Exercises 4 - MikØk2

Tobias Markeprand

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Exercise 1

Consider a monopolist who produces to a market with invers demand function

$$P(q) = a - q.$$

The monopolist does not know the parameter $a > 0$, but knows that it takes the values $\{5, 15\}$ with probability of $a = 5$ being $\frac{1}{2}$. The monopolist has constant marginal costs, $c = 1$.

a) Formulate the profit maximization problem when a is unknown.

b) Solve the profit maximization problem when a is unknown.

c) What is the maximal expected profit?

d) What is the optimal strategy when $a = 5$ and the monopolist observes this? resp. when $a = 15$? Find the corresponding profit levels.

e) If the monopolist could observe a before choosing q, what is then the expected profit?

f) What is the "informations cost" for the monopolist?

Exercise 1 - solution

a) The expected profit is

$$\mathrm{E}[\Pi] = \mathrm{E}[P(q)q - cq] = \frac{1}{2}(5 - q - 1)q + \frac{1}{2}(15 - q - 1)q$$

and thus the profitmaximization problem is to solve

$$\max_q / \frac{1}{2}(5 - q - 1)q + \frac{1}{2}(15 - q - 1)q$$

b) To solve the profit maximization problem we obtain the first order condition

$$18 - 4q^\star = 0$$

and hence $q^\star = \frac{9}{2}$.

c) The maximal expected profit is then $\mathrm{E}[\Pi^\star] = \frac{1}{2}(-\frac{9}{4}) + \frac{1}{2}(\frac{189}{4}) = \frac{45}{2} = 22.5$.

d) The optimal strategy when $a = 5$ and the monopolist observes this is to maximize

$$\max_{q_l} /(4 - q_l)q_l$$

which gives us $q_l = 2$ and the profit is $\Pi_l = 4$. When $a = 15$ the monopolist maximize

$$\max_{q_h} /(14 - q_h)q_h$$

which gives us $q_h = 7$ and the profit is $\Pi_h = 49$.

e) If the monopolist could observe a before choosing q, then the expected profit is

$$\mathrm{E}[\Pi \mid a] = \frac{1}{2}\Pi_l + \frac{1}{2}\Pi_h = 28$$

f) The "informations cost" for the monopolist is the cost he must obtain by not being able to observe the true demand, and thus setting an uniform quantity, and this is the loss in profit: $\mathrm{E}[\Pi] - \mathrm{E}[\Pi \mid a] = 5.5$.

Exercise 2

Consider 2 firms, firm 1 og 2, who are the only producers of a commodity with the invers demand function

$$P(q) = a - bQ$$

where Q is the total quantity of the commodity, and $a, b > 0$ are constants. Firm 1 has either marginal costs $c_H > c_L$, while firm 2's marginal costs c are publicly known. The firms compete ala Cournot, hence their strategic variable is the produced quantity.

a) Find the Nash equilibrium if $c = c_H$ and firm 2 knows this.

b) Find the Nash equilibirum if $c = c_L$ and firm 2 knows this.

Assume now that Firm 2 does not know the marginal cost c_1 of firm 1 but believs that firm 1 has $c = c_H$ with probability $\pi > 0$, and firm 1 knows this.

c) Formulate this as a Bayesian game

d) Show that in a Bayesian Nash equilibrium the strategies solves the equations

$$\begin{aligned} a - c_L - 2bq_1(c_L) - bq_2 &= 0 \\ a - c_H - 2bq_1(c_H) - bq_2 &= 0 \\ a - c - 2b(\pi q_1(c_L) + (1-\pi)q_1(c_H)) - bq_2 &= 0 \end{aligned}$$

e) Find the Bayesians Nash equilibrium in pure strategies.

Exercise 2 - solution

a) Find the Nash equilibrium if $c = c_H$ and firm 2 knows this. Firm 1 maximizes

$$\max_{q_1}(P(q_1 + q_2) - c_H)q_1$$

while firm 2 maximizes

$$\max_{q_2}(P(q_1 + q_2) - c)q_2$$

and obtainning the first order conditions

$$\begin{aligned} 2bq_1 + bq_2 &= a - c_H \\ 2bq_2 + bq_1 &= a - c \end{aligned}$$

and solving this we obtain

$$\begin{aligned} q_1 &= \frac{2(a - c_H)}{3b} - \frac{a - c}{3b} \\ q_2 &= \frac{2(a - c)}{3b} - \frac{a - c_H}{3b} \end{aligned}$$

b) Find the Nash equilibirum if $c = c_L$ and firm 2 knows this. Firm 1 maximizes

$$\max_{q_1}(P(q_1 + q_2) - c_L)q_1$$

while firm 2 maximizes

$$\max_{q_2}(P(q_1 + q_2) - c)q_2$$

and obtainning the first order conditions

$$\begin{aligned} 2bq_1 + bq_2 &= a - c_L \\ 2bq_2 + bq_1 &= a - c \end{aligned}$$

and solving this we obtain

$$\begin{aligned} q_1 &= \frac{2(a - c_L)}{3b} - \frac{a - c}{3b} \\ q_2 &= \frac{2(a - c)}{3b} - \frac{a - c_L}{3b} \end{aligned}$$

Assume now that Firm 2 does not know the marginal cost c_1 of firm 1 but believs that firm 1 has $c = c_H$ with probability $\pi > 0$, and firm 1 knows this.

c) Formulate this as a Bayesian game. We have that $N = \{1, 2\}$ is the set of players, the set of types is $\Theta = \{c_L, c_H\}$, the probabilities are $p(c_H) = \pi$

and $p(c_L) = 1 - \pi$, $S_i = \{q_i \geq 0\}$ is the strategy set of player i and the payoff functions are

$$\begin{aligned} u_1(q_1, q_2, c_L) &= (P(q_1 + q_2) - c_L)q_1 \\ u_1(q_1, q_2, c_H) &= (P(q_1 + q_2) - c_H)q_1 \\ u_2(q_1, q_2, c) &= (P(q_1 + q_2) - c)q_2 \end{aligned}$$

d) To find the Bayesian Nash equilibrium the set of strategies is $S_1 = \{(q_1^H, q_1^L) \geq 0\}$ while for player 2 it is $S_2 = \{q_2 \geq 0\}$. Then we can define the conditional, expected payoff

$$\begin{aligned} \tilde{u}_1(q_1, q_2, c_L) &= (P(q_1^L + q_2) - c_L)q_1^L \\ \tilde{u}_1(q_1, q_2, c_H) &= (P(q_1^H + q_2) - c_H)q_1^H \\ \tilde{u}_2(q_1, q_2, c) &= \mathrm{E}[(P(q_1 + q_2) - c)q_2] = (a - (\mathrm{E}[q_1] + q_2) - c)q_2 = (a - (\pi q_1^H + (1-\pi)q_1^H + q_2) - c)q_2 \end{aligned}$$

maximizing these with respect to q_1^H, q_1^L and q_2 respectively yields us the first order conditions

$$\begin{aligned} a - c_L - 2bq_1(c_L) - bq_2 &= 0 \\ a - c_H - 2bq_1(c_H) - bq_2 &= 0 \\ a - c - 2b(\pi q_1(c_L) + (1-\pi)q_1(c_H)) - bq_2 &= 0 \end{aligned}$$

e) Find the Bayesians Nash equilibrium in pure strategies. We find it by solving the first order conditions above and we obtain

$$\begin{aligned} q_1^L &= \frac{a - 2c_L + c}{3b} - \frac{\pi}{6b}(c_H - c_L) \\ q_1^H &= \frac{a - 2c_H + c}{3b} + \frac{1-\pi}{6b}(c_H - c_L) \\ q_2 &= \frac{a - 2c + \pi c_H + (1-\pi)c_L}{3b} \end{aligned}$$

Exercise 3

Consider 2 firms, firm 1 and 2, who are the only producers of a commodity with the invers demand function

$$P(q) = a - bQ$$

where Q is the total quantity of the commodity, $a, b > 0$ are constants. The firms have identical, constant marginal costs $c > 0$. The firms compete ala Cournot, hence their strategic variable is the produced quantity. Assume that they compete in T periods which are mere repeatitions of the static game. The firms have a common discount factor $\delta < 1$.

a) Find the Nash equilibrium in the stage-game and the corresponding stage-payoff, denote it by π^C.

b) What is the maximal total stage-payoff the firms could obtain? Denote it by π^M.

c) Let $T < \infty$. Show that $\pi = \pi^C$ is the unique SPE in this game. (*Hint: Use backward induction*)

d) Let $T = \infty$. Show that $\pi = \frac{\pi^M}{2}$ is possible to obtain in a SPE when δ is sufficiently close to 1.

e) Find a treshold for δ in question d). How does this treshold compare to the case of price-competition? Explain the difference.

Exercise 3 - suggested solution

a) Find the Nash equilibrium in the stage-game and the corresponding stage-payoff, denote it by π^C.

It is easy to find that $q_1^C = q_2^C = \frac{a-c}{3b} = q^C$ and the profit is $\pi_1^C = \pi_2^C = \pi^C = \frac{(a-c)^2}{9b}$.

b) What is the maximal total stage-payoff the firms could obtain? Denote it by π^M.

The maximal profit in total the firms can achieve is the monopoly profit, i.e., with $Q = Q^M = \frac{a-c}{2b}$ and the total profit is $\pi^M = \frac{(a-c)^2}{4b}$.

c) Let $T < \infty$. Show that $\pi = \pi^C$ is the unique SPE in this game. (*Hint: Use backward induction*)

In the stage T we have a unique Nash equilibrium: both plays q^C and obtains a profit π^C. Given that this is the outcome at stage T, at stage $T-1$ the maximal payoff firm 1 can obtain is

$$\max_{q_1} \delta(P(q_1 + q_2^C) - c)q_1 + \delta^2 \pi^C$$

which is exactly $q_1 = q^C$ by definition. Continuinning in this manner we have that $q_1 = q^C$ is the unique SPE in each stage.

d) Let $T = \infty$. Show that $\pi = \frac{\pi^M}{2}$ is possible to obtain in a SPE when δ is sufficiently close to 1.

We suggest the use of Trigger-strategies: for firm 1 the strategy is then for each stage t and history ξ^{t-1}

$$s_1^t(\xi^{t-1}) = \begin{cases} \frac{q^M}{2} & t = 1 \wedge \nexists \xi_i^s \neq \frac{q^M}{2} \\ q_1^C & \text{else} \end{cases}$$

We next consider a deviation by firm 1 at stage t: the best deviation of player 1 is the solution to the problem

$$\max_{q_1} (P(q_1 + \frac{q^M}{2}) - c)q_1$$

which is $\tilde{q}_1 = \frac{3}{8b}(a-c)$ and the stage profit is then $\tilde{\pi}_1 = \frac{9(a-c)^2}{64b}$ of firm 1 and $\tilde{\pi}_2 = \frac{3(a-c)^2}{32b}$ of firm 2. Then the payoff from deviating of firm 1 is thus

$$\tilde{U}_1 = \tilde{\pi}_1 + \sum_t^{\infty} \delta^t \pi^C = \tilde{\pi}_1 + \frac{\delta}{1-\delta}\pi^C$$

while the payoff from not deviating is

$$U_1^\star = \sum_t^{\infty} \delta^t \frac{\pi^M}{2} = \frac{\pi^M}{2(1-\delta)}$$

It is thus a SPE whenever it holds that $U_1^\star \geq \tilde{U}_1$ or

$$\frac{\pi^M}{2(1-\delta)} \geq \tilde{\pi}_1 + \frac{\delta}{1-\delta}\pi^C \Leftrightarrow \frac{\pi^M}{2} \geq (1-\delta)\tilde{\pi}_1 + \delta\pi^C$$

But as $\delta \to 1$ the righthand side of the equation above approches $\pi^C < \frac{\pi^M}{2}$ and thus for some $\underline{\delta} < 1$ for any $\delta \geq \underline{\delta}$ we have that $U_1^\star \geq \tilde{U}_1$.

e) Find a treshold for δ in question d). How does this treshold compare to the case of price-competition? Explain the difference.

By inserting the expressions of the profit into $\frac{\pi^M}{2} \geq (1-\underline{\delta})\tilde{\pi}_1 + \underline{\delta}\pi^C$ we obtain

$$\frac{(a-c)^2}{8b} = (1-\underline{\delta})\frac{9(a-c)^2}{64b} + \underline{\delta}\frac{(a-c)^2}{9b}$$

and solving this we obtain $\underline{\delta} = \frac{9}{17}$. We note that this threshold is higher than in the case of Bertrand competition. We reason is that the punishment in the Cournot competition is not as harsh as in the case of Bertrand. Recall that in Bertrand, a deviation implies that in all future the profit is $\pi^B = 0$, while here the punishment is $\pi^C > 0$. This implies that the discount rate needs to be higher in order to compensate for the weaker punishment in stage payoffs.