Exercises 2, suggested solutions - MikØk2

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1 Normal form games

Exercise 1

Consider the following situation:

There are 2 firms, an incumbent, player 1, and a potential entrant, player 2. The incumbent can choose to build a new factory "B" or not "D", while the potential entrant kan enter, "E", or remain passive, "D". The incumbent's construction costs, c, of building the new factory can either be high, c_H, or low, c_L, i.e. $c_H > c_L$.

If the construction costs are high the payoff will be as follows:

	E	D
B	0,-1	2,0
D	2,1	3,0

,

while low construction costs will imply that the payoffs are as follows:

	E	D
B	3,-1	5,0
D	2,1	3,0

.

a) What is the Nash equilibrium if $c = c_H$?

b) What is the Nash equilibrium if $c = c_L$?

Assume that only the incumbent knows his construction costs. The potential entrant believs that with p_1 the construction cost is high, $c = c_H$.

c) Formulate this as a Bayesian game

d) Find the Bayesian Nash equilibrium for $p_1 < \frac{1}{2}$

e) Find the Bayesian Nash equilibrium for $p_1 > \frac{1}{2}$

Exercise 1 - suggested solution

a) What is the Nash equilibrium if $c = c_H$?

It is easy to see that, B is a strictly dominated strategy of player 1 and given that firm 1 plays D it is optimal for firm 2 to play E. Hence the unique Nash equilibrium of the game of $c = c_H$ is for firm 1 ´´not to build" and firm 2 to ´´enter".

b) What is the Nash equilibrium if $c = c_L$?

Again, it is easy to see that, D is a strictly dominated strategy of player 1 and given that firm 1 plays B it is optimal for firm 2 to play D. Hence the unique Nash equilibrium of the game of $c = c_L$ is for firm 1 ´´to build" and firm 2 to ´´remain passive".

Assume that only the incumbent knows his construction costs. The potential entrant believs that with p_1 the construction cost is high, $c = c_H$.

c) Formulate this as a Bayesian game

A Bayesian game is defined by $(N, \Theta, p, (S_i, u_i)_{i \in N})$ and we have that

$$\begin{aligned} N &= \{1, 2\} \\ \Theta &= \Theta_1 \times \Theta_2 = \{\theta_1^a, \theta_1^b\} \times \{\theta_2\} = \{c_H, c_L\} \times \{\theta\} \\ p(c_H, \theta) &= p_1 \\ S_1 &= \{B, D\} \\ S_2 &= \{E, D\} \end{aligned}$$

and the payoffs are given in the bimatrix, ex $u_1(B, D, c_H) = 2$ and $u_1(B, D, c_L) = 5$.

d) Find the Bayesian Nash equilibrium for $p_1 < \frac{1}{2}$

As was shown during the lectures it is possible to define this in two equivalent ways. We shall consider the definition of ´´type-players": There are 3 players $\tilde{N} = \{(1, c_H), (1, c_L), (2, \theta)\}$ with strategy sets

$$\begin{aligned} S_{(1,c_H)} &= S_{(1,c_L)} = \{B, D\} \\ S_{(2,\theta)} &= \{E, D\} \end{aligned}$$

The expected payoff of player $(2, \theta)$ of playing E depends on the strategy of $(1, c_H)$ and $(1, c_L)$, we see that

$$\tilde{u}_2(E, BB, \theta) = p_1(-1) + (1 - p_1)(-1) = -1$$

where $(1, c_H)$ plays B and $(1, c_L)$ plays B, and

$$\tilde{u}_2(E, BD, \theta) = p_1(-1) + (1 - p_1)(1) = 1 - 2p_1$$

where $(1, c_H)$ plays B and $(1, c_L)$ plays D. More schematically, we obtain

	BB	BD	DB	DD
E	-1	$1 - 2p_1$	$2p_1 - 1$	1
D	0	0	0	0

However, firm 2 knows that $(1, c_H)$ will never play B, and he knows that $(1, c_L)$ will never play D, which then leaves firm 1 to choose between a payoff of $2p_1 - 1$ if he plays E and 0 if he plays D. Thus, whenever $p_1 < \frac{1}{2}$ the BNE is (DB, D).

e) Find the Bayesian Nash equilibrium for $p_1 > \frac{1}{2}$

Utilizing the scheme of question d) we obtain a BNE (DB, E), using the argumentation.

Exercise 2

Consider the following situation - called a "first-price, sealed bid auction":

There are 2 bidders, with a valuation v_i of bidder $i = 1, 2$ of a good. The good is indivisible and the supply is a single unit. The two bidders' valuation are independently, uniformly distributed on the interval $[0, 1]$. Each bidder only knows his own value of the good. If the bidder i obtains the good and pays a price of p, the value to the bidder is $v_i - p$. If he does not obtain the good the value is 0. The rules of the game is as follows: Each bidder simultanously submit a bid. The highest bidder is granted the good and pays the bid. If they bid the same the good is randomly allocated between the two, i.e., there is a probability of $\frac{1}{2}$ of each bidder getting the good.

a) Formulate this situation as a Bayesian game

b) Solve for a Bayesian Nash equilibrium (Hint: Assume that the strategy is of the affin form $b_i(v_i) = a_i + b_i v_i$. Solve then for a_i and b_i)

c) Do the bidders bid their value of the good?

Exercise 2 - suggested solution

a) Formulate this situation as a Bayesian game

A Bayesian game is defined by $(N, \Theta, p, (S_i, u_i)_{i\in N})$ and we have that

$$\begin{aligned} N &= \{1,2\} \\ \Theta &= \Theta_1 \times \Theta_2 = [0,1] \times [0,1] \\ p(v_1, v_2) &= 1 \\ S_1 &= \{b_1 \geq 0\} = \mathbb{R}_+ \\ S_2 &= \{b_2 \geq 0\} = \mathbb{R}_+ \end{aligned}$$

where p is the density function of the prior distribution on Θ. Denote by $P(v_1, v_2)$ the distribution function, and we have that $P(\bar{v}_1, \bar{v}_2) = \text{Prob}(v_1 \leq \bar{v}_1, v_2 \leq \bar{v}_2) = v_1 v_2$. The payoff function of 1 is

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1 & b_1 > b_2 \\ \frac{1}{2}(v_1 - b_1) & b_1 = b_2 \\ 0 & b_1 < b_2 \end{cases}$$

and symmetrically for player 2.

b) Solve for a Bayesian Nash equilibrium (Hint: Assume that the strategy is of the affin form $b_i(v_i) = \alpha_i + \beta_i v_i$. Solve then for α_i and β_i - I have substituted a_i by α_i and b_i by β_i to avoid notational confusion)

The payoff function for valuation v_i given a strategy $b_{-i} \colon [0,1] \to \mathbb{R}$ is given by

$$\tilde{u}_i(b_i, b_{-i}, v_i) = \text{Prob}\{b_i > b_{-i} \mid v_i\}(v_i - b_i) + \text{Prob}\{b_i = b_{-i} \mid v_i\}\frac{1}{2}(v_i - b_i)$$

But we have that

$$\text{Prob}\{b_i > b_{-i} \mid v_i\} = \text{Prob}\{b_i > \alpha_{-i}+\beta_{-i}v_{-i} \mid v_i\} = \int_0^{\frac{b_i-\alpha_{-i}}{\beta_{-i}}} p(v_{-i} \mid v_i)\, dv_{-i}$$

but we have that $p(v_{-i} \mid v_i) = \frac{p(v_i,v_{-i})}{\int_0^1 p(v_i,v_{-i})\ dv_{-i}} = \frac{1}{1} = 1$ and inserting this into the equation we obtain that

$$\tilde{u}_i(b_i, b_{-i}, v_i) = \text{Prob}\{b_i > b_{-i} \mid v_i\}(v_i - b_i) = (\frac{b_i - \alpha_{-i}}{\beta_{-i}})(v_i - b_i)$$

since $\text{Prob}\{b_i > b_{-i} \mid v_i\} = \text{Prob}\{b_i \geq b_{-i} \mid v_i\}$ and $\text{Prob}\{b_i = b_{-i} \mid v_i\} = 0$. We note that the following inequality holds:

$$\alpha_{-i} \leq b_i \leq \alpha_{-i} + \beta_{-i},$$

since it will never be optimal to bid below α_{-i}, which is the lowest bid of the other player, nor is it optimal to bid above $\alpha_{-i} + \beta_{-i}$, which is the highest bid of the other player. Maximizing this with respect to b_i gives us the first order condition

$$\frac{v_i - b_i}{\beta_{-i}} - \frac{b_i - \alpha_{-i}}{\beta_{-i}} = 0.$$

But then we have that $b_i = \frac{1}{2}(v_i + \alpha_{-i})$, and imposing the boundary condition of $b_i \geq \alpha_{-i}$ we have that the best respond of player $(1, v_i)$ is

$$b_i(v_i) = \begin{cases} \frac{1}{2}(v_i + \alpha_{-i}) & v_i \geq \alpha_{-i} \\ \alpha_{-i} & v_i < \alpha_{-i} \end{cases}$$

Since this holds for both players we have that

$$b_1(v_1) = \begin{cases} \frac{1}{2}(v_1 + \alpha_2) & v_1 \geq \alpha_2 \\ \alpha_2 & v_1 < \alpha_2 \end{cases}$$

$$b_2(v_2) = \begin{cases} \frac{1}{2}(v_2 + \alpha_1) & v_2 \geq \alpha_1 \\ \alpha_1 & v_2 < \alpha_1 \end{cases}$$

But since we have assumed that the strategies $b_i(v_i)$ are affin, this only holds if $\alpha_1 = \alpha_2 = 0$. But then the equilibrium strategy of player i is $b_i^\star(v_i) = \frac{v_i}{2}$.

c) Do the bidders bid their valuation of the good?

We note that $b_i^\star(v_i) < v_i$ for every v_i, and hence the bidders will always bid less than their valuation.

2 Extensive form games

Exercise 3

Consider the following description of a game - called “The truth game”:

There are 2 players, player 1 and 2, and a gamemaster. The gamemaster has a coin that is bent such that, flipped randomly, the coin will come up with “heads” 80% of the time. Both players know this. The game-master flips the coin and shows the result to player 1. Player 1 makes an announcement to player 2 what the result of the flip is, either “head” or “tail”. Player 2 having heard player 1’s announcement but not seen the result must guess what the result of the flip was. Player 1 has the following payoffs: if player 2 guess “head” payoff is 2 and payoff is 0 if the guess is “tail”. Moreover, if player 1’s announcement is correct player 1’s payoff is increased by 1. For player 2 the payoff is 1 if he guesses right and 0 if his guess is wrong.

a) Formulate this game as a game on extensivform. What is the game tree.

b) What is the normalform representation of this game?

c) Find all Nash equilibria of this game

Exercise 3 - suggested solution

a) Formulate this game as a game on extensivform. What is the game tree.

The extensive form of this game $(N, A, H, p, \alpha, \iota, I, \rho, (u_i)_{i\in N})$ is given as follows

$$\begin{aligned} N &= \{0,1,2\} \\ A &= \{H'', L'', H', L', H, L\} \\ H &= \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\} \end{aligned}$$

are the sets, then

$p(x_0) = \emptyset / p(x_1) = x_0 = p(x_2) / p(x_3) = p(x_4) = x_1 / p(x_5) = p(x_6) = x_2 / \ldots$

is the predecessor function,

$$\alpha(x_1) = H'' / \alpha(x_2) = L'' / \ldots$$

is the action function,

$$\iota(x_0) = 0/\iota(x_1) = \iota(x_2) = 1/\iota(x_3) = \iota(x_4) = \iota(x_5) = \iota(x_6) = 2$$

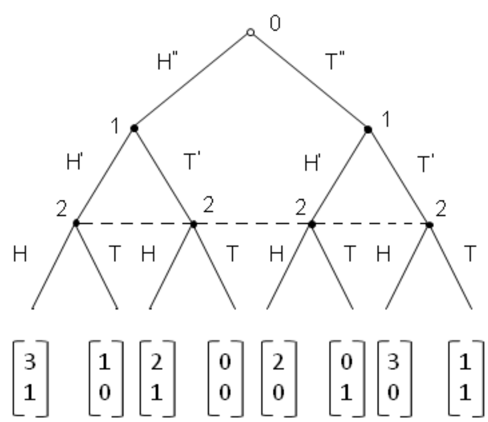
is the player function,

$$I(x_0) = \{x_0\}/I(x_1) = \{x_1\}/I(x_2) = \{x_2\}/I(x_3) = I(x_4) = I(x_5) = I(x_6) = \{x_3, x_4, x_5, x_6\}$$

is the information sets, and finally

$$\rho(x_0, H'') = 0.8$$

is the probability measure of nature. While the game tree is given as follows



b) What is the normalform representation of this game?

The set of players is $N = \{1, 2\}$, the strategy sets of the players is

$$\begin{aligned} S_1 &= \{H'H', H'T', T'H', T'T'\} \\ S_2 &= \{H, T\} \end{aligned}$$

where the strategy $H'T'$ of player 1 specifies that he will play H' if head is revealed to him and T' if tail is revealed to him. The bimatrices below contains the payoffs if H'' is chosen by nature

	H	T
$H'H'$	$(3, 1)$	$(1, 0)$
$H'T'$	$(3, 1)$	$(1, 0)$
$T'H'$	$(2, 1)$	$(0, 0)$
$T'T'$	$(2, 1)$	$(0, 0)$

and the payoffs if T'' is chosen by nature is

	H	T
$H'H'$	$(2,0)$	$(0,1)$
$H'T'$	$(3,0)$	$(1,1)$
$T'H'$	$(2,0)$	$(0,1)$
$T'T'$	$(3,0)$	$(1,1)$

c) Find all Nash equilibria of this game

We then see that the expected payoff from playing H is 0.8 disregarding the strategy of player 2, and the expected payoff from playing T is 0.2. Thus, T is a dominated strategy of player 2. Knowing this, the best respond of player 1 is then to play $H'T'$. There is then a unique Nash equilibrium, namely $(H'T', H)$.

Exercise 4

Consider the following description of a game:

There are 2 players, player 1 and 2. There are the following rules:

- Player 1 chooses the action l or r, where l ends the game with payoff $(2,0)$
- Player 2 observes player 1's action. If 1 chose r, the player 2 can choose between l' and r', where l' ends the game with payoff $(1,1)$
- Player 1 observes player 2's action and recalls his own choice. If there has been chosen r and r' then player 1 can choose between l'' and r'', both ending the game. If player 1 chooses l'' payoffs are $(3,0)$, while the payoffs from choosing r'' are $(0,2)$

a) What is the game tree of this game

b) Find the normalform representation

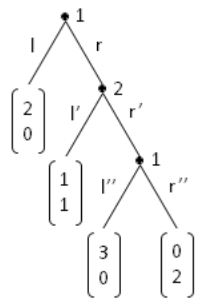
c) Find all Nash equilibria

d) Find the Subgame perfect Nash equilibrium.

Exercise 4 - suggested solution

a) What is the game tree of this game

The Game tree is as follows



b) Find the normalform representation The set of players is $N = \{1, 2\}$ and the strategy sets are

$$
\begin{aligned}
S_1 &= \{ll'', lr'', rl'', rr''\} \\
S_2 &= \{l', r'\}
\end{aligned}
$$

while the payoffs are represented in the bimatrix

	ll''	lr''	rl''	rr''
l'	$(2, 0)$	$(2, 0)$	$(1, 1)$	$(1, 1)$
r'	$(2, 0)$	$(2, 0)$	$(3, 0)$	$(0, 2)$

c) Find all Nash equilibria

The unique Nash equilibrium is (l', rr'').

d) Find the Subgame perfect Nash equilibrium.

Using Backwards induction we obtain that (l', rr'') is also the unique subgame perfect equilibrium.

Extra exercises

Exercise 5

Assume that there are 2 players, A and B. Consider the following game: They have to divide a dollar and they bargain using the following rules

- First, player A propose a division
- Second, player B is told the proposition, and can then accept or reject the proposition
- If B accepts the proposition the division is distributed to each and the payoff is the share. If B rejects neither gets a penny, and their payoff is zero.

There is no discounting of payoffs.

a) Write this as an Extensive form game

b) Show that any division is a Nash equilibrium of this game

c) Find the unique SPE equilibrium of the game

Exercise 5 - suggested solution

a) Write this as an Extensive form game

   The set of players is $N = \{1, 2\}$, there are 2 stages: in stage 1 player 1 chooses a division $x \in [0, 1]$, in stage 2 player 2 chooses between accepting, A, or rejecting, R. Thus, the set of outcomes is then $H = [0, 1] \times \{A, R\}$ and the payoffs are then $u_1(x, A) = x$, $u_1(x, R) = 0$, $u_2(x, A) = 1 - x$ and $u_2(x, R) = 0$.

b) Show that any division is a Nash equilibrium of this game

   The strategy set of each player is

$$\begin{aligned} S_1 &= [0, 1] \\ S_2 &= \{s_2 \colon [0, 1] \to \{A, R\}\}. \end{aligned}$$

Consider the following startegies

$$\begin{aligned} s_1 &= \bar{x} \\ s_2(x) &= \begin{cases} A & x \geq \bar{x} \\ R & x < \bar{x} \end{cases} \end{aligned}$$

We argue that this pair of strategies is a Nash equilibrium for any $\bar{x} \in [0, 1]$. To this end, we must show that it is not profitable to deviate from the strategy. Consider any $x < \bar{x}$, this would imply the outcome (x, A) given player 2's strategy, and thus the payoff $u_1(x, A) = x < \bar{x} = u_1(\bar{x}, A)$. On the other hand, for any $x > \bar{x}$, this would imply the outcome (x, R) given player 2's strategy, and thus the payoff $u_1(x, R) = 0 \leq \bar{x} = u_1(\bar{x}, A)$. Thus, it is not profitable for player 1 to deviate. Obviously, given $\bar{x}$, for player 2 to play $s_2(\bar{x}) = R$ would imply the outcome $(\bar{x}, R)$ and the payoff $u_2(\bar{x}, R) = 0 \leq \bar{x} = u_2(\bar{x}, A)$. Again, player 2 will not deviate.

c) Find the unique SPE equilibrium of the game

We find the unique SPE using backwards induction. Given a proposition $\bar{x} \in [0, 1]$, what is the best respond by player 2? Obviously, we obtain that $u_2(\bar{x}, A) = \bar{x} \geq 0 = u_2(\bar{x}, R)$, and, thus, backwards induction yields that $s_2^\star(x) = A$ for any $x \in [0, 1]$. Given that this strategy is the unique best respond in the last stage, the best respond of player 1 to play $x^\star = 1$.