Exercises 3, suggested solutions - MikØk2

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1 Repeated games

Exercise 1

Consider the following stage game

	O	F
O	3,3	0,4
F	4,0	1,1

Let $\frac{1}{2} < \delta < 1$ be the discount rate.

a) Draw the payoffs in a diagram with u_1 as the horisontal axis and u_2 as the vertical axis

b) What is the minimax payoff of player 1? of player 2?

c) What are the feasible average payoffs? What are the feasible rational payoffs?

d) Suggest a strategy pairs such that $(3, 3)$ is the average payoff of a SPE of the infinitely repeated game. Show that it is actually a SPE.

Assume that the players use the following strategies:

player 1 : start play O. In stage t, t odd, if the immediate predecessor is (F, O) then play O and else play F infinitely. In stage t, t even, if the immediate predecessor is (O, O) then play F and else play F infinitely.

player 2 : start play O. In stage t, t odd, if the immediate predecessor is (F, O) then play O and else play F infinitely. In stage t, t even, if the immediate predecessor is (O, O) then play O and else play F infinitely.

e) What is the discounted payoff to each player of the outcome

$$(O, O), (F, O), (O, O), (F, O), \ldots ?$$

f) What is the discounted payoff to each player of the outcome

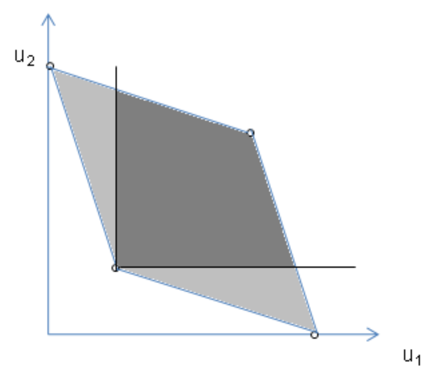
$$(F, O), (O, O), (F, O), (O, O), \ldots ?$$

g) Show that this is a SPE of the game when δ is sufficiently close to 1. Find an approximate value of the $\underline{\delta} < 1$ such that this is a SPE. What are the average payoffs of each player in this SPE?

Exercise 1 - suggested solution

- Draw the payoffs in a diagram with u_1 as the horisontal axis and u_2 as the vertical axis

  This is shown in figure X



b) What is the minimax payoff of player 1? of player 2?

We have that the minimax payoff for both players are $\pi_1 = \pi_2 = 1$, and it is the payoff the player can be guaranteed regardless of the other players' strategy.

c) What are the feasible average payoffs? What are the feasible rational payoffs?

The set of feasible average payoffs is the convex hull of the set of payoffs of the stage game, and it is illustrated in the figure above with grey. The set of rational payoffs is the set of feasible payoffs with payoffs greater than the minimax payoffs, they are also illustrated in the figure, with dark grey.

d) Suggest a strategy pairs such that $(3,3)$ is the average payoff of a SPE of the infinitely repeated game. Show that it is actually a SPE.

We could consider a pair of trigger strategies: for player i we have that

$$s_i^t\left(\xi^{t-1}\right) = \begin{cases} O & t = 1 \vee \exists s \leq t-1, i : \xi_i^s = F \\ F & else \end{cases}$$

then the payoff from the outcome $(O,O), (O,O), \ldots$ is

$$U_i = \sum_{t=1}^{\infty} 3\delta^t = \frac{\delta}{1-\delta}3$$

Could player 1 gain by deviating at time $t = 1$? The best deviation is playing F but then the outcome is $(F,O), (F,F), \ldots$ and the payoff is

$$\tilde{U}_i = \delta 4 + \sum_{t=2}^{\infty} \delta^t = \delta 4 + \frac{\delta^2}{1-\delta}$$

and thus $U_i \geq \tilde{U}_i$ iff

$$\frac{\delta}{1-\delta}3 \geq \delta 4 + \frac{\delta^2}{1-\delta} \Leftrightarrow \delta \geq \frac{1}{3}$$

Assume that the players use the following strategies:

player 1 : start play O. In stage t, t odd, if the immediate predecessor is (F,O) then play O and else play F infinitely. In stage t, t even, if the immediate predecessor is (O,O) then play F and else play F infinitely.

player 2 : start play O. In stage t, t odd, if the immediate predecessor is (F,O) then play O and else play F infinitely. In stage t, t even, if the immediate predecessor is (O,O) then play O and else play F infinitely.

e) What is the discounted payoff to each player of the outcome

$$(O,O),(F,O),(O,O),(F,O),\ldots?$$

We have that

$$\begin{aligned} U_1^o &= \delta 3+\delta^2 4+\delta^3 3+\cdots = \frac{\delta}{1-\delta^2}(3+\delta 4) \\ U_2^o &= \delta 3+\delta^2 0+\delta^3 3+\cdots = \frac{\delta}{1-\delta^2}3 \end{aligned}$$

where we have used the fact that any sum of the form $s=\delta a+\delta^2 b+\delta^3 a+\ldots$ is convergent and the sum equal to $s=\frac{\delta}{1-\delta^2}(a+\delta b)$.

f) What is the discounted payoff to each player of the outcome

$$(F,O),(O,O),(F,O),(O,O),\ldots?$$

Again we obtain the payoffs

$$\begin{aligned} U_1^e &= \delta 4+\delta^2 3+\delta^3 4+\cdots = \frac{\delta}{1-\delta^2}(4+\delta 3) \\ U_2^e &= \delta 0+\delta^2 3+\delta^3 0+\cdots = \frac{\delta^2}{1-\delta^2}3 \end{aligned}$$

g) Show that this is a SPE of the game when δ is sufficiently close to 1. Find an approximate value of the $\underline{\delta}<1$ such that this is a SPE. What are the average payoffs of each player in this SPE?

We use the one-stage deviation principle. Thus, we must show that for any stage t it is not beneficial by any player to deviate. Note that given the specified strategies the outcome of e) is the outcome of the subgame of any t odd, while the outcome of f) is the outcome of any subgame when t is even. First we note that if it is not beneficial to deviate by player 2 in the outcome of f) it will not be beneficial to deviate in outcome e), and also it is never optimal by player 1 to deviate since the discounted payoff is larger than in d). Thus, we only need to tjeck if player 2 has an incentive to deviate. But the best deviation in t even is to play F but then according to the strategies we obtain the outcome $(F,F),(F,F),\ldots$ but then the payoff to player 2 is

$$\tilde{U}_2=\sum\delta^t=\frac{\delta}{1-\delta}$$

and we have then that

$$U_2^e - \tilde{U}_2 = \frac{\delta^2}{1-\delta^2}3 - \frac{\delta}{1-\delta}$$

But since

$$\lim_{\delta \to 1} \frac{\frac{\delta^2}{1-\delta^2}}{\frac{\delta}{1-\delta}} = \lim_{\delta \to 1} \frac{\delta (1-\delta)}{1-\delta^2} = \lim_{\delta \to 1} \frac{2\delta - 1}{2\delta} = \frac{1}{2}$$

we have that for some δ close to 1 it must hold that $U_2^e \geq \tilde{U}_2$. Thus, a deviation is not beneficial. We obtain an approximation of $\bar{\delta}$ by solving $\frac{\delta^2}{1-\delta^2}3 - \frac{\delta}{1-\delta} = 0$, which gives us $\bar{\delta} = \frac{1}{2}$.

Exercise 2

Assume that there are 2 players, A and B. Consider the following game: They have to divide a dollar and they bargain using the following rules

- First, player A propose a division
- Second, player B is told the proposition, and can then accept or reject the proposition
- If B accepts the proposition the division is distributed to each and the payoff is the share. If B rejects neither gets a penny, and their payoff is zero.

There is no discounting of payoffs.

a) Write this as an Extensive form game

b) Show that any division is a Nash equilibrium of this game

c) Find the unique SPE equilibrium of the game

Exercise 3

Let there be 2 depositors each with a deposit of $D > 0$ in a bank. Then bank has invested the deposit of both depositors in a project which can be liquidated in the short- or long-run. If the project is liquidated in the short run the project is worth $2r$, while a liquidation in the long run yields a dividend of $2R$. We assume that $r < D < R$. The game proceed as follows:

stage 1:
- if both withdraw their deposit, the bank must liquidate and both gets a payoff of r, and the game ends
- if one withdraw her deposit, but the second keeps the deposit, the bank must liquidate and the withdrawer gets D, and the other gets $2r - D$, and the game ends
- if neither withdraws, the game proceed to the next stage

stage 2
- if both withdraw their deposit, the bank must liquidate and both gets a payoff of R, and the game ends
- if one withdraw her deposit, but the second keeps the deposit, the bank must liquidate and the withdrawer gets D, and the other gets $2R - D$, and the game ends
- if neither withdraws, their deposit is now R each

a) Find the bimatrix of the stage 2 game

b) What is the Nash equilibrium of the stage 2 game? What is the payoff in this outcome?

c) Find the bimatrix of the stage 1 game, in which the payoff of the action profile $(nowithdraw, nowithdraw)$ is the Nash equilibrium payoff of stage 2

d) Find the SPE of this game

e) If we interprete an early withdraw as a bank run, what does the model tells about the rationality of bank runs?

Exercise 3 - suggested solution

a) Find the bimatrix of the stage 2 game

We have that the bimatrix is given by

	W	N
W	(R, R)	$(D, 2R - D)$
N	$(2R - D, D)$	(R, R)

b) What is the Nash equilibrium of the stage 2 game? What is the payoff in this outcome?

We have the unique Nash equilibrium is (N, N) since $2R - D > R$ and the payoff of this outcome is (R, R).

c) Find the bimatrix of the stage 1 game, in which the payoff of the action profile $(nowithdraw, nowithdraw)$ is the Nash equilibrium payoff of stage 2

We obtain the bimatrix is givne by

	W	N
W	(r, r)	$(D, 2r - D)$
N	$(2r - D, D)$	(R, R)

d) Find the SPE of this game

We see that the Nash equilibrium of the bimatrix of stage 1 in c) is (W, W), since $r > 2r - D$. Note also that the strategy (N, N) is a Nash equilibrium. Thus, we have two Nash equilibria. That these are actually SPE of the dynamic game follows since the outcome have been found by using backwards induction. This also implies that there are two subgame perfect equilibria.

e) If we interprete an early withdraw as a bank run, what does the model tells about the rationality of bank runs?

We note that a SPE of the game involves the depositors withdrawing in the first stage. But then it is rational for having bank runs.

Exercise 4

Consider 2 firms, firm 1 and 2, who are the only producers of a commodity with the invers demand function

$$P(q) = a - bQ$$

where Q is the total quantity of the commodity, $a, b > 0$ are constants. The firms have identical, constant marginal costs $c > 0$. The firms compete ala Cournot, hence their strategic variable is the produced quantity. Assume that the firms can only produce in a few quantities, either the competitive q_i^c, the Cournot $q_i^\star$ or half the monopoly quantity $\frac{q^m}{2}$.

a) What is the normal form game and the bimatrix of this game?

b) Find the Nash equilibrium of the game

Exercise 4 - suggested solution

a) What is the normal form game and the bimatrix of this game?

We have that $N = \{1, 2\}$, $S_i = \{q_i^c, q_i^\star, \frac{q^m}{2}\}$ and using the results from MWG we obtain that these are actually given by

$$S_i = \left\{ \frac{a-c}{2b}, \frac{a-c}{3b}, \frac{a-c}{4b} \right\}$$

and the corresponding payoffs are given in the bimatrix

	q_2^c	$q_2^\star$	$\frac{q^m}{2}$
q_1^c	$(0,0)$	$\left(\frac{(a-c)^2}{12b}, \frac{(a-c)^2}{18b}\right)$	$\left(\frac{3(a-c)^2}{8b}, \frac{(a-c)^2}{16b}\right)$
$q_1^\star$	$\left(\frac{(a-c)^2}{18b}, \frac{(a-c)^2}{12b}\right)$	$\left(\frac{(a-c)^2}{9b}, \frac{(a-c)^2}{9b}\right)$	$\left(\frac{5(a-c)^2}{36b}, \frac{5(a-c)^2}{48b}\right)$
$\frac{q^m}{2}$	$\left(\frac{(a-c)^2}{16b}, \frac{3(a-c)^2}{8b}\right)$	$\left(\frac{5(a-c)^2}{48b}, \frac{5(a-c)^2}{36b}\right)$	$\left(\frac{(a-c)^2}{8b}, \frac{(a-c)^2}{8b}\right)$

b) Find the Nash equilibrium of the game

Note that by rescaling the payoffs the bimatrix is then

	q_2^c	$q_2^\star$	$\frac{q^m}{2}$
q_1^c	$(0,0)$	$\left(\frac{1}{12}, \frac{1}{18}\right)$	$\left(\frac{3}{8}, \frac{1}{16}\right)$
$q_1^\star$	$\left(\frac{1}{18}, \frac{1}{12}\right)$	$\left(\frac{1}{9}, \frac{1}{9}\right)$	$\left(\frac{5}{36}, \frac{5}{48}\right)$
$\frac{q^m}{2}$	$\left(\frac{1}{16}, \frac{3}{8}\right)$	$\left(\frac{5}{48}, \frac{5}{36}\right)$	$\left(\frac{1}{8}, \frac{1}{8}\right)$

$\approx$

	q_2^c	$q_2^\star$	$\frac{q^m}{2}$
q_1^c	$(0,0)$	$(0.08, 0.05)$	$(0.375, 0.065)$
$q_1^\star$	$(0.05, 0.08)$	$(0.11, 0.11)$	$(0.138, 0.104)$
$\frac{q^m}{2}$	$(0.065, 0.375)$	$(0.104, 0.138)$	$(0.125, 0.125)$

We have that the best responds of each player to

- q^c_{-1} is $\frac{q^m}{2}$
- $q^\star_{-i}$ is $q_i^\star$
- $\frac{q^m}{2}$ is q_i^c

Thus, there are three Nash equilibria. $NE = \{(q_1^c, \frac{q^m}{2}), (\frac{q^m}{2}, q_2^c), (q_1^\star, q_2^\star)\}$.