will have next period; (B.) the marginal value of utility from having extra (surviving) children.

To gain some more insight, consider the special case in which $\eta = 1-\sigma$. In this case, N is exactly like a capital good since the two utility effects of increasing N exactly cancel out. These two effects are: first, the direct benefit of having extra children in the utility function, $\frac{g(N)}{1-\sigma} = \frac{N^\eta}{1-\sigma}$; second, the direct cost of having children by diluting per capita consumption in the utility function, $\frac{\left[\frac{C}{N}\right]^{1-\sigma}}{1-\sigma}$. As we can see in the first order condition (4), two simplifications result. The first is that $(\eta + \sigma - 1) = 0$ and so term (B) disappears entirely, and second that $N_t^{\eta+\sigma-1} = N_{t+1}^{\eta+\sigma-1} = 1$, i.e., the marginal value of increased total consumption by the dynasty in periods t and $t+1$ no longer depend on the size of the dynasty in the period. Hence, equation (4) simplifies to:

$$\left[\frac{C_{t+1}}{C_t}\right]^\sigma = \beta\left[\frac{w_{t+1}}{\theta_{s,t}} + \frac{\theta_{s,t+1}}{\theta_{s,t}}\pi\right]. \tag{5}$$

This is the standard Euler Equation from an endogenous growth (Ak) model in terms of *aggregate consumption* and the stock of people, N, as the investment good with $(1-\pi)$ corresponding to depreciation, and time varying costs and benefits of flow investments (fertility), i.e., w_t and $\theta_{s,t}$. Equation (4) or (5) together with the feasibility constraint and the initial condition N_0 completely describe the equilibrium path.

4.2 Fertility measures

Next, we make an analytically and quantitatively useful distinction between several fertility measures. The first fertility measure we consider is the population growth rate between period t and period $t+1$ given by $\gamma_{N,t} = \frac{N_{t+1}}{N_t}$. Adjusting for the length of the period, this easily maps into annual population growth rates (net of immigration) observed in the data.

Further, we discuss two other measures of fertility related to common measures used in demography. These are: the Crude Birth Rate (CBR) and the Cohort Total Fertility Rate (CTFR). CBR is defined as the number of births in a period per person. Typically it is a yearly measure. Since our model period will be 20 years below, we adapt this to include those children born in the early part of a period and that have survived to the end of the period. Since it will be useful below to distinguish between overall births (CBR) and those births that actually survive to adulthood, we introduce one further concept, the Surviving Crude Birth Rate (CBR_s): $CBR_t = \frac{N_{b,t}}{N_t + N_{s,t}}$ and $CBR_{s,t} = \frac{N_{s,t}}{N_t+N_{s,t}}$.

Finally, the Cohort Total Fertility Rate (CTFR) is the number of children an adult woman in a particular birth cohort had over her lifetime. Since the model only makes predictions about how many children are born, not to whom they are born, further assumptions are required to develop a model analog. We therefore assume that only one-half (i.e., the females) of the surviving children from among those born in the previous period can have children in the current period. When a period is 20 years, which we will assume below, this assigns all births to women age 20 to 40. Under this assumption, there is a simple expression for the analog of CTFR in the model $CTFR_t = \frac{N_{bt}}{0.5 N_{st-1}}$.[14]

4.3 Balanced growth

One advantage of the version of the model with labor income only is that it delivers simple analytic comparative statics results across Balanced Growth Paths (BGPs).[15]

Assume that wages and costs of children grow at rate γ, i.e., $w_t = \gamma^t w$ and $\theta_{s,t} = \gamma^t \theta_s$. In this case, it can be shown directly that, both $\gamma_{N,t} = \frac{N_{t+1}}{N_t}$ and $\frac{C_t}{\gamma^t N_t}$ are independent of t (see Jones and Schoonbroodt (2007)). Because of this, it follows that $\gamma_{C,t} = \frac{C_{t+1}}{C_t}$ (the growth rate in aggregate consumption) and $\gamma_{c,t} = \frac{C_{t+1}/N_{t+1}}{C_t/N_t}$ (the growth rate in per capita consumption) are also independent of t and that $\gamma_c = \frac{\gamma_C}{\gamma_N} = \gamma$. Using this, the feasibility constraint and $\theta_s \equiv \theta_b/\pi_s$ in equation (4), after dividing both sides by $\gamma^{t+1} N_{t+1}^{\eta+\sigma-1} C_{t+1}^{-\sigma}$ and rearranging, we get

$$\frac{1}{\beta}\gamma_N^{1-\eta}\gamma^{\sigma-1} + \frac{(\eta+\sigma-1)}{(1-\sigma)}\gamma_N = \frac{\eta}{(1-\sigma)}\left[\frac{\pi_s w}{\theta_b} + \pi\right]. \tag{6}$$

Further, equation (6) simplifies considerably when $\eta = 1-\sigma$. In fact,

$$\gamma_N^{\sigma} = \beta\gamma^{1-\sigma}\left[\frac{\pi_s w}{\theta_b} + \pi\right]. \tag{7}$$

We use this version of the Euler equation in our quantitative analysis below.

Given parameters, equation (6) or (7) can be solved for the BGP population growth rate, γ_N. The other fertility measures are then given by $CBR_s = \frac{\gamma_N - \pi}{1+\gamma_N-\pi}$, $CBR = \frac{CBR_s}{\pi_s}$ and $CTFR = \frac{2\gamma_N}{\pi_s}$. Since these are all increasing functions of the population growth rate, γ_N, the comparative statics analysis below simplifies greatly.

[14] A more common measure in demography is the Total Fertility Rate (TFR). This is a point in time measure constructed by adding age specific fertility rates of woman of different cohorts alive fecund at a point in time. CTFR and TFR coincide when age specific fertility rates are constant over time. In the model, TFR and CTFR are the same since one model period also corresponds to the entire fertile period (e.g., from age 20 to age 40). In the data we will compare this object to CTFR whenever data is available.

[15] Similar results carry over to generalized versions of the model with capital or education. See Section 6 for a more detailed discussion.

4.4 Comparative statics

In this section, we derive analytic comparative statics across BGPs for all the fertility measures given above. We focus on three distinct changes leading to the quantitative experiments we explore in the next section, which correspond to three commonly discussed driving forces of the demographic transition over the period from 1800 to 1990: 1) the increased growth rate of labor productivity that came with industrialization (γ); 2) the substantial reduction in infant and youth mortality rates (π_s); and 3) the increase in (adult) life expectancy (π).

Note that the only endogenous variable in equation (6) is the population growth rate, γ_N, which only enters on the left-hand side. Moreover, the productivity growth rate, γ, only enters on the left-hand side while survival rates, π_s and π, only enter on the right-hand side. That is, holding $(\sigma, \beta, \eta, w, \theta_b)$ fixed, this equation is of the form: $LHS(\gamma_N; \gamma) = D(\pi_s, \pi)$, where $D(\pi_s, \pi) = \frac{\eta}{(1-\sigma)}\left[\frac{\pi_s w}{\theta_b} + \pi\right]$ (see Figure 1). $LHS(\gamma_N; \gamma)$ is increasing in γ_N, for all values of γ, since in both parameter configurations, AI and AII, we have that $\eta \in [-\infty, 1)$ and $\frac{(\eta+\sigma-1)}{(1-\sigma)} > 0$. Similarly, $D(\pi_s, \pi) > 0$ since $\frac{\eta}{(1-\sigma)} > 0$.

It is easy to see that in response to an increase in γ, $LHS(\gamma_N; \gamma)$ decreases (shifts right) if $\sigma < 1$ and increases (shifts left) if $\sigma > 1$. Hence, γ_N is increasing in γ if $\sigma < 1$ (AI) and decreasing in γ if $\sigma > 1$ (AII). That is, the *sign* of first order effect of productivity growth on population growth changes across the parameter configurations AI and AII. Since all other fertility measures are increasing functions of the population growth rate, one can see that whether CBR, CBR_s and $CTFR$ increase or decrease in response to an increase in γ analogously depends on the parameter configuration.

D, on the other hand, is increasing in both survival rates, π_s and π, for either parameter configuration, and hence, so is γ_N. That is, a decrease in either youth or adult mortality always leads to an increase in the population growth rate. The effect of mortality on the other fertility measures differs, however.

An increase in survival to adulthood, π_s, will also increase the crude birth rate in terms of surviving children, CBR_s, for AI and AII. The effects of youth mortality on total births, $CBR = \frac{CBR_s}{\pi_s}$, however, depends on the size of the response in CBR_s to an increase in π_s, relative to the increase in π_s itself. A similar relationship holds between the size of the change in γ_N and the sign of the response of $CTFR$. Similarly, even though an increase in π always causes γ_N and $CTFR$ to increase, whether or not it increases CBR_s and CBR depends on the size of $\frac{\partial \gamma_N}{\partial \pi}$.

While the *sign* of the effects of π_s and π on γ_N is the same under AI and AII, the parameter configuration matters for the *size* of these effects. To see this, note that the left-hand side of equation (6) is increasing in γ_N. Further, it is concave if $1 > \eta \geq 1-\sigma > 0$ (i.e.,

Figure 1: Comparative statics of γ_N

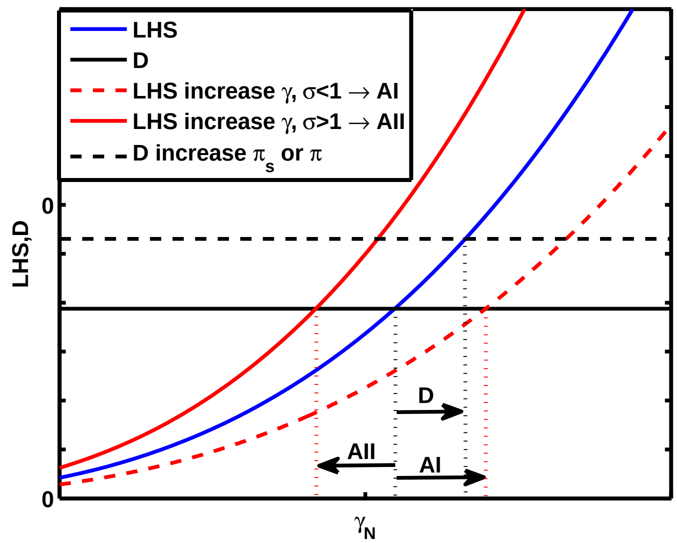
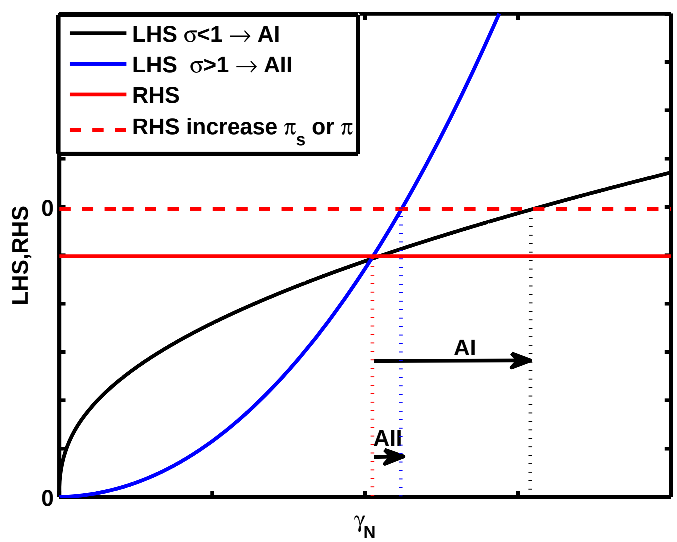


Figure 2: Increase in γ_N is smaller for AII than for AI



under AI), and convex in γ_N if $\eta \leq 1-\sigma < 0$ (i.e., under AII) (see Figure 2). This implies that in the range where $\gamma_N \geq 1$ the slope of LHS is larger under AII than it is under AI. Because of this, the response of γ_N to a given change in $D(\pi_s, \pi)$ is smaller under AII than AI.

More precisely, the empirically relevant ranges for parameters are when productivity growth is positive but small, $\gamma \approx 1$, and the same holds for γ_N. It is therefore of special interest to know the properties of the model when $\gamma = 1$ and $\gamma_N \geq 1$ (i.e., $\beta D \geq 1$). Suppose for simplicity that $\eta = 1-\sigma$. In this case, holding all parameters but σ fixed, it follows that a given size change in π_s and π has larger effects on γ_N in the low curvature case, AI, than in the high curvature case, AII. This is also true for CBR_s in response to a change in π_s. Because of this, it follows that when $\beta D = 1$, we have $\frac{\partial CBR}{\partial \pi_s}|_{\sigma<1} > \frac{\partial CBR}{\partial \pi_s}|_{\sigma>1}$. For example, if CBR falls from a given change in π_s when $\sigma < 1$, it falls more when $\sigma > 1$. It can also be shown that, in the special case where $\pi = 0$, $\frac{\partial CBR}{\partial \pi_s} < 0$ under both AI and AII.

In sum then, when γ and γ_N are both near one and π is not too large, an increase in π_s will cause CBR to fall under both AI and AII, and this fall will be larger when $\sigma > 1$ (AII). Similar results hold for $CTFR$.

Finally, we know that under AI, $\frac{\partial \gamma_N}{\partial \pi}$ is larger than under AII. Hence, as above, if parameters are such that CBR_s and CBR fall in response to an increase in adult survival, π, when AI holds, they fall more when AII holds. When $\gamma = 1$, $\beta D = 1$, $\eta = 1-\sigma$, and $\sigma > 1$, it can be shown that an increase in π causes CBR_s and CBR fall.

We summarize the results most relevant for the quantitative analysis in a proposition:

Proposition 1. *The following comparative statics results hold across BGPs :*

1. *If AI holds, then,* $\frac{\partial \gamma_N}{\partial \gamma}$, *and* $\frac{\partial CBR}{\partial \gamma}$ *are positive, while*
   if AII holds, then $\frac{\partial \gamma_N}{\partial \gamma}$, *and* $\frac{\partial CBR}{\partial \gamma}$ *are negative.*

2. (a) *In both cases AI and AII,* $\frac{\partial \gamma_N}{\partial \pi_s} > 0$. *However, when* $\gamma = 1$, $\eta = 1-\sigma$ *and* $\beta D \geq 1$ *(i.e.,* $\gamma_N \geq 1$*), then,* $\frac{\partial \gamma_N}{\partial \pi_s}|_{\sigma<1} > \frac{\partial \gamma_N}{\partial \pi_s}|_{\sigma>1}$.
   (b) *If* $\gamma = 1$, $\eta = 1-\sigma$ *and* $\beta D = 1$ *then,* $\frac{\partial CBR}{\partial \pi_s}|_{\sigma<1} > \frac{\partial CBR}{\partial \pi_s}|_{\sigma>1}$. *If in addition,* $\pi = 0$ *then,* $\frac{\partial CBR}{\partial \pi_s} < 0$ *for AI and AII.*

3. *In both cases AI and AII,* $\frac{\partial \gamma_N}{\partial \pi} > 0$. *However, when* $\gamma = 1$, $\eta = 1-\sigma$ *and* $\beta D \geq 1$ *(i.e.,* $\gamma_N \geq 1$*), then,* $\frac{\partial \gamma_N}{\partial \pi}|_{\sigma<1} > \frac{\partial \gamma_N}{\partial \pi}|_{\sigma>1}$ *and* $\frac{\partial CBR}{\partial \pi}|_{\sigma>1} < 0$.

It also follows that $\frac{\partial \gamma_N}{\partial \beta} > 0$ and $\frac{\partial \gamma_N}{\partial w} > 0$ if $\theta_b = a$ (goods cost) and $\frac{\partial \gamma_N}{\partial w} = 0$ if $\theta_b = bw$ (time cost). We do not emphasize these because they play no role in the quantitative discussion we focus on below.[16] Note, however, that this, like many models based on dynastic altruism, is a balanced growth model. Thus, increases in *income (wages)* do not, by themselves, trigger a change in fertility (at least, as long as costs change proportionally). Rather, it is a change in *growth rates* that is required. This feature brings focus to the hypothesis in demography that the fertility transition is in part caused by 'industrialization'. That is, in this particular model at least, it was the *acceleration* of productivity growth that mattered, not the change in *levels* per se.[17] Further, when $\eta = 1 - \sigma$ and $\sigma \to 1$, the utility function we use converges to $\sum \beta^t \log(C_t)$.[18] All of the results from this Proposition go through in this case. In particular there is no change in γ_N when γ is changed.[19]

These comparative statics have important implications for studying demographic transitions using this type of model. From the beginning of the 19th century to the end of the 20th century, CBR fell substantially, while population growth rates (net of immigration) decreased less dramatically. Since the rate of growth of productivity has increased over the period describing these demographic changes, it follows from the proposition that we would expect population growth rates to fall as a result as long as low IES and substitutability between number and utility of children are assumed (AII). Offsetting this effect is the fact that both youth and adult survival rates rose over this period. Thus, under AII, increases in population growth in response to changes in mortality declines (through π_s or π) will mitigate the negative effect from increases in productivity growth, but these effects are small. As we shall see, overall we see a decrease in γ_N in this case. Under AI, however, population growth rates increase in response to all, increased productivity growth, decreased youth mortality and longevity. Hence, the model has no shot at reproducing observed changes in population growth rates. With respect to CBR, the effect of productivity growth is the same as for population growth rates, while the effect of decreases in mortality could, in principle, be either an increase or a decrease. As we shall see, the model predicts, from this source alone, a small decrease in CBR under AI and hence, as in the proposition, a larger decrease under AII. Because of this, the combined effect will be to increase CBR when AI holds, and decrease CBR when AII holds.

[16] If all child costs are goods costs, increases in labor income taxes are equivalent to reductions in w. Thus, it follows that increasing the labor income tax rates will decrease both population growth rates and fertility on the BGP, see also Manuelli and Seshadri (2007). At the other extreme, if all costs are in terms of time ($\theta = bw$), fertility and population growth rates are independent of labor income taxes, short of child care allowances in the tax code.

[17] Galor and Weil (2000) and Greenwood and Seshadri (2002) generate the decrease in fertility by increased *levels* of income with non-homotheticities in utility.

[18] See Bar and Leukhina (2007) for an explicit derivation of Barro-Becker preferences in this case.

[19] A similar result for low versus high IES holds when physical capital is added to the model. The threshold IES is typically below one, however. In particular, γ_N is typically decreasing in γ in the log case. See Section 6 for details.

5 The U.S. experience 1800-1990

Our next objective is to extend the insights gained above to get quantitative estimates of the model predictions in terms of population growth rate and CBR in response to permanent (unanticipated) changes in productivity growth (γ), longevity (π), and youth mortality (π_s) as observed in the data.

First, we discuss how the model can be calibrated to match (current) levels of fertility and/or population growth rates in the U.S.. Most of the parameters needed can be taken directly from the quantitative growth and RBC literatures. The exceptions to this are θ, the cost of raising a child, and η the utility parameter governing curvature over dynasty size. For simplicity we set $\eta = 1 - \sigma$ throughout.[20] One puzzle in the quantitative growth literature with endogenous fertility has been that calibrated costs of children were unrealistically high. We find that when $\sigma > 1$ (AII), the needed calibrated costs to match any given fertility level are substantially lower.

Next, we simulate the predicted effects on fertility from changing all three of the above driving forces with the appropriate timing of events in the U.S. over the last 200 years. We find that how well the model does at reproducing the historical facts depends critically on the size of the IES. When the IES is high, as is commonly assumed in the fertility literature, the model (counterfactually) predicts large increases in both CBR and γ_N. On the other hand when the IES is low, the model captures two thirds of the decrease in CBR observed in the data and about half of the decrease in population growth rates—and with similar timing.

Finally, we decompose the contribution of each factor separately for the low IES case and find that the increase in the growth rate of productivity accounts for about 90 percent of the predicted fall in fertility before 1880, and changes in (youth) mortality account for 90 percent of the predicted change from 1880 to 1990.

Overall, these results are consistent with the qualitative comparative statics results given in Proposition 1. This must be interpreted with caution however since the results in the Proposition hold $\frac{\theta}{w}$ fixed, while here we are recalibrating this so as to match final levels of fertility for different values of σ.

5.1 Calibration: fertility levels and costs of children

In this subsection we use the Euler equation in (7) to calibrate costs of children given all other parameters and targeting 0.65 annual population growth. This target roughly corresponds

[20] Note that without this assumption, given our two admissible parameter configurations, η would have to satisfy $\eta < 1 - \sigma$ whenever $\sigma > 1$ or $\eta > 1 - \sigma$ whenever $\sigma < 1$. Hence, the assumption that $\eta = 1 - \sigma$ makes results more readily comparable. In this case, the two utility effects of increasing dynasty size cancel out—independently of σ. We report sensitivity results below.

to choosing a cost of children that matches the U.S. fertility experience (i.e., about 17 births per 1000 population[21] or 2.3 children per woman) over the 1970 to 1980 period.

Suppose there are T years in a period. Then we can rewrite equation (7) in annual terms as (the subscript $_{ann}$ denotes annual values):

$$\gamma_{N,ann}^{T} = \left[\beta_{ann}^{T}\left[\frac{w}{\theta_s} + \pi_{ann}^{T}\right]\right]^{1/\sigma}\left[\gamma_{ann}^{T}\right]^{1/\sigma-1}.$$

Solving for $\frac{\theta_s}{w}$ gives

$$\frac{\theta_s}{w} = \frac{1}{\left[\frac{\gamma_{N,ann}^{\sigma}}{\beta_{ann}\gamma_{ann}^{(1-\sigma)}}\right]^{T} - \pi_{ann}^{T}}.$$

where the relevant π_{ann}^{T} is the probability of an adult surviving (in the workforce) for T years. We choose $T = 20$ and hence $\pi_{ann} = 0.977$ to match an expected adult (working) life, $\frac{T}{1-\pi}$, of 53 years. Moreover, we assume an annual discount factor of $\beta_{ann} = 0.96$ and compute an annual productivity growth rate of $\gamma_{ann} = 1.018$ from wage data. Note that over the period 1970 to 1990 survival to adulthood is close to certain, i.e., $\pi_s \approx 1$ and hence $\theta_s \approx \theta_b$. We introduce additional layers of complexity below so that we can use detailed data on infant, child and youth mortality rates in the time series experiments in the next subsection. Throughout, we consider values of $\sigma \in \{0.5, 1, 3\}$.

We report our results in Table 1. The units of $\frac{\theta_s}{w}$ are the fraction of one periods per capita output that it costs to raise one child. To get a sense of how costly children must be in the model in order to match realistic growth rates of population, it is more convenient to express it in terms of the number of *years* of output that are required to raise a child, i.e., $T \times \frac{\theta_s}{w}$. Our choice of $T = 20$ also corresponds to assuming that it takes 20 years for a newborn to become a productive worker. In addition, we report the maximal number of children a two parent household can (feasibly) have over one (fertile) period ($Max\ CTFR = 2T/\left(T \times \frac{\theta_s}{w}\right) = 2\frac{w}{\theta_s}$).

Table 1: U.S. Costs of children in 1990, Time Series Experiment

σ	$\frac{\theta_s}{w}$	$T \times \frac{\theta_s}{w}$	$Max\ CTFR$
0.5	0.72	14.32	2.80
1.0	0.51	10.25	3.90
3.0	0.16	3.24	12.34

[21] Note that reasonable life lengths ($\pi > 0$) are crucial to bring population growth rates in line with CBR in the calibration. With $\pi = 0$ population growth of 0.65% per year would correspond to a CBR of 26.6.

Clearly, the values of the costs of raising children—in the range of 3 to 14 years of one person's output—are very sensitive to the choice of the IES. The intuition for this is as follows. When σ is high (i.e., there is a strong desire to smooth consumption), high growth in aggregate consumption is not valuable (since $\eta = 1 - \sigma$ and the two utility effects of dynasty size in equation 4 cancel out this is the only effect that matters). Aggregate consumption grows at rate $(\gamma\gamma_N)$. Hence, everything else equal, the population growth rate is decreasing in the desire to smooth consumption (σ). Vice versa, to get a *given* population growth rate (i.e., fertility level) to be the optimal choice, one needs higher costs of children when σ is lower. With $\sigma = 0.5$, a two parent household can maximally have 2.8 children during one 20 year period, while with $\sigma = 3.0$ as many as 12.34 are feasible. This shows that with low IES, costs of children lie in a much more reasonable range.

In the historical experiments we perform below we will examine how these costs change over time due to changes in mortalities. Since age specific mortality rates changed unevenly since 1800, we follow Doepke (2005) and assume that there are three relevant subperiods of childhood: infancy (up to age 1), childhood (ages 1 to 5) and youth (from age 5 to adulthood at age 20). Let π_i, π_{ic} and π_{cy} and θ_i, θ_c, and θ_y denote the three conditional survival rates and costs for these sub-periods. Then, the cost of producing one surviving child is:

$$\frac{\theta_s}{w} = \frac{\theta_i/w}{\pi_i \pi_{ic} \pi_{cy}} + \frac{\theta_c/w}{\pi_{ic} \pi_{cy}} + \frac{\theta_y/w}{\pi_{cy}}. \tag{8}$$

As in Doepke (2005) we assume that $\theta_y = 0$ and costs are the same for every year age 0 to 5, i.e., $\frac{\theta_c}{\theta_i} = 4$. Given these assumptions, the calibrated values of θ_s/w given σ, and historical data on mortality rates, we can back out θ_i/w and θ_c/w from equation (8) for each value of σ. Over time, we assume that the base costs as a fraction of wages $(\theta_i/w, \theta_c/w, \theta_y/w)$ remained constant, while survival rates $(\pi_i, \pi_{ic}, \pi_{cy})$ have increased—decreasing the effective cost of surviving children, θ_s/w.

5.2 U.S. Fertility decline: model versus data

Next, we examine the model predictions for fertility and population growth from changing γ, θ_s (through changes in π_i, π_{ic}, and π_{cy}) and π in line with the experience of the U.S. over the period from 1800 to 1990. These predictions are then to be compared with their data counterparts. All data series and sources can be found in Table A.1 in the Appendix.

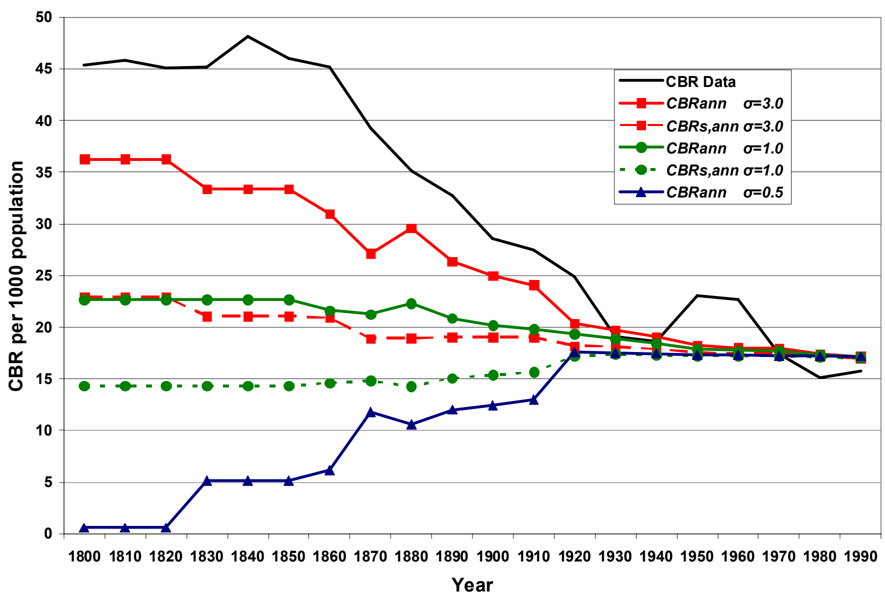
In the data provided by Hacker (2003), CBR was roughly constant at 45 births per 1000 population from 1800 to 1860, then decreased to 19 in 1930.[22] From Haines (1994b), we get

[22] Note that earlier estimates of CBR in the 19th century were higher, at about 55 births per 1000 population in 1800 to 45 in 1850 (e.g. Haines (1994b)). The values reported here are backward projections

that from 1930, the bottom of the pre–WWII baby bust, CBR increased to 23 in the 1950s and 1960s, the peak of the post–WWII baby boom, and finally fell to 15.8 in 1990.[23] CTFR and TFR show similar patterns. Finally, also from Haines (1994b), population growth rates (net of immigration) decreased from 2.6 percent population growth per year to 0.65 percent per year, again with a down then upward swing from 1930 to 1960.

We use data on productivity growth, survival probabilities to adulthood and expected lifetimes in the U.S. from 1800 to 1990. This data implies that productivity growth, γ_{ann}, increases from zero to 1.8 percent per year, adult survival, π_{ann}, increases from 0.964 to 0.977 and survival to adulthood, $\pi_{s,ann}$, increases from 0.632 to 0.989. Note that, in the model, agents assume that current values of these parameters will prevail forever.[24]

Figure 3: The U.S. experience from 1800 to 1990, CBR_{ann} and $CBR_{s,ann}$



The results of this experiment are shown in Table 2, while Figure 3 plots data and model predictions for CBR_{ann} and $CBR_{s,ann}$ to illustrate the different effects on surviving fertility

from 1850 Census data using better mortality estimates than previously assumed. See Hacker (2003) for a complete discussion.

[23]For an application of a stochastic version of the present model to address these fertility fluctuations, see Jones and Schoonbroodt (2007).

[24]The assumption that agents believe that productivity growth, mortality and longevity will be constant forever is an extreme one but greatly simplifies the analysis. An alternative extreme would be to assume that agents perfectly foresee the exact future path of parameter changes. The results of this exercise for $\sigma = 3$ can be found in Jones and Schoonbroodt (2008). The predictions are virtually identical.

Table 2: Time Series Experiment: Data versus Model, for several values of σ

	$\gamma_{N,ann}$			CBR_{ann}			CTFR		
	1800	1880	1990	1800	1880	1990	1800	1880	1990
Data	1.027	1.016	1.006	45.4	35.2	15.8	7.04	4.90	1.97
$\sigma = 0.5$	0.965	0.979	1.006	0.59	10.6	17.2	1.54	2.04	2.30
$\sigma = 1.0$	0.994	.995	1.006	22.7	22.3	17.2	2.79	2.79	2.30
$\sigma = 3.0$	1.014	1.005	1.006	36.3	29.6	17.2	4.20	3.45	2.30

versus total births on a similar scale.[25] As can be seen, for high IES ($\sigma = 0.5$) the model predicts a substantial increase in $CTFR$ and CBR. The model also predicts a large increase in population growth rates in this case. These predictions are in part due to the increase in survival to adulthood and in part due to the increased growth rate in productivity, both of which always imply an increase in population growth rates for $\sigma < 1$. This finding is consistent with Doepke (2005) (who also uses $\sigma = 0.5$ and labor income only) and is the reason for his conclusion that the basic Barro-Becker model does not fit the facts (even though a more sophisticated model with sequential fertility choice performs better).[26]

For $\sigma = 1$, the model predicts a sizable fall in $CTFR$ and CBR. However, since the probability of surviving to adulthood is increasing over the period, the number of surviving children (CBR_s) increases even though the number of all births (CBR) falls. This property of CBR_s is directly reflected in the predicted time path of population growth rates (see Table 2). This is partly due to the fact that changes in productivity growth do not affect CBR_s or population growth rates when $\sigma = 1$. These results are in keeping with Bar and Leukhina (2007) who use a more elaborate version of this model with a utility specification equivalent to $\sigma = 1$ and find that changes in productivity have only a small effect on fertility while changes in mortality have a relatively large effect.

Finally, for low IES ($\sigma = 3$), the overall changes give rise to a predicted fall in $CTFR$ from 4.2 in 1800 to 3.8 in 1850 (the 1826 birth cohort) and 2.3 in 1980 (the 1958 birth cohort) and a predicted fall in CBR from 36.3 in 1800 to 17.2 in 1990. In this case, CBR_s

[25] Annual values per 1000 population are computed as $CBR_{ann} = 1000 \times CBR/T$ and $CBR_{s,ann} = 1000 \times CBR_s/T$. For plots of detailed predicted time paths for all other measures of fertility, see Jones and Schoonbroodt (2008).

[26] The present version of the model overstates the poor performance when $\sigma = 0.5$ to some degree. When physical capital is also included, there is only a small increase in CBR over the period from about 15.7 to 17.2. See the discussion in Section 6.

is decreasing over the period (see Figure 3). This is mainly due to increased productivity growth. In terms of population growth rates, the model predicts a fall from 1.4 to 0.6 percent per year. This follows from the effect of changing productivity growth which overturns the effect of decreased mortality. Hence, changes in CBR predicted by the model capture about two-thirds of changes observed in the data, while model predictions of changes in $CTFR$ and population growth rates, γ_N, capture about one half of observed changes.[27,28]

5.3 Productivity versus mortality: decomposition

Next, we decompose the sources of these changes (for $\sigma = 3$) into the components separately. Since the effects of changing π are quantitatively quite small we don't emphasize those changes. That is, we ask what the model would predict for $CTFR$, CBR and γ_N if two of the three forcing variables had stayed at their 1800 levels, while the other changed as per the experiment above. The results from these calculations are shown in Table 3.

We find that, taken one at a time, both changes productivity growth (γ) and changes in survival probability to adulthood (π_s) have sizable impacts on both $CTFR$ and CBR. Changes in expected lifetime (π) generate a hardly noticeable increase in $CTFR$ and account for a relatively small decrease in CBR. As can be seen, the effect is largest for changes in π_s which by itself shows a decrease in $CTFR$ from 4.2 to 2.9 and in CBR from 36.3 to 24.7. The effect of changing γ alone is smaller but still significant, reducing $CTFR$ from 4.2 to 3.3 and CBR from 36.3 to 28.6. The effect of a change in π is substantially smaller, causing a reduction in CBR from 36.3 to 33.0.

On the other hand, the changes in both π_s and π cause population growth, γ_N, to increase. It is the increase in productivity growth, γ, that alleviates this so that, in sum, the effect of the three changes taken together on γ_N is negative, i.e., the increase in γ_N from increases in π_s and π is more than offset by the decrease in γ_N resulting from the increase in γ. These findings are consistent with Mateos-Planas (2002) who focuses on population growth rates (rate of natural increase) only and therefore understates the importance of mortality to understand the facts relating to births of the demographic transition.[30]

[27] We have also experimented with even higher values for σ (not shown here). Although the implied levels for the CBR are even higher in the earlier periods, this change is not large, and even levels of σ close to 1,000 do not generate the entire change seen in the data.

[28] We performed sensitivity with respect to η as well and found that, for $\eta < 1 - \sigma < 0$, results are not very sensitive to the value of η, holding $\sigma = 3$. For the case, $\eta = 0.8 > 1 - \sigma = 0.5$, the effect of mortality on CBR starts to be negative, so that the model predicts an upward hump until 1880 and a slight decrease thereafter.

[30] Here, we have assumed that the base costs (relative to wage income) of raising a child to adulthood have been unchanged over the period. However, when one adopts a broad view of what determines these costs—e.g., subtracting out any direct input from the child on a farm—this is clearly a strong assumption. Indeed, the relative availability of land in the U.S. and the resulting implications for the size of net costs of

Table 3: Decomposition: Productivity (γ) vs. Mortality (π_s) and Longevity (π)

$\sigma = 3$	$\gamma_{N,ann}$			CBR_{ann}			CTFR		
	1800	1880	1990	1800	1880	1990	1800	1880	1990
Data	1.027	1.016	1.006	45.4	35.2	15.8	7.04	4.90	1.97
Productivity (γ)	1.014	1.005	1.002	36.3	30.3	28.6	4.20	3.49	3.30
Mortality (π_s)	1.014	1.014	1.018	36.3	35.9	24.7	4.20	4.15	2.90
Longevity (π)[29]	1.014	1.014	1.015	36.3	35.9	33.0	4.20	4.20	4.23

The results of the decomposition exercise are also interesting because of their implication about the timing of fertility decline. Previous authors have criticized the hypothesis that the fertility decline was a byproduct of a reduction of infant mortality rates because of questions about the relative timing of these two changes (see for example, Van de Walle (1986), Doepke (2005), Fogel (1990)). The model predictions here also show that the decrease in youth mortality has very little effect before 1880. Then, from 1880 to 1990, changes in youth mortality (from survival to age 20 of $\pi_s = 0.64$ in 1880 to $\pi_s = 0.99$ in 1990) account for 90 percent of the total model predicted change in CBR.[31] However, changes in productivity growth rates (from 0 in 1800 to 1.4 percent per year in 1880) account for about 90 percent of the total model predicted change in CBR for the early period.[32]

In sum then, both, changes in youth mortality and productivity growth rates, are quantitatively important in understanding the model predictions about the history of fertility and population growth over the last 200 years. An important requirement for the success of the model is low values for the IES in consumption.[33]

children may be one of the key reasons why the model predicts too low fertility for the early years. Mateos-Planas (2002) adjusts base costs residually in order to match the entire path of population growth rates in several European countries and finds large increases in these costs since 1900. A similar exercise in the present model with U.S. data would require base costs, θ_b/w, to have increased threefold over and above the assumed increase proportional to wages to capture the full change in CBR or twofold to capture the full change in population growth rates when $\sigma = 3$. Clearly the analysis would benefit greatly from a more direct accounting of the costs of children along these dimensions but is beyond the scope of this paper.

[31] Using Tables 2 and 3, we get $\frac{35.9-24.7}{29.6-17.2} \approx 90\%$.

[32] Using Tables 2 and 3, we get $\frac{36.3-30.3}{36.3-29.6} \approx 90\%$.

[33] We performed the same experiment as in Section 5 using data for the United Kingdom (England and Wales for the most part; we thank Michael Bar and Oksana Leukhina for help with data sources). The fertility experience in the U.K. over the past 200 years is similar to that of the U.S., except that fertility levels in 1800 were lower already. Mortality was also lower in the U.K. than it was in the U.S. around that time. Finally, our estimates of productivity growth suggest the latter was higher in 1800. Since, fertility, mortality and productivity growth are very similar in the two countries in 1990, one would expect the model to capture a smaller *absolute* decrease but the same *fraction* of changes in fertility and population growth.

6 Investments in children

In the model laid out so far, parents have no control over children's well being or quality, U_{t+1}. This simplification allowed us to derive simple intuitions and analytic comparative statics. There are various ways, however, in which parents can affect children's initial conditions. Three ways addressed frequently in the literature using dynastic models are bequests (e.g., Becker and Barro (1988), Barro and Becker (1989)), human capital investments (e.g. Becker, Murphy, and Tamura (1990), Manuelli and Seshadri (2007)) and investments in children's health (e.g., Cigno (1998), De la Croix and Licandro (2007), Manuelli and Seshadri (2007)).

In this section, we show that even when these choices and more elaborate production functions are introduced, the effects of increased productivity growth and decreased mortality on fertility still depend on the choice of the IES. While in the model with labor income only, the most relevant threshold was an IES above or below unity, this threshold may well be different depending on the exact assumptions about the aggregate production function. The basic result goes through, however: there exists a threshold, σ^*, such that, below this value comparative statics are in line with case AI in Proposition 1, while above this threshold comparative statics follow case AII.

Consider the model analyzed so far but assume that households can invest in some assets (e.g., leave bequests or educate children). That is, the household's budget constraint becomes:

$$c_t + \theta_{s,t} n_{s,t} + \pi_{x,t} x_t \leq w_t + r_t q_t;$$
$$x_t \leq (n_{s,t} + \pi) q_{t+1} - (1 - \delta) q_t.$$

where x_t is parent's investment in next period's quality per surviving adult (worker), q_t. Multiplying by N_t, the dynasty budget constraint and aggregate law of motion for quality are given by

$$C_t + \theta_{s,t} N_{s,t} + \pi_{x,t} X_t \leq w_t N_t + r_t Q_t;$$
$$Q_{t+1} \leq (1 - \delta) Q_t + X_t.$$

Note that the resulting Planner's problem is well defined under both assumption AI and AII derived in Section 2.[34] To close the model, prices are determined in equilibrium by

Indeed, we capture about two-thirds of the change in CBR and one half of the change in population growth. See Jones and Schoonbroodt (2008) for details.

[34] To ensure interiority in partial equilibrium, we have to (1) either rule out $\eta = 1 - \sigma$ or, (2) if $\eta = 1 - \sigma$, make the necessary parameter assumptions so that rates of returns to children and capital are equalized. In

a representative firm hiring labor and quality to maximize profits with a constant returns to scale production function and labor augmenting technological progress, $F(Q_t, Z_t)$, where $Z_t = \gamma^t N_t$ denotes effective labor.

6.1 Complements versus substitutes

In the model above, quality, q_t, can be interpreted either as physical capital (bequests) in which case, $\pi_{x,t} = 1$ and r_t is the interest rate on physical capital,[35] or, if $\pi = 0$, as human capital investments, in which case, $w_t + r_t k_t$ can be interpreted as the wage per unit of labor where $\pi_{x,t} \leq 1$ takes into account that some human capital already invested may die with a non-surviving child.[36] The problem of maximizing the dynastic head's utility in expression (3) subject to aggregate feasibility and laws of motion can be written recursively as a Bellman equation with two state variables: the detrended aggregate stock of quality, $\widehat{Q}_t = Q_t/\gamma^t$, and the stock of population, N_t. The value function, $V(\widehat{Q}, N)$ can be shown to be homogeneous of degree η in $(\widehat{Q}, N)$. Thus, $V(Q, N) = N^\eta V(Q/N, 1) \equiv N^\eta v(q)$. Below we show that the stock of population (fertility), N, and quality per survivor, q, are still complements in the utility of the parent under AI, the high IES case, and substitutes under AII, the low elasticity case.

First, it is straightforward to show that $V > 0$ under AI and $V < 0$ under AII, while $V_1 = v' > 0$ and $V_2 = \eta N^{\eta-1} v(q) > 0$ under both AI and AII. It follows that $V_{21} = \eta N^{\eta-1} v'(q) > 0$ under AI because $\eta > 0$ and $V_{21} = \eta N^{\eta-1} v'(q) < 0$ under AII because $\eta < 0$.

This alone is not enough however to fully determine the effect of productivity growth on population growth as before. The properties of the production function also play a role.

As to empirical evidence about the "right" utility specification, findings are mixed (see Schultz (2005) for a useful summary). While the negative relationship between family size and various measures of child quality—in terms of investments or outcomes—is clearly negative, whether this is a missing variables problem or a true quantity-quality trade-off is still controversial. One regularity seems to carry through most studies using twin births as exogenous variations in family size: the negative relationship between number and quality of children is more strongly negative in developing countries (e.g., Rosenzweig and Wolpin (1980) for India, Li, Zhang, and Zhu (2007) for rural China) than it is in more advanced societies (e.g., Angrist, Lavy, and Schlosser (2005) for Israel, Black, Devereux, and Salvanes

general equilibrium, prices will adjust to achieve this.

[35] Notice that since as before one implicit assumption is that all alive adults are identical in this model, we have to assume that parents share their assets equally among all surviving family members (including themselves), so that initial conditions of all households are the same.

[36] If adults live for more than one period, $\pi > 0$, and π_s different from π, the model becomes a vintage human capital model with many state variables.

(2005) for Norway). Yet, Cáceres-Delpiano (2006) finds that a twin on a later birth reduces the likelihood that older children attend private school in U.S. Census data from 1980. One reason for the discrepancies between rich and poor countries might be the availability of high quality public schools in developed countries. For example, De la Croix and Doepke (Forthcoming) find that the negative relationship between fertility and private schooling choices flattens as the quality of public schooling goes up.

In sum then, in the context of the demographic transition which happened alongside the process of industrialization and economic development, the substitutes assumption can be supported, but more work is needed to empirically disentangle utility from production function parameters, both of which can generate a negative quantity-quality relationship in the data.

6.2 Physical capital and bequests

Here we interpret quality Q as physical capital and let $K \equiv Q$. Quality per child can then be interpreted as bequests, $k \equiv q$. We distinguish between the partial equilibrium version as in Becker and Barro (1988) and general equilibrium version where interest rates and wages are determined in equilibrium as in Barro and Becker (1989). The findings shed light on the original BB papers and extend findings from Section 4.

Quantitatively, with a Cobb-Douglas production function, labor augmenting technological change and an IES of one-third, effects are slightly larger than without capital—still leaving room for costs of children and education to matter.

6.2.1 Partial versus general equilibrium

The first order condition with respect to C_t, K_{t+1} and N_{t+1} for the dynasty and the firm together with the budget constraint and the laws of motion for capital and population fully describe the equilibrium path. On a BGP, we have $\gamma_{c,t} = \gamma_c = \gamma$, $\gamma_{N,t} = \gamma_N$, $\gamma_C = \gamma_K = \gamma\gamma_N$, wages grow at γ and interest rates are constant. From the first-order conditions for C_t, C_{t+1} and K_{t+1}, we get the usual equation relating population growth and the interest rate:[37]

$$\gamma_N = \frac{[\beta(r+1-\delta)]^{\frac{1}{1-\eta}}}{\gamma^{\frac{\sigma}{1-\eta}}} \tag{9}$$

From equation (9), it is clear that—as usual in Barro-Becker type models—higher interest rates are typically associated with higher population growth. In light of the comparative statics results above, suppose the rate of productivity growth, γ, increases. Holding the interest rate fixed (i.e., in partial equilibrium), this decreases the population growth rate.[38]

[37] See Jones and Schoonbroodt (2008) for details.

[38] See also Becker and Barro (1988), p. 19, and Barro and Becker (1989), p.494.

However, the interest rate may be endogenous. In response to an increase in productivity growth the BGP level of the capital to effective labor ratio, $\widehat{k} = (K_t/Z_t)$, decreases (unless the production function is of the form $F(K_t, Z_t) = \min\{K_t, Z_t\}$) and hence the interest rate, r, increases which tends to increase population growth. Whether the population growth rate increases or decreases depends on all parameters, in particular the side of the parameter space one chooses:

I. Under AI, the positive effect from the interest rate is large since $\frac{1}{1-\eta} > 1$. However, since in this case, we also have $\frac{\sigma}{1-\eta} \geq 1$, the direct negative effect from an increase in productivity growth is also large.

II. Under AII, the positive effect from the interest rate is small since $\frac{1}{1-\eta} < 1$. However, since in this case, we also have $\frac{\sigma}{1-\eta} \leq 1$, the direct negative effect from an increase in productivity growth is also small.

Again, consider the case special case where $\eta = 1 - \sigma$. Then, as long as K and Z are somewhat substitutable in production, there exists a threshold σ^* such that for $\sigma > \sigma^*$ the direct negative effect of an increase in γ dominates the indirect positive effect through r so that $\frac{\partial \gamma_N}{\partial \gamma} < 0$ while for $\sigma < \sigma^*$ the opposite is true. In addition, the more substitutable K and Z are in production, the larger is the decrease in $\widehat{k}$ in response to an increase in γ and the larger the increase in r. Hence, the threshold σ^* is increasing in the degree of substitutability in production. Finally, σ^* is decreasing in the capital share.

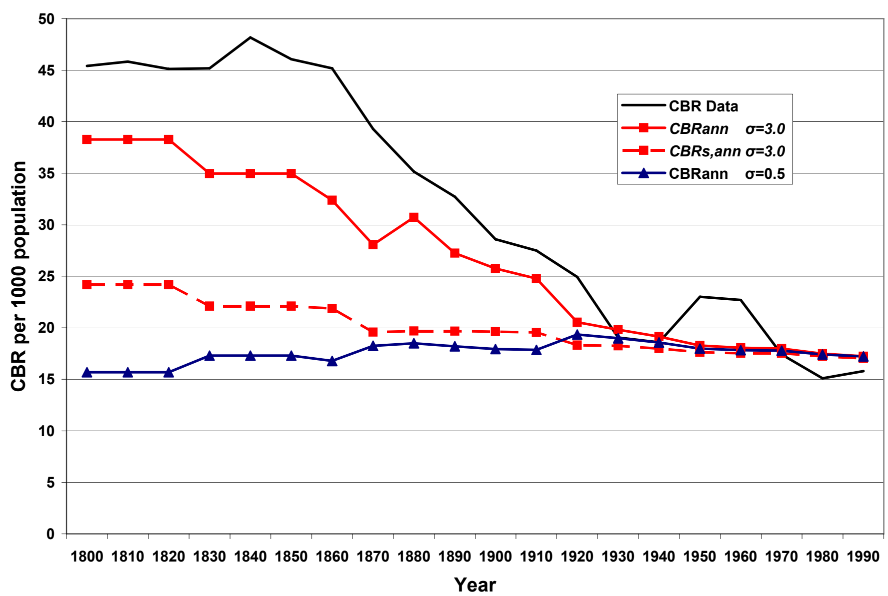
Decreases in mortality tend to increase the rate of return on population, N, relative to capital, K, and hence decreases the capital to effective labor ratio. Again, this causes an increase in the interest rate. From equation (9) and as in Proposition 1, the effect from mortality on population growth rates is larger under AI than under AII.

To gain more intuition, consider the following special case. Let $\eta = 1 - \sigma$ and let the production function be of the Cobb-Douglas variety, $F(K_t, Z_t) = AK_t^{\alpha}(\gamma^t N_t)^{1-\alpha}$. A sufficient condition for the population growth rate to be decreasing in productivity growth is $\sigma > 1 - \alpha$. This condition is necessary and sufficient in the case where there is full depreciation in physical capital and people live for only one period, i.e., $1 - \delta = \gamma\pi = 0$ (i.e., $\sigma^* = 1 - \alpha$). Since $\alpha > 0$, it is possible that the net effect on population growth from an increase in productivity is negative even if $\sigma < 1$. Nevertheless, the typical values used in the fertility literature, $\sigma \approx 0.5$ and $\pi = 0$, with a capital share of $\alpha \approx 0.3$ do not satisfy this condition. On the other hand, the log utility case, $\eta = 1 - \sigma$ and $\sigma \to 1$, often used in the growth and business cycle literatures will produce a decrease in γ_N in response to an increase in γ in this model.

6.2.2 Quantitative experiments: the U.S. experience

Quantitative experiments analogous to those in Section 5 (calibrating to 1990 and changing youth mortality, longevity and productivity growth as observed in the U.S. since 1800), give results that are quite similar in magnitude as before. Let $T = 20$, $\eta = 1 - \sigma$ and $\sigma \in \{0.5, 3\}$ and fix survival rates, $\pi_s = \pi_i \pi_{ic} \pi_{cy}$ and π, and productivity growth, γ, as observed in the data in 1990 (see Table A.1). The values for $\delta = 1-(1-0.088)^T$ and $\beta = 0.96^T$ are taken from McGrattan (1994) and adjusted for period length, while A, θ_s and α are chosen to match an annual population growth rate of 0.65 percent, annual interest rate net of depreciation of roughly 4 percent and an annual capital-output ratio of 2.7. The latter two targets are also taken from McGrattan (1994). For $\sigma = 3$, we find $\theta_s = 0.144$, $A = 3.7$, $\alpha = 0.33$.

Figure 4: The U.S. experience from 1800 to 1990, CBR_{ann} and $CBR_{s,ann}$ (Model with capital)



Changing productivity growth and mortality as per the experiment in Section 5, the model predicts a fall in CBR from 38.3 to 17.2 (compared to 36.3 to 17.2 in the model without capital). When $\sigma = 0.5$, we recalibrate costs to $\theta_s = 1.19$ to match the population growth rate. In this case, CBR increases from 12.6 to 17.2 (compared to 2 to 17.2 in the model without capital). However, the capital-output ratio rises to 12.2 while the interest rate drops to -2 per cent. Recalibrating θ_s, β and A to reproduce the three targets again, gives $\theta_s = 0.115$, $\beta = 0.903^T$ and $A = 3.48$. In this case, CBR increases only slightly, from 15.7 to 17.19.

For population growth rates, the effects are also larger than in the experiment without capital. When $\sigma = 3$, the population growth rate decreases from 1.75 percent per year to 0.65 percent and increases from -1.99 percent to 0.65 percent per year when $\sigma = 0.5$.[39]

Hence, the choice of IES is just as crucial for qualitative and quantitative predictions of dynastic models when parents have some control over children's initial conditions (through bequests in this case) and hence, their well-being, U_{t+1}. However, the model still does not capture the full extend of the demographic transition.

6.3 Human and health capital

Many of the authors who have worked on the history of fertility have emphasized the role of changes in education over the last 200 years. Examples include Becker, Murphy, and Tamura (1990), Rosenzweig (1990), Benhabib and Nishimura (1993), Galor and Weil (2000), Fernandez-Villaverde (2001), Galor and Moav (2002), Doepke (2004), Soares (2005), Manuelli and Seshadri (2007) and many others. These channels may well be important to account for the entire demographic transition in this model. However, even if these channels are taken into account, qualitative and quantitative results still depend on the size of the IES. To see this, consider the model above but assume that households can invest in human capital of their children in addition to physical capital. Now, as long as physical capital is an alternative, the first order conditions still generate equation (9). Hence, the same intuitions about the interest rate will go through here. However, the threshold σ^* now depends on the complementarities/substitutabilities and input shares in the production functions for human capital, for effective labor and for final output. An important and interesting exercise suggested by the results in this paper is to revisit channels analyzed by others with low elasticity. In particular, the necessary conditions on technology to generate a decrease in population growth and fertility in these models may well be weaker than in Becker, Murphy, and Tamura (1990), for example, once low elasticity is allowed.

Furthermore, the results suggest that focusing on the *acceleration* of productivity growth rates in conjunction with *low IES* may be an alternative theory to those based on increases in productivity *levels* and *non-homotheticities* in utility as in Galor and Weil (2000) and Greenwood and Seshadri (2002). Also, the quantitative importance of policies including the kinds of regulatory changes that have taken place, e.g., child labor laws and compulsory schooling (see Doepke (2005) for example) might well change.

Finally, in our analysis, we have assumed that survival probabilities are exogenous to the

[39] The fact that both the results for $\sigma's$ are better than before may be due to the fact that $\sigma = 3$ is further above the threshold $\sigma^* = 0.66$ for which the effect of γ is nil, while $\sigma = 0.5$ is closer from below, than in Section 5, where $\sigma^* = 1$.

decision maker. While this is probably a reasonable assumption about many of the improvements in health over the period (the development of the germ theory of disease, the advent of pasteurization of milk and vaccinations), many aspects of health that affect these probabilities are in fact chosen. Indeed, some authors have explicitly modeled this choice (e.g., Cigno (1998), Fernandez-Villaverde (2001), Kalemli-Ozcan (2002, 2003), Manuelli and Seshadri (2007), De la Croix and Licandro (2007), Hall and Jones (2007)). Revisiting previous results along the lines of this paper would probably also be fruitful.

7 Conclusion

In this paper, we have studied a class of dynastic models a la Barro-Becker and have shown that it can be used as the basis of a quantitative theory of the Demographic Transition if the appropriate preference parameter configuration is adopted. This requires a low IES in flow utility over consumption as is used in the growth and business cycle literatures. The success of the models is tightly connected to a utility driven quantity-quality trade-off because number and quality of children are substitutes rather than complements. In versions of the model in which child quality (e.g., bequests or education) is not under the control of the parent, this requires that family size and descendant utility are directly substitutes in the utility function. In generalizations in which child quality is chosen by the parent, this condition follows directly from assumptions on the flow utility function over consumption.

Thus, this approach gives an alternative, complementary, route to building a quantitatively successful model of the Demographic Transition. It differs from earlier approaches in that different driving forces are important. For example, here important factors are changes in the survival rates of children at different ages and the overall pace of growth. In others, it is changes in income levels in the presence of non-homotheticicities in utility or choice sets. This suggests new ways to look at the data with these differences in mind.

As we have emphasized, the key issue here is whether quantity and quality are complements or substitutes in the value function of a parent. Existing reduced form evidence does not shed much light on this question. Because of this, structural estimates are needed to determine whether the demographic transition was driven by preferences or properties of technology, such as increasing returns to human capital or capital-skill complementarities.

It is not until these questions are answered more definitively that the welfare benefits of various development policy recommendations (such as education subsidies, child-labor and fertility restrictions (e.g. China's one-child policy)) can be reliably addressed by using models like those developed in this paper.

A Appendix

A.1 Time series used in Section 5

Table A.1: Annual Data Used for the Time Series Experiment in Section 5, U.S.

a *Year*	b γ	c π_i	d π_{ic}	e π_{cy}	f $\frac{T}{1-\pi}$ EL	g π_{ann} EL	h CBR data	i PG data	j $CTFR$ data	k BC
1800	1.000	0.771	0.895	0.916	38.41	0.964	45.40	1.027	*7.04*	—
1810	1.000	0.771	0.895	0.916	38.41	0.964	45.82	1.025	*6.92*	—
1820	1.000	0.771	0.895	0.916	38.41	0.964	45.12	1.027	*6.73*	—
1830	1.007	0.771	0.895	0.916	38.41	0.964	45.18	1.024	*6.55*	—
1840	1.007	0.771	0.895	0.916	38.41	0.964	48.17	1.023	*6.14*	—
1850	1.007	0.771	0.895	0.916	38.41	0.964	46.05	1.020	5.59	1828
1860	1.007	0.803	0.909	0.926	40.32	0.966	45.19	1.018	5.49	1838
1870	1.014	0.816	0.917	0.933	41.14	0.967	39.30	1.017	5.36	1848
1880	1.014	0.775	0.899	0.920	39.65	0.966	35.16	1.016	4.90	1858
1890	1.014	0.840	0.921	0.934	40.96	0.967	32.71	1.013	4.50	1868
1900	1.014	0.871	0.930	0.940	41.73	0.968	28.58	1.013	3.25	1878
1910	1.014	0.885	0.947	0.940	42.63	0.969	27.50	1.011	3.15	1888
1920	1.018	0.908	0.990	0.992	44.45	0.971	24.92	1.012	2.82	1898
1930	1.018	0.931	0.994	0.995	45.14	0.971	19.05	1.007	2.30	1908
1940	1.018	0.945	0.997	0.997	46.77	0.973	18.60	1.012	2.59	1918
1950	1.018	0.967	0.998	0.998	49.01	0.974	23.00	1.015	3.11	1928
1960	1.018	0.973	0.999	0.998	49.65	0.975	22.70	1.011	3.01	1938
1970	1.018	0.977	0.999	0.998	49.63	0.975	17.40	1.006	2.22	1948
1980	1.018	0.987	0.999	0.999	51.73	0.976	15.10	1.006	1.80	1958
1990	1.018	0.990	1.000	0.999	52.95	0.977	15.80	1.006	*1.97*	—

Data Sources for Table A.1:

- Column b, γ (annual productivity growth rate): is from the data on real wages in Greenwood and Vandenbroucke (2005), from 1830 to 1988, for the period from 1800 to 1830, we assumed $\gamma = 1.00$;
- Column c, π_i (survival probability from age 0 to age 1): is derived from data on Infant Mortality Rates, IMRs from:
  1850 to 1900 are from Haines (1994a), U.S. Model, Total Population Both Sexes;
  1800 to 1840 are assumed to be the same as 1850;

1910 is taken from U.S. Department of Commerce, Bureau of the Census (1910);
1920 to 1990 are from National Center for Health Statistics (1998);

- Column d, π_{ic} (survival probability to age 5, conditional on surviving to age 1):
  1850 to 1900 are derived from year by year death rates in Haines (1994a), U.S. Model, Total Population Both Sexes;
  1800 to 1840 are assumed to be the same as 1850;
  1910 is derived from year by year death rates from U.S. Department of Commerce, Bureau of the Census (1910);
  1920 to 1990 are from National Center for Health Statistics (1998);
- Column e, π_{cy}(survival probability to age 20,conditional on surviving to age 5):
  1850 to 1900 are derived from year by year death rates in Haines (1994a), U.S. Model, Total Population Both Sexes;
  1800 to 1840 are assumed to be the same as 1850;
  1910 is derived from year by year death rates from U.S. Department of Commerce, Bureau of the Census (1910);
  1920 to 1990 are from National Center for Health Statistics (1998);
- Column f, $\frac{T}{1-\pi}$ (EL) (expectation of life at age 20): is taken from Lee (2001), Column B;
- Column g, π_{ann} (EL) (annual adult survival rate): is derived from Column f;
- Column h, CBR (crude birth rate, annual):
  1800 to 1930 are taken from Hacker (2003), Figure 1,
  1940 to 1990 are from Haines (1994b), Table 3;
- Column i, PG (population growth rate, annual): are taken from Haines (1994b), Table 1, RNI=rate of natural increase (net of immigration).
- Column j, $CTFR$ (cohort total fertility rate):
  1800 to 1840 TFR taken from Haines (1994b), Table 3, Whites
  1850 to 1980 CEB taken from Jones and Tertilt (2006), Table A6;
  1990 TFR taken from Haines (1994b), Table 3, Weighted average Whites-Blacks;
- Column j, BC (birth cohort): birth year of mothers, year midpoint for 5-year cohorts.

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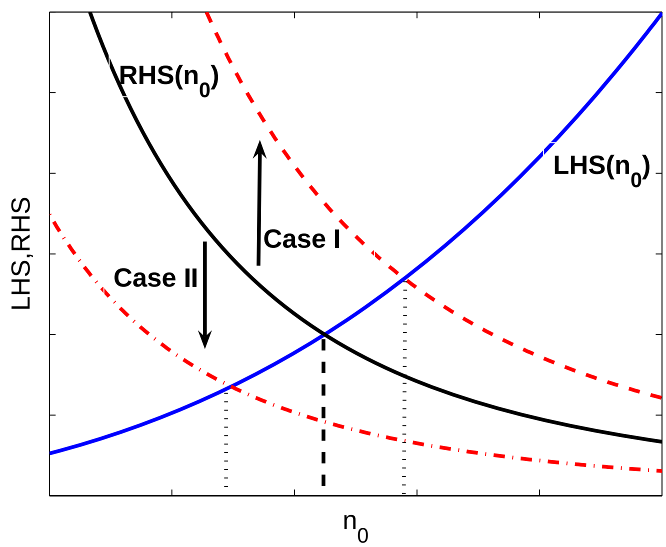
B Additional Appendix

B.1 Illustration of Equation (1)

Figure A.1: Comparative Statics with respect to U_1



B.2 Altruism toward unborn children

This section concerns the utility of unborn children and altruism toward them. That is, one interpretation of the fact that children that are not born do not enter the calculation of time-t utility is that they are assigned $U = 0$. This interpretation is fine when $\sigma < 1$, but causes difficulty when $\sigma > 1$. When utility is negative we can assume that unborn children also get negative utility, and even less than that received by born children ($\bar{u}_{unborn} < \bar{u}_{born}$) and that parents are altruistic towards these children too. To see this, let the utility of the parent be given by:

$U_t = u(c_t) + g(n_t)\bar{u}_{born} + h(n_p - n_t)\bar{u}_{unborn}$

where n_p is the number of potential children and n_t is the number of children born. As can be seen from this, one interpretation of the preferences we use is that $h = 0$, not that $\bar{u}_{unborn} = 0$. Under this interpretation, parents are only weakly altruistic toward their children however.

Strict altruism with respect to the level of utilities holding the number of births fixed requires $g(\cdot) > 0$ and $h(\cdot) > 0$. Since $\bar{u}_{unborn} < \bar{u}_{born}$, strict altruism also requires that increasing n_t strictly increases U_t. This can be written as:

$\frac{d}{dn}\left[g(n)\bar{u}_{born} + h(n_p - n)\bar{u}_{unborn}\right] > 0.$

This condition is not necessarily satisfied. In particular, it is important that the marginal utility from the unborn increases more slowly than the marginal utility from born children decreases as children move from the unborn to the born state. To gain some intuition, consider the case where utilities are isoelastic and the same—$g(n) = h(n) = n^{\eta}$. Then the condition above becomes:

$$1 < (\frac{n_p}{n_t} - 1)\left[\frac{\bar{u}_{unborn}}{\bar{u}_{born}}\right]^{1/(\eta-1)}.$$

One simple way to satisfy this condition is to assume that the number of children that can feasibly be born, $\bar{n}_t = \frac{w_t}{\theta_t}$, is small relative to the number of potential children, n_p, i.e., as $\frac{\bar{n}_t}{n_p} \rightarrow 0$, the additive term representing the unborn in parent's utility is more or less independent of parent's choices. In this case, the decisions made are approximately the same as the ones made with the utility functions used throughout the paper.

B.3 Quantitative comparative statics

Here we give some simple comparative statics of our measures of fertility for various values of σ for the range of relevant parameter values. We calibrate θ_s/w to match $\gamma_N = 1$ using $\gamma = 1.02$, $\pi = 1.0$, $\pi_s = 1.0$. Table A.2 shows the results of changing only γ from $\gamma = 1.00$ to $\gamma = 1.02$ while Table A.3 does the same for $\pi_s = 0.6$ to $\pi_s = 1$. Finally, Table A.4 examines changes in π corresponding to an expected lifetime at age twenty ranging from 25 to 45 years.

Table A.2: Changing productivity growth

Productivity Growth	$\sigma = 0.5$		$\sigma = 1.0$		$\sigma = 3.0$	
γ_{ann}	$\gamma_{N,ann}$	$CBR_{s,ann}$	$\gamma_{N,ann}$	$CBR_{s,ann}$	$\gamma_{N,ann}$	$CBR_{s,ann}$
1.00	0.98	5.25	1.00	15.38	1.013	21.37
1.01	0.99	10.49	1.00	15.38	1.007	18.45
1.02	1.00	15.38	1.00	15.38	1.00	15.38

Table A.3: Changing survival to adulthood (STA)

	π_{cy} (STA)	θ_s/w	$\gamma_{N,ann}$	CBR_{ann}	$CBR_{s,ann}$
	0.6	0.85	0.993	19.68	11.81
$\sigma = 0.5$	0.8	0.80	0.997	17.51	14.01
	1.0	0.77	1.00	15.38	15.38
	0.6	0.65	0.996	22.46	13.48
$\sigma = 1.0$	0.8	0.61	0.998	18.31	14.65
	1.0	0.59	1.00	15.38	15.38
	0.6	0.26	0.998	24.16	14.50
$\sigma = 3.0$	0.8	0.24	0.999	18.80	15.04
	1.0	0.23	1.00	15.38	15.38

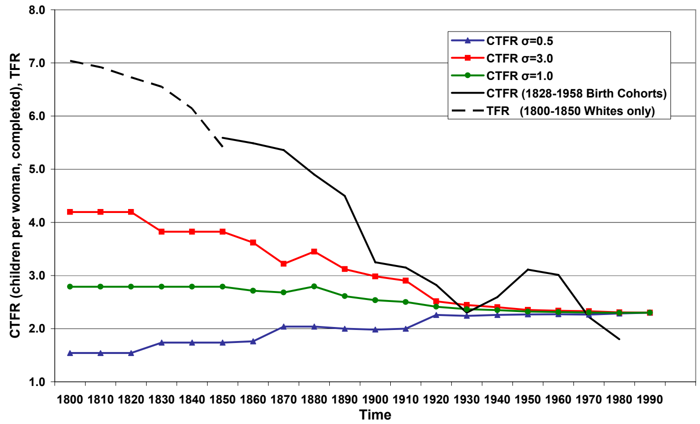
Table A.4: Changing expected lifetime (EL)

	π_{ann}	$\frac{T}{1-\pi}$ (EL)	$\gamma_{N,ann}$	CBR_s
	0.923	25	0.979	15.60
$\sigma = 0.5$	0.959	35	0.993	15.26
	0.971	45	1.00	15.38
	0.923	25	0.991	19.56
$\sigma = 1.0$	0.959	35	0.997	17.00
	0.971	45	1.00	15.38
	0.923	25	0.998	21.84
$\sigma = 3.0$	0.959	35	0.999	18.01
	0.971	45	1.00	15.38

B.4 Historical experiment: auxiliary figures

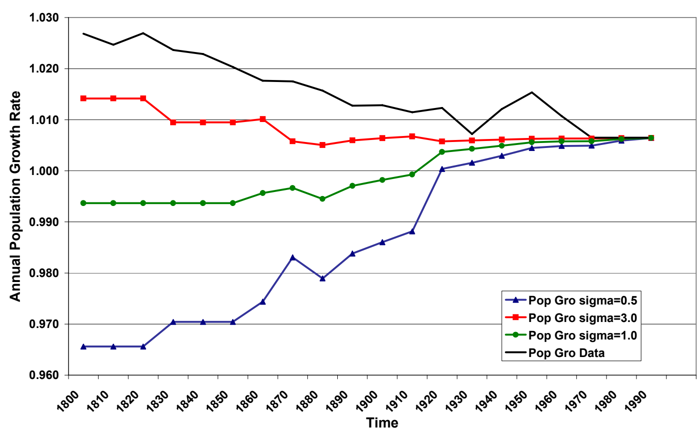
Here we give figures—like the one given in the main text for CBR in Section 5—for the other measures of fertility we have discussed, $CTFR$ and γ_N. They show a similar pattern overall

Figure A.2: The U.S. experience from 1800 to 1990, $CTFR$ and TFR



with the model capturing significant fractions of the overall changes seen in the data—about half (see also Table 2).

Figure A.3: The U.S. experience from 1800 to 1990, annual population growth rate



Next, we depict the decomposition exercise in Section 5 of the overall changes in these three measures into the three components, changes in γ only, changes in π only and changes in π_s only (see also Table 3).

Figure A.4: Decomposition, $CTFR$

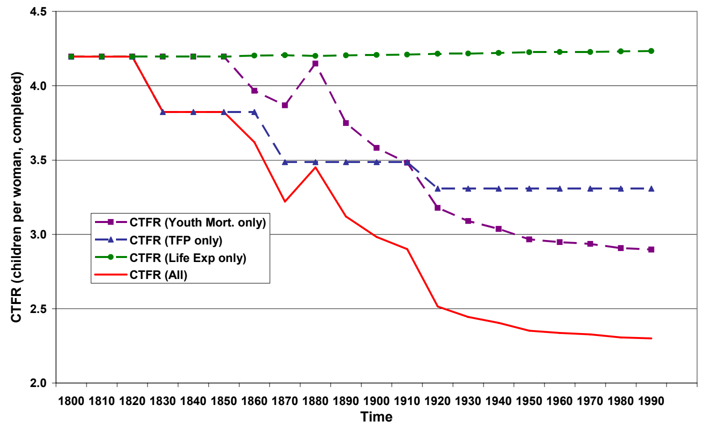


Figure A.5: Decomposition, CBR

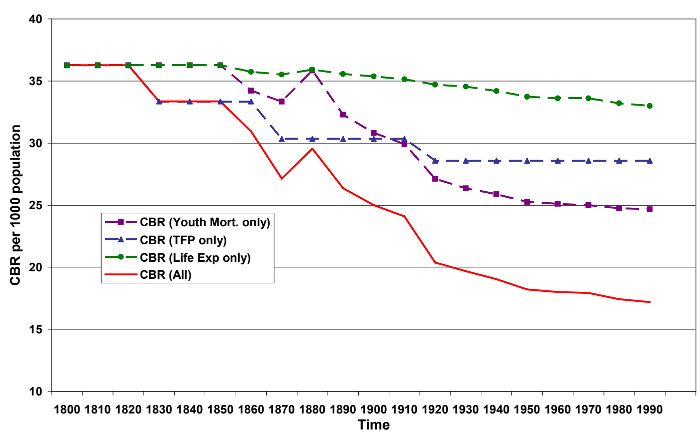
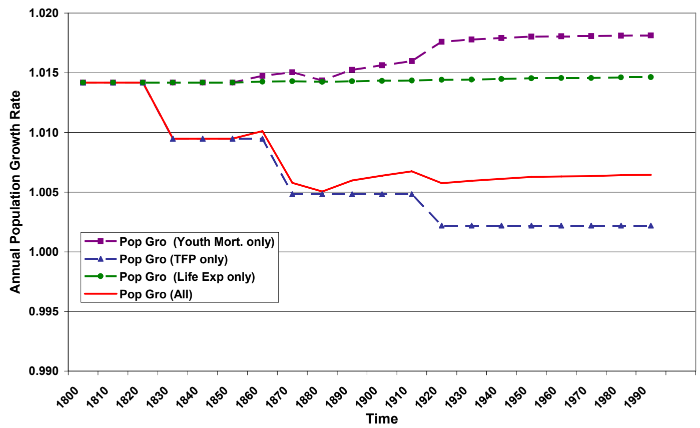


Figure A.6: Decomposition, annual population growth rate



B.5 Using expected working life, instead of life expectancy

In this section we compare results when expected working lifetime instead of expected lifetime conditional on reaching age 20 is used in the experiments. The range for survival probabilities conditional on reaching age 20, π, in the main text are deduced from measures of expectation of life (EL) at age 20 (see Table A.1, column g). EL increased from 38.5 to 53 years. One issue related to this is that expected time in retirement has increased dramatically over the past 150 years (see Lee (2001)). We performed the same experiment using expected working life at age 20 (EWL) (i.e. the difference between expected lifetime and expected years of retirement) which implied an increase from 36.7 to 40.3 years (taken from Lee (2001), Column C). We obtained very similar results since the effect of longevity on births and population growth rates are small in the experiments above. In particular, we find a decrease in $CTFR$ from 4.21 to 2.3, a decrease in CBR from 36.95 to 19.6 and a decrease in population growth from 1.44 percent per year to 0.65 percent per year. The trade-off between using either one of these measures is that on the one hand, EL overstates the benefits in terms of income from having children, while EWL understates the benefits from dynasty size (since retirement is analogous to death in this case).

B.6 Deviating from $\eta = 1 - \sigma$

In all the quantitative experiments, we assume that $\eta = 1 - \sigma$. Without this assumption, given our two admissible parameter configurations, AI and AII, η would have to satisfy $\eta < 1 - \sigma$ whenever $\sigma > 1$ or $\eta > 1 - \sigma$ whenever $\sigma < 1$. Hence, the assumption that $\eta = 1 - \sigma$ makes results more readily comparable. In this case, the two utility effects of increasing dynasty size cancel out—independently of σ.

We performed sensitivity with respect to η and found that, for $\eta < 1 - \sigma < 0$, results are not very sensitive to the value of η, holding $\sigma = 3$ (see FigureA.7). For the case, $\eta = 0.8 > 1 - \sigma = 0.5$, the effect of mortality on CBR starts to be negative, so that the model predicts an upward hump until 1880 and a slight decrease thereafter (see FigureA.8).

B.7 Perfect Foresight Transition Paths to BGP

In this section, we discuss an alternative to the calculation given in section 5. There, the simulated data were calculated assuming that the agents believed that their current circumstances, in terms of child costs, productivity growth rates and survival probabilities, would prevail indefinitely into the future when making their decisions—i.e., the calculations are BGP to BGP. The weakness of this is that it assumes that agents act as if circumstances will not change in the future, even though they actually will. At the other extreme, one could assume that agents in a give period t, anticipate exactly all future changes that will occur—i.e., there is perfect foresight with respect to future values of γ, π and θ_s. Here we give the calculations for the model under this alternative assumption. We find that this makes very little difference in the end.

B.7.1 Solving the model with perfect foresight

From the Planner's problem in Section 3.3, the first– order condition for N_{t+1} is:

$$\theta_{s,t} N_t^{\eta+\sigma-1} C_t^{-\sigma} = \beta \left[\tfrac{(\eta+\sigma-1)}{(1-\sigma)} N_{t+1}^{\eta+\sigma-2} C_{t+1}^{1-\sigma} + \left[w_{t+1} + \theta_{s,t+1}\pi_{t+1} \right] N_{t+1}^{\eta+\sigma-1} C_{t+1}^{-\sigma} \right]$$

The other equation determining the solution is:

$$C_t = \left[w_t + \theta_{s,t}\pi_t \right] N_t - \theta_{s,t} N_{t+1}.$$

After some algebra, these two equations can be rewritten as:

$$\gamma_{Nt}^{1-\eta} \left[\frac{\left[\frac{w_{t+1}}{\theta_{s,t+1}} + \pi_{t+1} \right] - \gamma_{Nt+1}}{\left[\frac{w_t}{\theta_{s,t}} + \pi_t \right] - \gamma_{Nt}} \right]^{\sigma}$$

$$= \beta \left[\tfrac{\theta_{s,t+1}}{\theta_{s,t}} \right]^{1-\sigma} \left[\tfrac{(\eta+\sigma-1)}{(1-\sigma)} \left[\left[\tfrac{w_{t+1}}{\theta_{s,t+1}} + \pi_{t+1} \right] - \gamma_{Nt+1} \right] + \left[\tfrac{w_{t+1}}{\theta_{s,t+1}} + \pi_{t+1} \right] \right]$$

Figure A.7: Sensitivity with respect to $\eta < 1 - \sigma$, for $\sigma = 3$

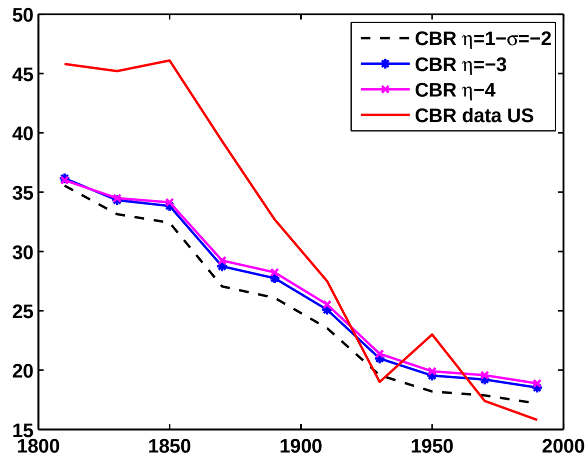
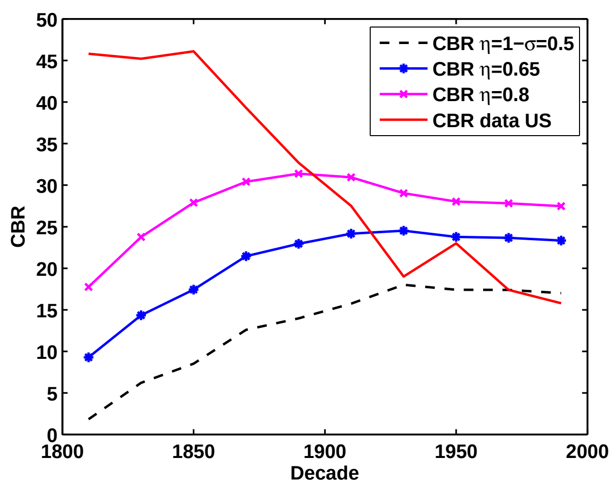


Figure A.8: Sensitivity with respect to $\eta > 1 - \sigma$, for $\sigma = 0.5$



$$\frac{C_t}{N_t}\frac{1}{\theta_{s,t}} = \left[\frac{w_t}{\theta_{s,t}} + \pi_t\right] - \gamma_{Nt}.$$

The first of these is a first order difference equation in γ_N. It has time varying coefficients however.

If $(\theta_{s,t}, w_t, \pi_t)$ converge in the sense that $\frac{w_t}{\theta_{s,t}} \to \frac{w}{\theta_s}$, $\frac{\theta_{s,t+1}}{\theta_{s,t}} \to \gamma$, $\pi_t \to \pi$, it can be shown that the solution to the model converges to the BGP determined by $\frac{w}{\theta_s}$, γ, and π. Further, assuming that $\frac{w_t}{\theta_{s,t}} = \frac{w}{\theta_s}$, $\frac{\theta_{s,t+1}}{\theta_{s,t}} = \gamma$, and $\pi_t = \pi$ for all $t \geq t^*$ for some t^*, it can be shown that all of the relevant variables, measured in per capita terms, are constant after date t^*. Because of this, the model can be solved backwards from t^* in this case. Thus, suppose that the sequence of exogenous parameters is given by:

$$(\theta_{s,0}, w_0, \pi_0, ..., \theta_{s,t^*}, w_{t^*}, \pi_{t^*}, \gamma\theta_{s,t^*}, \gamma w_{t^*}, \pi_{t^*}, ...).$$

Then, the solution to the perfect foresight infinite horizon problem is of the form:

$$(C_0, N_0, ..., C_{t^*}, N_{t^*}, C_{t^*+1}, N_{t^*+1}, ...)$$

where:

1) for $t \geq t^* + 1$, $N_{t+1} = \gamma_N N_t$ with γ_N given by the solution to:

$$\gamma_N^{1-\eta} = \beta\gamma^{1-\sigma}\left[\frac{(\eta+\sigma-1)}{(1-\sigma)}\left[\left[\frac{w_{t^*}}{\theta_{s,t^*}} + \pi_{t^*}\right] - \gamma_N\right] + \left[\frac{w_{t^*}}{\theta_{s,t^*}} + \pi_{t^*}\right]\right];$$

2) $\frac{C_{t^*}}{N_{t^*}}\frac{1}{\theta_{s,t^*}} = \frac{C^*_{t^*}}{N_{t^*}}\frac{1}{\theta_{s,t^*}} = \left[\frac{w_{t^*}}{\theta_{s,t^*}} + \pi_{t^*}\right] - \gamma_N$;

3) for $s \geq 1$,

$$\frac{C^*_{t^*}}{N_{t^*}}\frac{1}{\theta_{s,t^*}} = \left[\frac{w_{t^*}}{\theta_{s,t^*}} + \pi_{t^*}\right] - \gamma_N \Leftrightarrow \frac{C_{t^*+s}}{N_{t^*+s}}\frac{1}{\gamma^s\theta_{s,t^*}} = \left[\frac{w_{t^*}}{\theta_{s,t^*}} + \pi_{t^*}\right] - \gamma_N;$$

4) For $t < t^*$, γ_{Nt} evolves according to the difference equation:

$$\gamma_{Nt}^{1-\eta}\left[\frac{\left[\frac{w_{t+1}}{\theta_{s,t+1}}+\pi_{t+1}\right]-\gamma_{Nt+1}}{\left[\frac{w_t}{\theta_{s,t}}+\pi_t\right]-\gamma_{Nt}}\right]^\sigma$$

$$= \beta\left[\frac{\theta_{s,t+1}}{\theta_{s,t}}\right]^{1-\sigma}\left[\frac{(\eta+\sigma-1)}{(1-\sigma)}\left[\left[\frac{w_{t+1}}{\theta_{s,t+1}} + \pi_{t+1}\right] - \gamma_{Nt+1}\right] + \left[\frac{w_{t+1}}{\theta_{s,t+1}} + \pi_{t+1}\right]\right];$$

5) For $t < t^*$, $\frac{C_t}{N_t}$ is given by:

$$\frac{C_t}{N_t}\frac{1}{\theta_{s,t}} = \left[\frac{w_t}{\theta_{s,t}} + \pi_t\right] - \gamma_{Nt}.$$

B.7.2 Numerical Implementation

We keep the length of a period at $T = 20$ years. Suppose from $t^* = 1990$ on the growth rate in productivity, γ, infant, child and youth mortality $(\pi_i, \pi_{ic}, \pi_{cy})$ (and hence, detrended costs of raising surviving children, θ_s) and adult mortality (longevity), π, are constant. Then, we can use 1) above to solve for the population growth rate, γ_N, on the balanced growth path using parameter values for 1990. We can then use 4) to solve backward for γ_{Nt}, $t = 1970$ using $\gamma_{Nt+1} = \gamma_{Nt^*}$ $t^* = 1990$ and so on. To do this, we have to make one additional assumption (similar to the balanced growth assumption), namely that base costs of raising children, $(\theta_i, \theta_c, \theta_y)$ grow at the same rate as wages every period but are otherwise constant while the cost of raising a surviving child, θ_s, may vary additionally because youth mortality varies.

As in Section 5, we assume that base costs are constant fractions of calibrated costs when children survive with certainty. The results from this experiment are almost indistinguishable from the BGP to BGP experiment in Section 5. This is not surprising since changes in mortality and productivity growth are very smooth. That is, knowing that mortality and productivity change slightly in the next few periods induces very similar choices to the setting in which people believe today's parameters will prevail forever. Moreover, the length of a period being 20 years implies large discounts of future utility (children's utility) and hence changes expected in the future do not affect current decisions very much.

Figure A.9: Perfect Foresight versus Balanced Growth to Balanced Growth, CBR

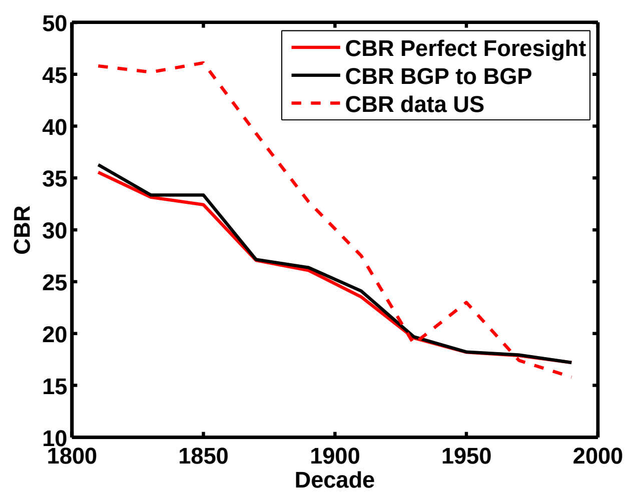
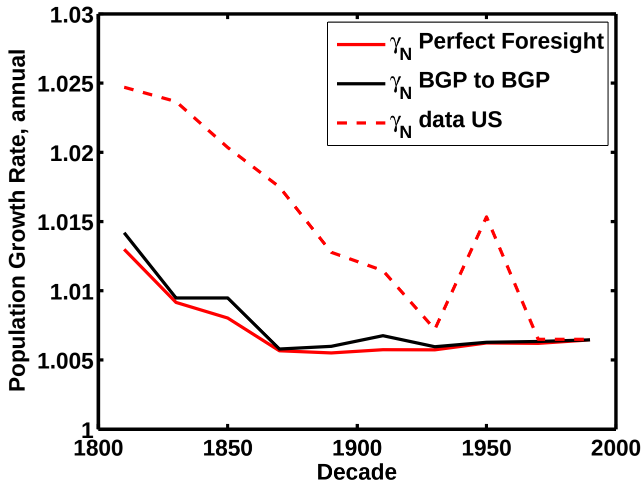


Figure A.10: Perfect Foresight versus Balanced Growth to Balanced Growth, γ_N



B.8 The case of the U.K.

In this section, we perform the same experiment as in Section 5 but using data for the United Kingdom (England and Wales for the most part). The results are quite similar: we capture about two-thirds of the change in CBR and one half of the change in population growth.

The fertility experience in the U.K. over the past 200 years is similar to that of the U.S., except that levels in 1800 were lower already. Because of this the decrease, both in CBR and population growth, was smaller. Mortality was also lower in the U.K. than it was in the U.S. around that time. Finally, our estimates of productivity growth suggest the latter was higher in 1800 as well. Since, fertility, mortality and productivity growth are very similar in the two countries in 1990, all these observations are consistent with our theory and one would expect the model to capture the same fraction of changes in fertility and population growth.

Table A.5: U.K. Costs of children in 1990, Time Series Experiment

σ	$\frac{\theta_s}{w}$	$T \times \frac{\theta_s}{w}$	$Max\ CTFR$
0.5	0.76	15.19	2.63
1.0	0.56	11.27	3.55
3.0	0.20	4.04	9.90

Table A.6: U.K. Time Series Experiment: Data versus Model, for several values of σ

	$\gamma_{N,ann}$			CBR_{ann}			CTFR		
	1801	1881	1986	1801	1881	1986	1801	1881	1986
Data	1.013	1.012	1.003	37.6	33.9	13.2	—	—	—
$\sigma = 0.5$	0.967	0.974	1.003	1.65	5.31	15.1	1.46	1.74	2.17
$\sigma = 1.0$	0.989	0.992	1.003	18.1	18.2	15.1	2.43	2.46	2.17
$\sigma = 3.0$	1.007	1.005	1.003	29.8	26.2	15.1	3.44	3.13	2.17

Table A.7: U.K. Decomposition: Productivity (γ) , Mortality (π_s) and Longevity (π)

$\sigma = 3$	$\gamma_{N,ann}$			CBR_{ann}			CTFR		
	1800	1880	1990	1800	1880	1990	1800	1880	1990
Data	1.013	1.012	1.003	37.6	33.9	13.2	—	—	—
Productivity (γ)	1.007	1.003	0.998	29.8	27.9	24.7	3.44	3.24	2.94
Mortality (π_s)	1.007	1.007	1.011	29.8	28.8	21.8	3.44	3.32	2.51
Longevity $(\pi)^{40}$	1.007	1.006	1.007	29.8	29.0	25.4	3.44	3.35	3.48

Figure A.11: The U.K. experience from 1800 to 1990, CBR

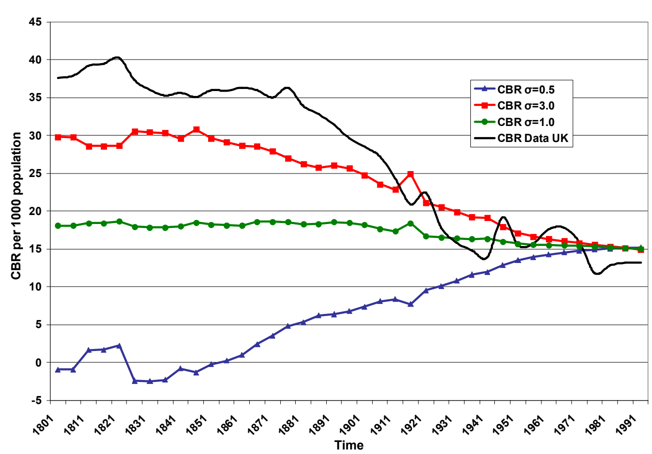


Table A.8: Annual Data Used for the Time Series Experiment in Section B.8, U.K.

a $Year$	b γ	c π_i	d π_{ic}	e π_{cy}	f $\frac{T}{1-\pi}$ EL	g π_{ann} EL	h CBR data	i CBR HP	j PG data	k PG HP
1801	1.006	0.863	0.884	0.864	39.00	0.965	37.60	39.23	1.014	1.014
1806	1.006	0.863	0.884	0.864	39.00	0.965	37.90	38.98	1.013	1.014
1811	1.006	0.867	0.870	0.925	39.00	0.965	39.18	38.72	1.015	1.014
1816	1.006	0.867	0.870	0.925	39.00	0.965	39.48	38.46	1.015	1.014
1821	1.006	0.855	0.864	0.943	39.00	0.965	40.22	38.17	1.016	1.014
1826	1.006	0.855	0.864	0.857	39.00	0.965	37.30	37.87	1.014	1.014
1831	1.006	0.860	0.877	0.840	39.00	0.965	36.03	37.55	1.012	1.014
1836	1.007	0.860	0.877	0.840	39.00	0.965	35.27	37.21	1.012	1.014
1841	1.007	0.838	0.859	0.894	40.57	0.967	35.61	36.86	1.011	1.013
1846	1.007	0.826	0.847	0.886	38.99	0.965	35.06	36.47	1.012	1.013
1851	1.008	0.829	0.858	0.893	40.19	0.966	35.98	36.05	1.016	1.013
1856	1.008	0.830	0.856	0.898	40.63	0.967	35.89	35.57	1.012	1.013
1861	1.009	0.836	0.854	0.902	40.48	0.967	36.30	35.02	1.012	1.013
1866	1.009	0.826	0.864	0.909	39.93	0.966	35.95	34.39	1.012	1.012
1871	1.010	0.830	0.874	0.913	40.01	0.966	35.00	33.67	1.015	1.012
1876	1.010	0.839	0.882	0.926	40.29	0.966	36.30	32.84	1.015	1.012
1881	1.010	0.845	0.889	0.930	41.33	0.967	33.90	31.91	1.012	1.011
1886	1.010	0.844	0.898	0.941	41.79	0.968	32.80	30.88	1.010	1.011
1891	1.010	0.837	0.900	0.942	41.37	0.968	31.40	29.76	1.016	1.010
1896	1.010	0.825	0.907	0.950	42.85	0.969	29.60	28.57	1.013	1.009
1901	1.010	0.843	0.919	0.953	43.43	0.970	28.50	27.32	1.010	1.009
1906	1.010	0.868	0.932	0.957	44.59	0.971	27.20	26.04	1.010	1.008
1911	1.011	0.883	0.939	0.958	45.37	0.971	24.30	24.74	1.006	1.008
1916	1.011	0.900	0.936	0.919	38.3	0.964	20.90	23.47	0.984	1.007
1921	1.011	0.918	0.958	0.964	47.54	0.973	22.40	22.23	1.010	1.007
1926	1.011	0.927	0.963	0.968	47.85	0.973	17.80	21.06	1.005	1.007
1931	1.012	0.935	0.972	0.970	48.54	0.974	15.80	19.98	1.004	1.006
1936	1.013	0.942	0.981	0.975	49.32	0.974	14.80	18.99	1.005	1.006
1941	1.013	0.946	0.984	0.968	48.33	0.974	13.90	18.09	0.979	1.006
1946	1.014	0.960	0.992	0.982	50.76	0.975	19.20	17.29	1.050	1.006
1951	1.015	0.972	0.995	0.992	52.39	0.976	15.50	16.55	1.003	1.006
1956	1.015	0.976	0.996	0.993	53.17	0.977	15.70	15.87	1.005	1.006
1961	1.016	0.978	0.997	0.993	53.48	0.977	17.60	15.22	1.010	1.005
1966	1.017	0.981	0.997	0.993	53.97	0.977	17.80	14.60	1.006	1.005
1971	1.017	0.983	0.997	0.994	54.26	0.977	15.90	13.98	1.005	1.005
1976	1.017	0.986	0.998	0.994	54.79	0.978	11.80	13.36	1.000	1.004
1981	1.018	0.989	0.998	0.995	55.69	0.978	12.80	12.75	1.000	1.004
1986	1.018	0.991	0.998	0.996	56.46	0.978	13.20	12.15	1.003	1.003
1991	1.018	0.993	0.999	0.996	57.39	0.979	13.20	12.15	1.004	1.003

Data Sources for Table A.8:[41]

- Column b, γ (annual productivity growth rate) from annual growth rate in GDP per capita, (log GDP HP filtered $\lambda = 400$):
  1800 to 1865 from Clark (2001),
  1850 to 1990 from Maddison (1995), p. 194, rescaled to match Clark in 1850;
- Column c,d,e, $(\pi_i, \pi_{ic}, \pi_{cy})$ (survival rates from age specific mortality rates)
  1800 to 1837 from Wrigley et al. (1997), Table 6.1, p.215,
  1841 to 1990 from Human Mortality Database;
- Column f, $\frac{T}{1-\pi}$ (EL) (expectation of life at age 20):
  1841 to 1990 from Human Mortality Database,
  1800 to 1836 set constant at 39 years;
- Column g, π_{ann} (EL) (annual adult survival rate): derived from Column f;
- Column h, CBR (crude birth rate, annual):
  1800 to 1871 from Wrigley et al. (1997),
  1871 to 1986 from Mitchell (1998)
- Column i, CBR HP filtered (crude birth rate, annual): Column h HP filtered, $\lambda = 400$;
- Column j, PG (population growth rate, annual):
  1800 to 1837 from Wrigley et al. (1997), Table 6.1, p.215;
  1841 to 1990 from Human Mortality Database;
- Column k, PG HP filtered (population growth rate, annual): Column j HP filtered, $\lambda = 400$.

B.9 Details on the model with capital

B.9.1 Solution of the model with physical capital

The representative dynasty problem we are interested in is given by

$$
\begin{aligned}
& Max_{\{C_t,N_t,K_t\}} \qquad U_0(\{C_t, N_t, K_t\}) = \textstyle\sum_{t=0}^{\infty} \beta^t N_t^{\eta+\sigma-1} \frac{C_t^{1-\sigma}}{1-\sigma} \\
\text{s.t.} \quad & C_t + \theta_{st} N_{st} + X_t \leq w_t N_t + r_t K_t \\
& K_{t+1} \leq (1-\delta) K_t + X_t, \\
& N_{t+1} \leq \pi N_t + N_{st} \\
& (N_0, K_0) \text{ given.}
\end{aligned}
$$

Note that this problem is well defined under both assumption AI and AII derived in Section 2 as long as $\eta \neq 1 - \sigma$. The first order condition with respect to K_{t+1} and N_{t+1} together with the budget constraint, boil down to the following system of equations governing the solution to this (partial equilibrium) problem:

$$\gamma_{ct}^{\sigma} \gamma_{Nt}^{1-\eta} = \beta(r_{t+1} + 1 - \delta)$$

$$\theta_t(r_{t+1} + 1 - \delta) = \left[\frac{(\eta+\sigma-1)}{(1-\sigma)} \frac{C_{t+1}}{N_{t+1}} + [w_{t+1} + \theta_{t+1}\pi]\right]$$

[41] We thank Michael Bar and Oksana Leukhina for help with data sources.

$$\frac{C_t}{N_t} + \theta_t \gamma_{Nt} + \frac{K_{t+1}}{N_{t+1}} \gamma_{Nt} = [w_t + \pi\theta_t] + (r_t + 1 - \delta)\frac{K_t}{N_t}.$$

To ensure interiority in partial equilibrium, we have to (1) either rule out $\eta = 1-\sigma$ or, (2) if $\eta = 1-\sigma$, make the necessary parameter assumptions so that rates of returns to children and capital are equalized. In general equilibrium, prices will adjust to achieve this. To close the model, wages and interest rates are determined in equilibrium by a firm hiring labor and capital to maximize profits with a constant returns to scale—Cobb-Douglas—production function, $F(K_t, \gamma^t N_t) = AK_t^{\alpha}(\gamma^t N_t)^{1-\alpha}$. That is,

$$r_t = F_K(K_t, \gamma^t N_t)$$

$$w_t = F_N(K_t, \gamma^t N_t).$$

On a balanced growth path, we have $\gamma_{c,t} = \gamma_c = \gamma$, $\gamma_{N,t} = \gamma_N$, $\gamma_C = \gamma_K = \gamma\gamma_N$, wages grow at γ and interest rates are constant. Denoting detrended variables by $\widehat{x}$ the above equations become:

$$\gamma_N = \frac{[\beta(r+1-\delta)]^{\frac{1}{1-\eta}}}{\gamma^{\frac{\sigma}{1-\eta}}}$$

$$\widehat{c}^* = [\widehat{w} + \pi\theta_s] + \gamma\,(r+1-\delta)\,\widehat{k}^* - \theta_s\gamma_N - \widehat{k}^*\gamma_N\gamma$$

$$r + 1 - \delta = \frac{\gamma}{\theta_s}\frac{(\eta+\sigma-1)}{(1-\sigma)}\widehat{c}^* + \frac{\gamma}{\theta_s}\widehat{w} + \pi\gamma$$

$$r = \alpha A\widehat{k}^{*\alpha-1}$$

$$\widehat{w} = (1-\alpha)A\widehat{k}^{*\alpha}.$$

These five equations together with initial conditions completely characterize the equilibrium path.

B.9.2 Numerical implementation: auxiliary figures

Here, we show figures of the calculations for the history of U.S. fertility given in section 5 using the version of the model including physical capital discussed in Section 6. We include only the calculations for $\sigma = 3.0$ and $\sigma = 0.5$. We find that the results are very similar as described in the main text. The decomposition confirms our previous finding about timing of events.

Figure A.12: CBR Model with K vs. Data $\sigma = 3$

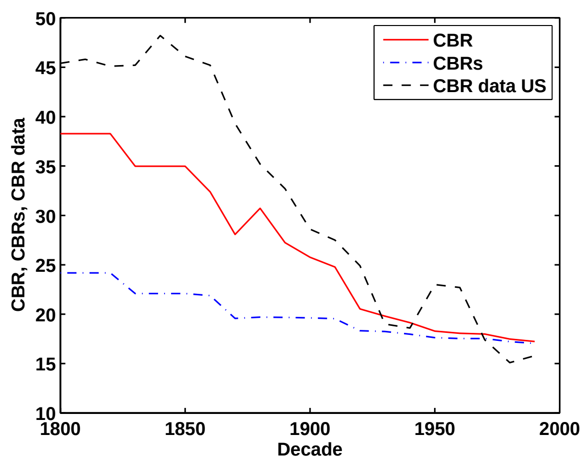


Figure A.13: CBR Model with K vs. Data $\sigma = 0.5$

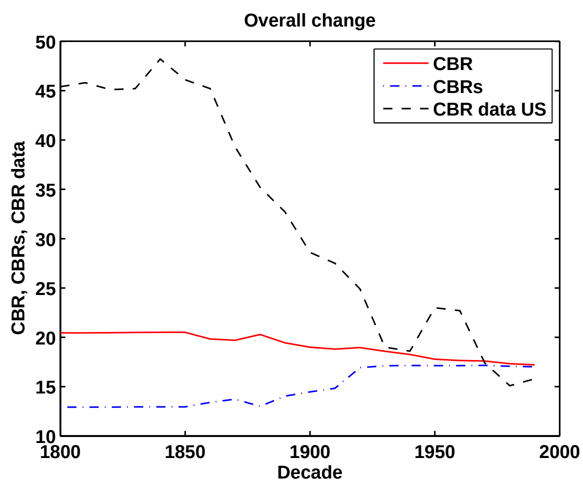


Table A.9: Decomposition: Productivity (γ) vs. Mortality (π_s) and Longevity (π)

$\sigma = 3$	$\gamma_{N,ann}$			CBR_{ann}			CTFR		
	1800	1880	1990	1800	1880	1990	1800	1880	1990
Data	1.027	1.016	1.006	45.4	35.2	15.8	7.04	4.90	1.97
Productivity (γ)	1.018	1.007	1.004	38.3	31.5	29.5	4.48	3.62	3.40
Mortality (π_s)	1.018	1.018	1.020	38.3	37.8	25.4	4.48	4.43	3.02
Longevity (π)	1.018	1.018	1.018	38.3	37.9	35.2	4.48	4.48	4.50

Figure A.14: CBR: Decomposition, Model with K, $\sigma = 3$

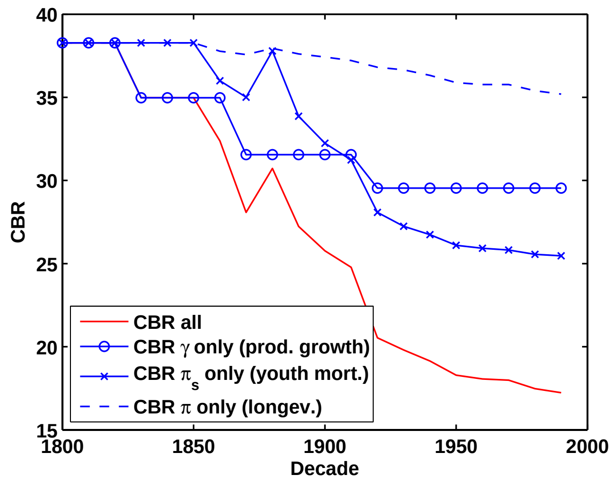


Figure A.15: Population Growth Rate: Decomposition, Model with K, $\sigma = 3$