A note on the stability of collusion in differentiated oligopolies

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Received 8 October 2001; accepted 18 June 2002

Abstract

Stability of collusion in differentiated oligopolies is studied without symmetry restrictions on the available strategies. It is demonstrated that if the number of firms is sufficiently large, two-phase stick-and-carrot punishment schemes apply at the highest possible discount rate with respect to collusion on the joint profit-maximizing output. If stick-and-carrot punishment schemes are used, collusive stability of the joint profit-maximizing output improves monotonically with the degree of product differentiation. The conclusions contrast with those obtained by Wernerfelt [Economics Lett. 29 (1989) 303].

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JEL classification: C72; L13

Keywords: Collusion; Product differentiation; Stick-and-carrot

1. Introduction

The impact of product differentiation on collusive behavior is an issue of vital importance for firms operating in oligopolistic markets and for regulatory authorities. Is tacit collusion, ceteris paribus, more or less likely to occur in markets with differentiated products than in markets with homogeneous products? In the literature, this issue has been studied in various dynamic models. Focus has primarily been on models of spatial competition à la Hotelling (Chang, 1991; Ross, 1992; Häckner, 1996; Rath, 1998) and on models with multiproduct demand structure and production costs normalized to zero...
(Deneckere, 1983, 1984; Majerus, 1988; Wernerfelt, 1989; Lambertini and Sasaki, 1999; Tyagi, 1999). This note is concerned with the latter class of models. Attention is not restricted in advance to any particular subclass of the credible (subgame perfect) strategies, an approach also taken by Wernerfelt (1989).¹ Tyagi (1999) shows that the relationship between product substitutability and collusive stability is likely to be sensitive to the functional form of demand. In this note, it is demonstrated that the specific choice of feasible strategies and the choice of collusive output may as well affect the results.

Stability of collusion depends on the ability to punish deviators. For a given discount rate, we may, therefore, want to make use of the optimal punishments, i.e. the worst possible subgame perfect equilibria (w.r.t. discounted profit for the punished firm). These were analyzed by Abreu (1986) in the oligopolistic Cournot model with homogeneous products. Abreu introduced the symmetric ‘stick-and-carrot’ punishments, which consist of one punishment stage possibly with negative profits (the stick) followed by stationary collusive output in infinitely many stages (the carrot). Abreu showed that the optimal stick-and-carrot punishments are optimal punishments within the class of symmetric punishments. The class of symmetric punishments is very restrictive: in every stage, all firms supply the same volume of output. It seems not unreasonable that punishments involve asymmetry between the punished firm and the punishers. However, Abreu showed that if the discount rate is sufficiently low, the globally optimal punishment can be constructed by stick-and-carrot, i.e. there exists a stick-and-carrot punishment which is optimal among the class of all subgame perfect punishments—symmetric or not. But, after all, there is only one purpose of optimal punishments: supporting collusive behavior under critical circumstances, i.e. when discounting is high. At ‘sufficiently low’ discount rates, collusive paths are in general easy to sustain by a variety of punishment schemes. In other words, as long as the discount rate is low, optimality of the punishment is superfluous. Since the stick-and-carrot punishments are only globally optimal for low discount rates it remains thereby unanswered when the stick-and-carrot paths are indispensable in the sense of supporting collusive behavior.

The papers by Abreu (1986, 1988) investigate the structure of the optimal punishments for a fixed discount rate. However, for an analysis of cartel stability in an oligopoly, the natural starting point seems perhaps not to be some fixed discount rate but rather a certain cooperative output (for example, the joint profit-maximizing output). This note addresses the following question. Given a collusive output, which subgame perfect punishment scheme can support collusion at the highest possible discount rate? Such punishment will be denoted a maximal punishment and the corresponding highest possible discount rate for which collusion can be sustained is then a maximal discount rate. In Section 3, necessary definitions are given and some useful properties of maximal punishments are demonstrated.

The demand model is similar to that of Wernerfelt (1989) who compares collusive stability among a fixed supply level between markets with different degrees of product differentiation, concluding that the effect of product differentiation on collusive stability can go either way, depending on the number of firms and the degree of substitution.

However, the analysis by Wernerfelt does not take into account that when the degree of differentiation increases, the aggregate demand also increases. Thereby, the strategic effect of more differentiated products is mixed with the strategic effect caused by higher aggregate demand. Moreover, the results obtained by Wernerfelt only apply to a subset of the relevant parameter space, delimited by a rather complicated pair of inequalities, essentially because—in the terminology of this note—the maximal punishments cannot be constructed by stick-and-carrot schemes for all relevant parameters. A normalized collusive output is therefore suggested in Section 4: simply the joint profit-maximizing output (or more general, some fixed fraction of it). It turns out that, when stick-and-carrot strategies are maximal, stability of collusion increases monotonically with the degree of product differentiation. Furthermore, the characterization of the relevant parameter space turns out to be very transparent: the maximal punishments can be constructed by stick-and-carrot if the number of firms is sufficiently large. On the other hand, if there are only a few firms, there is no particular reason for applying stick-and-carrot strategies for the purpose of sustaining collusion on the joint profit-maximizing output.

To sum up, the main purposes of this note are (1) to establish a framework for measuring the stability of collusion in terms of maximal discount rates, (2) to reconsider Wernerfelt’s findings on cartel stability, and (3) to demonstrate that for the joint profit-maximizing output, stick-and-carrot schemes are applicable at (near-) maximal discount rates only when the number of firms is sufficiently large.

2. The model

Following Wernerfelt, a differentiated oligopoly with \( n + 1 \) firms is considered. Let \( N = \{1, \ldots, n + 1\} \) denote the set of firms, and let \( q_i \in [0, \infty) \) be the output produced by firm \( i \). The goods are substitutes, and the price net of (constant) marginal production costs, \( p_i \), is determined by a linear inverse demand function

\[
p_i = 1 - \beta q_i - (1 - \beta) \sum_{j \in N \setminus \{i\}} q_j, \quad i \in N,
\]

where higher values of \( \beta \in [\frac{1}{2}, 1) \) correspond to more product differentiation. Note that as the marginal production cost is normalized to zero, a negative price is not ruled out.\(^3\) Firm \( i \)'s profit is \( \pi_i(q) = p_i q_i = (1 - \beta q_i - (1 - \beta) \sum_{j \neq i} q_j) q_i \) when output is \( q \). Let \( \pi_i^*(q_{-i}) \) be firm \( i \)'s best response profit when the others firm’s outputs are given by \( q_{-i} \).

The firms repeat infinitely the stage game described above and discount future profits with a common discount rate \( r \in (0, \infty) \). A path \( Q = \{q(t)\}_{t=0}^{\infty} \) is an infinite sequence of outputs. Define

\[
v_i(r, Q) = \sum_{t=0}^{\infty} \frac{1}{(1 + r)^t} \pi_i(q(t)).
\]

\(^2\) Determining the degree of product differentiation, the discount rate, and the number of firms.

\(^3\) It is, however, implicitly assumed that the actual (absolute) price is non-negative (Lambertini and Sasaki, 2001).
Let $v_i(r)$ be the discounted profit for firm $i$ under an optimal punishment, i.e. $v_i(r)$ is the lowest possible profit for firm $i$ in a subgame perfect equilibrium at the discount rate $r$.

3. Maximal punishments: definition and basic properties

Consider collusion along a path $Q^* = \{q^*_i(t)\}_{t=0}^\infty$. Assume throughout, to avoid a trivial case, that $q^*_i(t)$ is not a Nash equilibrium of the stage game for all $t$. Let $\mathcal{Q} = \{Q^*_i\}_{i=1}^N$ be a punishment profile. If firm $i$ deviates from the prescribed output $q^*_i(t)$ at stage $t$, then the punishment path $Q^*_i$ is started at stage $t + 1$. Any further deviation from a firm $j$ triggers the start of punishment path $Q^*_j$, etc. A pair $(Q^*, \mathcal{Q})$ defines a strategy profile with $Q^*$ as the collusive path followed if no deviations occur and with $\mathcal{Q}$ as the punishment profile.

**Definition 1.** A punishment profile $\mathcal{Q}$ supports collusion on a path $Q^*$ at discount rate $r$ if $(Q^*, \mathcal{Q})$ forms a subgame perfect strategy profile at $r$.

A collusive path $Q^*$ is then supported at the discount rate $r$ (by $\mathcal{Q}$) if there is a punishment profile $\mathcal{Q}$ that supports collusion in the sense of Definition 1. In general, the collusive path and the punishment paths can be non-stationary between stages and/or asymmetric between firms.

**Definition 2.** For a collusive path $Q^*$, a punishment profile $\mathcal{Q}$ is a maximal punishment profile if for some $r > 0$, $Q^*$ is supported by $\mathcal{Q}$, and for all $r' > r$ if $Q^*$ is supported by any other punishment profile then $Q^*$ is also supported by $\mathcal{Q}$.

Accordingly, $r^*$ is a maximal discount rate if a punishment profile supports collusion from $Q^*$ at $r^*$, and if there exists no punishment profile which supports collusion on $Q^*$ at any $r > r^*$. The maximal discount rate is thus the critical discount rate for the maximal punishment profile. Let $S \subseteq (0, \infty)$ denote the set of discount rates for which $Q^*$ is supported by $\mathcal{Q}$. Although the set of discount rates supporting a given collusive path may not be an interval, critical discount rates do always exist.

**Lemma 1.** For any collusive path $Q^*$ and punishment profile $\mathcal{Q}$, if $S$ is non-empty then there exists a highest (‘critical’) discount rate for which $Q^*$ is supported by $\mathcal{Q}$.

The proof of Lemma 1 and all subsequent propositions are placed in Appendix A. With the definitions and the lemma in place, it can be shown that for any collusive path, a maximal punishment profile exists.

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4. If more than one firm deviate simultaneously, then the deviator with the lowest number is punished.

5. In fact, it is possible to construct examples showing that asymmetric payoff distributions may only be supported if the collusive output follows a non-stationary path.

6. The critical discount rate is the highest discount rate for which a given punishment profile (not necessarily maximal) supports a given collusive path.
Proposition 1. Let a collusive path $Q^*$ be supported at some discount rate. Then there exists a maximal punishment profile $\mathcal{Q}$.

Thus, by Lemma 1 and Proposition 1 non-existence of maximal discount rates is ruled out. For this model, a useful connection between the maximal punishment profile and the optimal punishment profile at the maximal discount rate may now be established.

Proposition 2. Let a collusive path $Q^*$ be supported at some discount rate. Then a maximal punishment profile contains, for at least one firm, a punishment which is an optimal punishment at the maximal discount rate.

With respect to collusion on symmetric output (such as the joint profit-maximizing output studied in Section 4), Proposition 2 thus implies that a maximal punishment profile consists of punishments which are all optimal at the maximal discount rate.

4. Collusion on the joint profit-maximizing output

In this section, cooperation on the joint profit-maximizing output is examined. When output is symmetric, let $x$ (without subscript) denote the per firm output.

The firms maximize aggregate profits $\sum_{i \in N} \pi_i$ by the symmetric output $x^m = 1/[2n(1-\beta) + 2\beta]$. This leads to the symmetric price $p_i^m = 1/2$ and profits $\pi_i^m = p_i^m x^m = 1/[4n(1-\beta) + 4\beta]$. $x^m$ is the lower bound for meaningful cooperation, since for $x < x^m$ profit is reduced and temptation to deviate is increased.

The unique Nash equilibrium of the stage game is $x^c_n = 1/[n(1-\beta) + 2\beta]$. This output is the upper bound for meaningful cooperation, since by cooperating on some $x > x^c_n$ profits are even lower and $x^c_n$ is always a possible level of cooperation regardless of the discount rate. Note that if $x \geq 1/[1-\beta]n \equiv x^n$ then $\pi_i^d = 0$.

Except for $n = 1$ (duopoly), increasing degree of product differentiation does not only change the substitutability of the goods, but also expands aggregate demand (and profit), in the sense that

$$\forall x \forall n : \beta < \beta' \Rightarrow p(\beta, x, n) < p(\beta', x, n).$$

Since increasing $\beta$ expands aggregate demand in absolute terms, it seems perhaps not clear why one should compare maximal discount rates for the same absolute value of $x^*$ for different degrees of product differentiation. It may seem more natural to examine whether collusion on $x^m$ can be sustained (or slightly more general, collusion on $\alpha x^m$). Cooperation on $\alpha x^m$ is meaningful as long as $x^m \leq \alpha x^m \leq x^c_n$ (the trivial case $\alpha x^m = x^c_n$ is excluded in the following, so $1 \leq \alpha < [2\beta + 2(1-\beta)n]/[2\beta + (1-\beta)n]$). $\alpha = 1$ is the natural choice as long as the discount rate is low enough.

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7 If $\beta \neq 1/2$, $\arg \max_{\pi} \{\sum_{i \in N} \pi_i(q)\}$ is unique. For $\beta = 1/2$, $\arg \max_{\pi} \{\sum_{i \in N} \pi_i(q)\}$ is no longer unique, but $x^m$ can still be considered as the natural choice, because $(x^m, \ldots, x^m) = \lim_{\beta \rightarrow 1/2} \arg \max_{\pi} \{\sum_{i \in N} \pi_i(q)\}$. 
Suppose that all firms cooperate on a symmetric collusive output $\alpha x^m$. Collusive profit, denoted $\pi_i^{cm}$, is

$$\pi_i^{cm} = \frac{(2 - \alpha)\alpha}{4n(1 - \beta) + 4\beta}.$$ 

The optimal deviation quantity from the collusive output $\alpha x^m$ is then

$$q_i^{ad} = \frac{1 - [(1 - \beta)n\alpha]/[2n(1 - \beta) + 2\beta]}{2\beta},$$

and the corresponding deviation profit will be

$$\pi_i^{ad} = \frac{(2n - 2n\beta + 2\beta + n\alpha\beta - n\alpha)^2}{16\beta(n\beta - n - \beta)^2}.$$ 

Note that $\pi_i^{ad}$ is non-monotonic in $\beta$.

By Proposition 2, at the maximal discount rate, the maximal punishment is also optimal. In principle, one could try to derive the optimal punishment for every possible discount rate, and then find the critical one. However, note that

$$\pi_i^{ad} \leq \pi_i^{cm} + \pi_i^{om},$$

since otherwise it would be profitable to deviate from the collusive phase even if the deviator obtains zero profit in the subsequent stages. The maximal discount rate is thus bounded from above by the inequality

$$r^s \leq \frac{\pi_i^{om}}{\pi_i^{ad} - \pi_i^{om}}. \tag{1}$$

A stick-and-carrot punishment consists of a single stage where every firm supplies a punishment output $x^s$ after which the collusive output $x^p$ is supplied by all firms in all subsequent stages. If a stick-and-carrot punishment exists with a critical discount rate satisfying inequality (1) with equality, it is a maximal punishment. For $r^s = \pi_i^{om}/[\pi_i^{ad} - \pi_i^{om}]$, the ‘stick’ $x^s$ solving

$$\pi_i(x^s, \ldots, x^s) + \frac{1}{r^s} \pi_i^{om} = 0 \tag{2}$$

should satisfy

$$x^s \succeq x^a, \tag{3}$$

i.e. a firm should obtain zero profit by conforming to the punishment (assured when Eq. (2) holds) and deviations from the punishment stage should also be unprofitable (assured by inequality (3)).

It is now possible to characterize the domain of the parameter space where the maximal punishments have a stick-and-carrot structure, by examining for which triples $(n, \beta, \alpha)$ inequality (3) is satisfied. What remains is to isolate $x^s$ in Eq. (2), and then testing inequality (3). It turns out that inequality (3) holds when the number of firms is large (see
proof of Proposition 3 in Appendix A). Let \( r^* = r^*(n, \beta, \alpha) \) be the maximal discount rate given the parameters \( n, \beta, \) and \( \alpha \).

**Proposition 3.** For any \( \alpha \) and \( \beta, \) if the number of firms \( n \) is sufficiently large, the maximal punishments can be constructed by stick-and-carrot schemes and \( y(r^*) = 0 \) at the maximal discount rate \( r^* \).

For a triple \((n, \beta, \alpha)\), it may be the case that the maximal discount rate is higher than the largest discount rate supported by a stick-and-carrot punishment. However, by Proposition 3 the stick-and-carrot paths ‘break the surface’ and become maximal punishments when the number of firms is sufficiently large.

As an example, with \( \beta = \frac{1}{2} \) and \( \alpha = 1 \) the maximal punishments can be constructed by stick-and-carrot if \( n \geq 5 \). It can be checked that the number of firms sufficient for obtaining the maximal punishment by stick-and-carrot schemes is increasing in \( \beta \) and decreasing in \( \alpha \).

Using inequality (1) with equality, a monotonic relationship between the product differentiation and the maximal discount rate can now be established (see Appendix A for a proof).

**Proposition 4.** On the domain \( D = \{(n, \beta, \alpha) | y(r^*) = 0\} \) where the maximal punishments can be constructed by stick-and-carrot schemes, the maximal discount rate \( r^* \) increases monotonically with the degree of product differentiation.

As mentioned in Section 1, this result contrasts with the mixed conclusions of Wernerfelt (1989). The difference is due to the fact that in Wernerfelt’s study, the collusive output is not normalized with respect to the marked size, which varies with the degree of substitution in this model (except for the duopoly case).

### 5. Concluding remarks

Proposition 3 states that if the number of firms is large, it is sufficient to restrict attention to stick-and-carrot punishment schemes. If there are only a few firms however, stick-and-carrot schemes are insufficient. To see this, consider for example the case of homogeneous products. If the number of firms is low, the discounted profit at the optimal stick-and-carrot punishment is greater than zero at the maximal discount rate (see proof of Proposition 3).

Moreover, by Proposition 2, punishments supporting the maximal discount rate are indeed optimal at this rate. Then it follows from Abreu (1986, Theorem 25) that the optimal punishments are asymmetric and non-stationary. In particular, for the duopoly case widely studied in the literature it turns out that with respect to collusion on profit-maximizing output, the maximal punishments cannot be constructed by stick-and-carrot schemes.

Apparently, there exists no complete characterization of the optimal punishments in Cournot models, which by Proposition 2 is required for an analytical expression of the maximal punishments if \( n \) is not sufficiently large in the sense of Proposition 3. Abreu, Pearce and Stacchetti (1990) develop an abstract fixed point characterization of the set of
subgame perfect equilibrium profits in a model of imperfect monitoring, which also applies to settings with perfect monitoring. See Cronshaw (1997) for numerical results along these lines for a linear Cournot model with homogeneous products.8 However, up to now, a complete characterization of the maximal punishments remains unknown. Finally, it may be observed (the details are omitted here for brevity) that Propositions 1–4 apply to any demand model of the form
\[ A(\beta)p_i = B(\beta) - C(\beta)q_i - D(\beta) \sum_{j \in N \setminus \{i\}} q_j, \quad i \in N, \]
where \( A, B, C, D \) are positive functions, \( C \) increasing and \( D \) decreasing in the degree of substitution \( \beta \). The models studied by Deneckere (1983), Majerus (1988), and Lambertini and Sasaki (1999) are of this form, and indeed the results obtained here apply directly to their models. The rough intuition for this is as follows. The terms \( A(\beta) \) and \( B(\beta) \) only affect the unit of measurement and nominal marked size respectively and does not have any impact on the maximal discount rate for collusion on the joint profit-maximizing output. Accordingly, the relevant aspects of the market game are captured solely by ratio \( C(\beta)/D(\beta) \) (Albæk and Lambertini, 1998).9

Acknowledgements

I am grateful to Jens Leth Hougaard, Juan de Dios Moreno-Ternero and an anonymous referee for valuable comments. The usual disclaimer applies.

Appendix A

Proof of Lemma 1. Since discounted profits are continuous in \( r \), \( S \) consists of a union of at most countable many disjoint and closed intervals. By assumption, at some \( t : \pi^d_t(q^*(t)) > \pi_t(q^*(t)) \), hence \( S \) is bounded; so it remains to verify that \( \max\{r | r \in S\} \) exists. For this, assume that this is not the case. Then for any \( r' < \sup\{r | r \in S\} \) there exists some \( r'' \) such that \( r' < r'' < \sup\{r | r \in S\} \) and such that \( Q^* \) is not supported by \( \mathcal{A} \) at \( r'' \). On the other hand, if \( Q^* \) is not supported by \( \mathcal{A} \) at \( \sup\{r | r \in S\} \) then, since profits are continuous in \( r \), there exists an interval, \( \{r | r \in S\} - \epsilon, \sup\{r | r \in S\} \} \), where \( Q^* \) is not supported by \( \mathcal{A} \). Thus a contradiction is obtained.

Proof of Proposition 1. First note that by Abreu (1988, Proposition 2), an optimal punishment profile exists for any fixed discount rate \( r \). Now assume that a maximal punishment profile does not exist. Then there exists \( \hat{r} \) such that \( Q^* \) is not supported at \( \hat{r} \), and

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8 A public observable randomizing device is required.
9 This line of reasoning depends on a normalization of the collusive output; hence the results obtained by Wernerfelt (1989) are sensitive to the specific functional form of demand.
\( \forall r < \hat{r} \) there exists \( r' \), \( r < r' < \hat{r} \), such that \( Q^* \) is supported at \( r' \). Let \( \mathcal{Q} = \{ Q' \}_i \in \mathbb{N} \) be an optimal punishment profile at \( \hat{r} \). Let \( \{ \mathcal{Q}_k \}^\infty_{k=1} \) be a sequence of punishment profiles, and let \( \{ r_k \}^\infty_{k=1} \) be a growing sequence converging to \( \hat{r} \), so that \( \mathcal{Q}_k \) supports collusion on \( Q^* \) at discount rate \( r_k \). Let \( \mathcal{Q}_k^l \) be the punishment path for firm \( i \) at \( \mathcal{Q}_k \). Then in some stage \( t \) there is an output \( q^l(k)(t) \) along the collusive path \( (l = * ) \) or along a punishment path for some firm \( j \) \( (l = j) \) such that for a firm \( i \)

\[
\pi_i^l(q^l(k)(t)) + \frac{1}{1 + r_k} v_i(r_k, \mathcal{Q}_k^l) \leq \sum_{s=t}^{\infty} \frac{1}{(1 + r_k)^{s-t}} \pi_i(q^l(s)),
\]

for all \( k \), and

\[
\pi_i^l(q^l_{-i}(t)) + \frac{1}{1 + \hat{r}} v_i(\hat{r}, \mathcal{Q}^l) > \sum_{s=t}^{\infty} \frac{1}{(1 + \hat{r})^{s-t}} \pi_i(q^l(s)),
\]

where \( q^l(t) \) is output along the collusive path \( Q^* \) \( (l = * ) \) or along a punishment path \( Q^l \) for firm \( j \) \( (l = j) \). Since \( \inf_k \{ r_k \} = r_1 > 0 \), there is some finite number \( \bar{s} > 0 \) such that \( q^l_k(t) \in [0, \bar{s}] \) for any output \( q^l_k(t) \) from the sequence \( \{ q^l_k \}^\infty_{k=1} \). Thus, it is possible to choose \( \{ \mathcal{Q}_k \}^\infty_{k=1} \) converging with \( \lim_{k \to \infty} \mathcal{Q}_k = \hat{Q} \). Now, by continuity of \( \pi_i \)

\[
\pi_i^l(q^l_{-i}(t)) + \frac{1}{1 + \hat{r}} v_i(\hat{r}, \mathcal{Q}^l) \leq \sum_{s=t}^{\infty} \frac{1}{(1 + \hat{r})^{s-t}} \pi_i(q^l(s)),
\]

for all \( t, l, i \). Hence \( \hat{Q} \) is a subgame perfect punishment profile at \( \hat{r} \) which supports \( Q^* \), and a contradiction is obtained. \( \square \)

**Proof of Proposition 2.** Let \( r^* \) denote the maximal discount rate for the collusive path \( Q^* \). Let \( \hat{r} > 0 \) denote the critical discount rate where a punishment profile \( \mathcal{Q} = \{ Q' \}_i \in \mathbb{N} \) supports collusion on \( Q^* \), and assume that \( Q^* \) is inoptimal for all \( i \in N \) at \( \hat{r} \). Then it must be verified that \( \hat{r} < r^* \). By Abreu (1988, Proposition 2) there exists a punishment profile \( \mathcal{Q} \) consisting of optimal punishments only at the discount rate \( \hat{r} \), so \( v_i(\hat{r}, Q^*) < v_i(\hat{r}, Q^l) \) for all \( i \in N \). Now, replace the punishment profile \( \mathcal{Q} \) by \( \mathcal{Q} \). Then the incentive constraints relax for all firms at all stages along the collusive path \( Q^* \); that is

\[
\forall t \forall i : \pi^l_i(q^l_{-i}(t)) + \frac{1}{1 + \hat{r}} v_i(\hat{r}, Q^l) < \sum_{s=t}^{\infty} \frac{1}{(1 + \hat{r})^{s-t}} \pi_i(q^*(s)).
\]

Since

\[
\frac{1}{1 + \hat{r}} v_i(\hat{r}, Q^l), \quad \text{and} \quad \sum_{s=t}^{\infty} \frac{1}{(1 + \hat{r})^{s-t}} \pi_i(q^*(s))
\]

are continuous in \( r \) for all \( i \) and \( t \), there exists \( \tilde{r} > \hat{r} \) such that

\[
\forall t \forall i : \pi^l_i(q^l_{-i}(t)) + \frac{1}{1 + \tilde{r}} v_i(\tilde{r}, Q^l) \leq \sum_{s=t}^{\infty} \frac{1}{(1 + \tilde{r})^{s-t}} \pi_i(q^*(s)),
\]

implying that \( \mathcal{Q} \) is not a maximal punishment profile. \( \square \)
Proof of Proposition 3. Writing out Eq. (2) yields

\[ x^4 - (x^3)^2(\beta + (1 - \beta)n) + \frac{1}{4\alpha(2 - \alpha)} \frac{\beta(\beta + n - \beta)}{(2n - n\alpha - 2n\beta + 2\beta - 2\alpha\beta + n\alpha\beta)^2} \]

\[ \times \frac{(2 - \alpha)\alpha}{4n(1 - \beta) + 4\beta} \]

\[ = 0 \]

or

\[ x^4 - (x^3)^2(\beta + (1 - \beta)n) + \frac{(2n - n\alpha - 2n\beta + 2\beta - 2\alpha\beta + n\alpha\beta)^2}{16(n\beta - n - \beta)^2\beta} = 0. \]

The relevant solution to this second degree polynomial is

\[ x^4 = \frac{1}{2(n - n\beta + \beta)} \left( 1 + \frac{1}{2} \sqrt{\frac{E}{\beta(n\beta - n - \beta)}} \right), \]

where

\[ E = F n^2 + G n + H, \quad F = -(\beta - 1)^2(\alpha - 2)^2, \]

\[ G = 4\beta(3 + \alpha^2 - 3\alpha)(\beta - 1), \quad H = 4\beta^2(2\alpha - \alpha^2 - 2). \]

The stick-and-carrot punishment is subgame perfect if \( x^4 \geq x^u \), i.e. if

\[ \frac{1}{2(n - n\beta + \beta)} \left( 1 + \frac{1}{2} \sqrt{\frac{E}{\beta(n\beta - n - \beta)}} \right) \geq \frac{1}{(1 - \beta)n} \]

or equivalently if

\[ \left( 1 + \frac{1}{2} \sqrt{\frac{E}{\beta(n\beta - n - \beta)}} \right) \geq \frac{2(n(1 - \beta) + \beta)}{(1 - \beta)n}. \]

Since

\[ 1 \leq \alpha < \frac{2\beta + 2(1 - \beta)n}{2\beta + (1 - \beta)n} < 2 \quad \text{and} \quad \frac{1}{2} \leq \beta < 1, \]

it follows that \( F < 0 \) and \( \beta(n\beta - n - \beta) < 0 \). Thus

\[ \lim_{n \to \infty} \left( 1 + \frac{1}{2} \sqrt{\frac{E}{\beta(n\beta - n - \beta)}} \right) = \infty, \]
and since
\[
\lim_{n \to \infty} \frac{2(n(1 - \beta) + \beta)}{(1 - \beta)n} = 2,
\]
it appears that \( x^s \geq x^u \) when \( n \) is sufficiently large. \( \square \)

**Proof of Proposition 4.** Note that inequality (1) holds with equality on the domain \( D \). Writing out the right hand side of inequality (1) yields
\[
r^x = \frac{(2 - \alpha)\alpha}{4n(1 - \beta) + 4\beta} - \frac{(2 - \alpha)\alpha}{16\beta(n\beta - n - \beta)^2} \quad \frac{4\alpha(2 - \alpha)\beta(n - n\beta)}{(2n - n\alpha - 2n\beta + 2\beta - 2\alpha\beta + n\alpha\beta)^2}.
\]
The first order derivative w.r.t. \( \beta \) is then
\[
\frac{\partial r^x}{\partial \beta} = 4\alpha(2 - \alpha) \left( \frac{(2\beta + n - 2n\beta)(2n - n\alpha - 2n\beta + 2\beta - 2\alpha\beta + n\alpha\beta)^2}{(2n - n\alpha - 2n\beta + 2\beta - 2\alpha\beta + n\alpha\beta)^4} \right.
- \frac{\beta(\beta + n - n\beta)2(2n - n\alpha - 2n\beta + 2\beta - 2\alpha\beta + n\alpha\beta)(-2n + 2 + 2\alpha + n\alpha)}{(2n - n\alpha - 2n\beta + 2\beta - 2\alpha\beta + n\alpha\beta)^4} \bigg) \\
= 4n\alpha(2 - \alpha) \frac{n(\alpha - 2)(\beta - 1) + 2\beta}{(n(\alpha - 2)(\beta - 1) - 2\beta(\alpha - 1))^2}.
\]
Since
\[
1 \leq \alpha < \frac{2\beta + 2(1 - \beta)n}{2\beta + (1 - \beta)n} < 2,
\]
it follows that \( 4n\alpha(2 - \alpha) > 0 \) and \( n(\alpha - 2)(\beta - 1) + 2\beta > 0 \). Moreover,
\[
n(\alpha - 2)(\beta - 1) - 2\beta(\alpha - 1) > 0 \Leftrightarrow \alpha < \frac{2\beta + 2n(1 - \beta)}{2\beta + n(1 - \beta)},
\]
which is satisfied by assumption. Thus
\[
\frac{\partial r^x}{\partial \beta} > 0.
\]
\( \square \)

**References**
