Exercises 7 (suggested solution) - MikØk2

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Exercise 1

Consider the following situation: There are two firms, a fisherman, denoted f, and a steel mill, denoted s located in the same area. The fisher man make his daily catch in a medium sized lake, which the steel mill is located at the shores about. The steel mill discharges chemicals and waste into the lake as a by-product of its activity.

Both firms sell their output on competitive markets, letting p_f be the price of fish and p_s the price of steel.

The steal mill in producing an amount $x \geq 0$ of steel and discharging an amount $h \geq 0$ of chemicals and waste, has a cost of $c_s(x,h) \geq 0$, with the following assumptions: $\frac{\partial c_s(x,h)}{\partial x} > 0$, $\frac{\partial c_s(x,h)}{\partial h} \leq 0$, $\frac{\partial^2 c_s(x,h)}{\partial x^2} \leq 0$, $\frac{\partial^2 c_s(x,h)}{\partial h^2} \leq 0$ and $\frac{\partial^2 c_s(x,h)}{\partial h \partial x} < 0$ for every $x, h \geq 0$. Also, for every x there exists $\bar{h}$ such that $\frac{\partial c_s(x,\bar{h})}{\partial h} = 0$.

The fisherman in producing an amount of $f \geq 0$ of fish while the discharging of $h \geq 0$ amount of chemicals and waste, endures a cost of $c_f(f,h) \geq 0$, with the following assumptions: $\frac{\partial c_f(f,h)}{\partial f} > 0$, $\frac{\partial c_f(f,h)}{\partial h} > 0$ and $\frac{\partial^2 c_f(f,h)}{\partial f^2} < 0$ for every $f, h \geq 0$.

a) State the profit maximization problem of the fisherman and the steel mill respectively

b) State the necessary first order conditions characterizing the solution of $x^\star$, $h^\star$ and $f^\star$. Are they also sufficient conditions?

c) Find conditions which characterize the solution $x^{\star\star}$, $h^{\star\star}$ and $f^{\star\star}$ that maximize the total profits of both the fisherman and the steel mill.

d) Show that $h^{\star\star} < h^\star$. Argue that the laiss-afaire situation is not Pareto efficient.

Assume next that the government knows the true cost functions, $c_s(\cdot,\cdot)$ and $c_f(\cdot,\cdot)$, and they levi a tax $t \geq 0$ on the dischargement of chemicals and waste. The revenue is paid out to the fisherman as a compensation for the damage.

e) State the profit maximization problem of the fisherman and the steel mill respectively

f) State the necessary first order conditions characterizing the solution of x^t, h^t and f^t.

g) How should t be chosen so that $h^t = h^{\star\star}$?

Assume next that the government setup a market for polution permissions, and that they grant the steel mill the right to issue these polution permissions. Assume that they both firms know the other firm's cost function. The trade of polution permissions is conducted by means of bargaining between the steel mill and the fisherman. The bargining takes place as follows:

1. The steel mill propose a contract, (q, z) where the fisherman pays $q \geq 0$ per polution permission and he obtains z polution permissions
2. The fisherman then accept or rejects. If the fisherman accepts the transfer of money is carried through and the steel mill cannot polute more than z. If the fisherman rejects the proposal the steel mill can freely polute any amount.

h) Which contracts (q, z) will the fisherman accept?

i) Show that it will always be optimal to offer a contract which gives the fisherman exactly a profit equal to $p_f f^\star - c(f^\star, h^\star)$.

j) Show that in any contract (q, z) in a SPE we must have that $z = h^{\star\star}$.

Exercise 1 - suggested solution

a) The profitmax of the steel mill

$$\max_{x,h} p_s x - c_s(h, x)$$

while the profitmax of the fisher is

$$\max_{f} p_f f - c_f(f, h)$$

b) The first order conditions of the steel mill

$$\begin{aligned} p_s - \frac{\partial c_s(x^\star, h^\star)}{\partial x} &= 0 \\ -\frac{\partial c_s(x^\star, h^\star)}{\partial h} &= 0 \end{aligned}$$

while the FOC of the fisherman is

$$p_f - \frac{\partial c_f(f^\star, h^\star)}{\partial f} = 0,$$

and they characterize the solutions $x^\star$, $h^\star$ and $f^\star$. They are sufficient since the second order conditions are meet

$$\begin{aligned}
\frac{\partial^2 c_s}{\partial x^2} &\leq 0 \\
-\frac{\partial^2 c_s}{\partial h^2} &\leq 0 \\
-\frac{\partial^2 c_s}{\partial x^2}\frac{\partial^2 c_s}{\partial h^2} - 2\frac{\partial^2 c_s}{\partial x \partial f} &\leq 0 \\
-\frac{\partial^2 c_f}{\partial f^2} &\leq 0
\end{aligned}$$

hence the profit functions are concave.

c) The total profits are

$$\max_{x,f,h} p_s x + p_f f - c_s(x,h) - c_f(f,h)$$

while the conditions which characterize the solution $x^{\star\star}$, $h^{\star\star}$ and $f^{\star\star}$ that maximize the total profits of both the fisherman and the steel mill are given by

$$\begin{aligned}
p_s - \frac{\partial c_s(x^{\star\star}, h^{\star\star})}{\partial x} &= 0 \\
p_f - \frac{\partial c_f(f^{\star\star}, h^{\star\star})}{\partial f} &= 0 \\
-\frac{\partial c_s(x^{\star\star}, h^{\star\star})}{\partial h} - \frac{\partial c_f(f^{\star\star}, h^{\star\star})}{\partial h} &= 0
\end{aligned}$$

d) According to c) we have that

$$-\frac{\partial c_s(x^{\star\star}, h^{\star\star})}{\partial h} = \frac{\partial c_f(f^{\star\star}, h^{\star\star})}{\partial f} > 0$$

and thus we conclude that

$$-\frac{\partial c_s(x^{\star\star}, h^{\star\star})}{\partial h} > -\frac{\partial c_s(x^\star, h^\star)}{\partial h}.$$

Hence $h^{\star\star} < h^\star$ since $\frac{\partial^2 c_s}{\partial h^2} > 0$. Thus, the laiss-afaire situation is not Pareto efficient since the owners of the steel mill and the fisherman could merge and make both owners better of.

Assume next that the government knows the true cost functions, $c_s(\cdot,\cdot)$ and $c_f(\cdot,\cdot)$, and they levi a tax $t \geq 0$ on the dischargement of chemicals and waste. The revenue is paid out to the fisherman as a compensation for the damage.

e) The profitmax of the steel mill

$$\max_{x,h} p_s x - th - c_s(h, x)$$

while the profitmax of the fisher is

$$\max_{f} p_f f + th - c_f(f, h)$$

f) The first order conditions of the steel mill

$$\begin{aligned} p_s - \frac{\partial c_s(x^t, h^t)}{\partial x} &= 0 \\ -t - \frac{\partial c_s(x^t, h^t)}{\partial h} &= 0 \end{aligned}$$

while the FOC of the fisherman is

$$\begin{aligned} p_f - \frac{\partial c_f(f^t, h^t)}{\partial f} &= 0 \\ t - \frac{\partial c_f(f^t, h^t)}{\partial h} &= 0 \end{aligned}$$

and they characterize the solution of x^t, h^t and f^t.

g) We chose t such that

$$t^{OP} = \frac{\partial c_f(f^{\star\star}, h^{\star\star})}{\partial h}$$

because then we have that

$$\frac{\partial c_s(x^t, h^t)}{\partial h} = -t = -\frac{\partial c_f(f^{\star\star}, h^{\star\star})}{\partial h}$$

so that $h^t = h^{\star\star}$.

Assume next that the government setup a market for polution permissions, and that they grant the steel mill the right to issue these polution permissions. Assume that they both firms know the other firm's cost function. The trade of polution permissions is conducted by means of bargaining between the steel mill and the fisherman. The bargining takes place as follows:

1. The steel mill propose a contract, (q, z) where the fisherman pays $q \geq 0$ per polution permission and he obtains z polution permissions

2. The fisherman then accept or rejects. If the fisherman accepts the transfer of money is carried through and the steel mill cannot polute more than z. If the fisherman rejects the proposal the steel mill can freely polute any amount.

h) The fisherman will accept any contracts (q, z) such that

$$p_f f - qz - c(f, z) \geq p_f f^\star - c(f^\star, h^\star)$$

i) If the steel mill offers any contract (q, z) such that $p_f f - qz - c(f, z) > p_f f^\star - c(f^\star, h^\star)$, then by offering $q - \epsilon < q$ the fisherman will accept this, and the profit of the steel mill increases.

j) Finding the SPE of the game we can solve the problem

$$\begin{gathered} \max_{x,q,z} p_s x + qz - c(x, z) \\ \text{s.t.} \\ p_f f - qz - c(f, z) = p_f f^\star - c(f^\star, h^\star) \end{gathered}$$

but then by FOC we obtain that

$$\begin{aligned} -\frac{\partial c_s}{\partial h} + \lambda \frac{\partial c_f}{\partial h} &= 0 \\ z + \lambda z &= 0 \end{aligned}$$

where $\lambda > 0$ is the lagrange multiplier of the problem such that

$$-\frac{\partial c_s}{\partial h} + \frac{\partial c_f}{\partial h} = 0$$

and we conclude that $z = h^{\star\star}$.

Exercise 2

Consider 2 firms. Firm 1 produces an output x which it sells in a competitive market. However, the production of x imposes a cost $e(x)$ on firm 2. The production of x output units costs firm 2 an amount $c(x)$. We assume that both $e(\cdot)$ and $c(\cdot)$ are contiuous differentiable, strictly increasing and convex. Let p be the price of the output of firm 1. Firm 2 obtains a profit gros polution of $\bar{\pi}_2$.

a) What is the profit of each of the firms

b) Find the FOC that maximizes firm 1's profit, and denote by x_q the optimal solution

Assume that the social planner wishes to maximize the total profit of the firms

c) State the problem of the social planner formally and determine the FOC, and denote x_e the efficient amount

d) Show that $x_q > x_e$

Assume that the social planner in order to obtain the efficient outcome introduce a tax, $t > 0$.

e) What is the profit of each of the firms

f) Find the FOC that maximizes firm 1's profit with taxes t

g) Find the level of t that implements the Pareto efficient amount x_e

Exercise 2 - suggested solution

a) The profit of firm 1 is

$$\pi_1 = px - c(x)$$

while the profit of firm 2 is

$$\pi_2 = \bar{\pi}_2 - e(x)$$

b) The FOC that maximizes firm 1's profit is

$$p - \frac{dc(x_q)}{dx} = 0$$

Assume that the social planner wishes to maximize the total profit of the firms

c) The problem of the social planner is to maximize

$$\pi_1 + \pi_2 = px - c(x) + \bar{\pi}_2 - e(x)$$

by choosing x such that the FOC in x_e, the efficient amount, is

$$p - \frac{dc(x_e)}{dx} - \frac{de(x_e)}{dx} = 0$$

d) According to b) and c) we have that

$$p - \frac{dc(x_e)}{dx} = \frac{de(x_e)}{dx} > 0 = p - \frac{dc(x_q)}{dx}$$

such that $x_q > x_e$ since $c(\cdot)$ is convex.

Assume that the social planner in order to obtain the efficient outcome introduce a tax, $t > 0$.

e) The profit of firm 1 is

$$\pi_1 = px - tx - c(x)$$

while the profit of firm 2 is

$$\pi_2 = \bar{\pi}_2 - e(x)$$

f) The FOC that maximizes firm 1's profit with taxes t is

$$p - t - \frac{dc}{dx} = 0$$

g) By letting t be equal to $t^{OP} = \frac{de(x_e)}{dx}$ one easily sees that this implements the Pareto efficient amount x_e.