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Monotonicity of social welfare optima

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Abstract

This paper considers the problem of maximizing social welfare subject to participation constraints. It is shown that for an income allocation method that maximizes a social welfare function there is a monotonic relationship between the incomes allocated to individual agents in a given coalition (with at least three members) and its participation constraint if and only if the aggregate income to that coalition is always maximized. An impossibility result demonstrates that there is no welfare maximizing allocation method in which agents’ individual incomes monotonically increase in society’s income. Thus, for any such allocation method, there are situations where some agents have incentives to prevent society in becoming richer.
Keywords: Income Allocation, Monotonicity, Core, Social Welfare, Cooperative Game.

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1 Introduction

A central problem of political philosophy and welfare economics is that of allocating societal income given claims of various groups of individuals. Such a claim may be interpreted as the minimal amount of total income a group would need in order to survive or remain reasonably satisfied, or it may represent an economic participation constraint that specifies what a group can obtain on its own if it chooses not to cooperate with any other in society.

To model the problem of allocating income in society we use the framework of a cooperative game, which specifies society’s total income and a participation constraint for each group of individuals (coalition) in monetary units. Social values are represented by a social welfare function. The problem therefore becomes a question of allocating income in a way that maximizes social welfare, but does so under various participation constraints (core constraints).

Maximizing social value (or minimizing social inequality) subject to various types of stability constraints is a theme that has lately drawn much attention in the literature. Pioneered by the work of Dutta and Ray [7] on egalitarian allocations, the problem has been further investigated in a number of recent papers, e.g., Dutta [6], Dutta and Ray [8], Klijn et al. [20], Arin and Inarra [1], Hougaard et al. [13], Koster [21], Hokari [12], Jaffray and Mongin [16], Arin et al. [2, 3] and Hougaard et al. [14]. In this literature social value is represented by the Lorenz partial ordering of income allocations or weighted generalizations of this concept. To model social value by a (differentiable strictly concave) social welfare function, as in the present paper, represents a general approach to these types of allocation methods.
which encompasses a broad range of income inequality measures.

Our approach differs from traditional allocation methods for cooperative games, which, for example, consist of maximizing the smallest excess of any coalition as in the nucleolus (Schmeidler [27]) or taking a weighted average of marginal contributions to the participation constraints (coalitional worth) as in the Shapley value (Shapley [28]). Traditional allocation methods typically depend on the entire specification of participations constraints, whereas for allocation methods maximizing social value, the role of the participation constraints is limited to determining the set of socially stable allocations.

The present paper investigates conflicts of interests between individual members of a given group and the group as a whole, in a society which maximizes welfare subject to participation constraints. Conflicts may arise if some members of a coalition have incentives to weaken its power (i.e. to reduce the coalition’s participation constraint). In order to avoid conflicts, the income to members of the coalition should therefore monotonically increase when the coalition’s participation constraint increases (holding constant all other participation constraints). In particular, for the society as a whole, the income of all members should monotonically increase when society’s total income increases (for all other participation constraints held constant).

Monotonicity conditions have a prominent place in the game theory literature, see, e.g., Megiddo [23], Kalai and Smorodinsky [19], Thomson and Myerson [31] for early studies. The particular form studied in this paper, i.e. monotonicity in the participation constraint of a given coalition, was first studied by Kalai [17] in the context of two-agent cooperative games without side-payments and later by Kalai and Samet [18] in the context of
n-agent cooperative games without side-payments. For cooperative games with side-payments (as studied in this paper), a well-known result by Young [32] shows that no core allocation method satisfies such monotonicity conditions for all coalitions and all games with non-empty core (disregarding games of very small size). This result demonstrates that generally there is a trade-off between social stability (that is, upholding participation constraints) and monotonicity with respect to all coalitions. In our context of welfare maximization, we will see that potential conflicts of interests between individual members of a group and the group as a whole are the rule rather than the exception.

1.1 Summary of the results

We are concerned with the family of income allocation methods that maximize social welfare subject to participation constraints, and examine the relationship between agents’ incomes and variations in participation constraints as well as society’s total income for games with a non-empty core (i.e., games where there is at least one income allocation which upholds all participation constraints).

The first result shows that if a welfare maximizing allocation method is monotonic with respect to a given coalition $S$ with at least three members, then the marginal social desirability of transferring income to members of this group is higher than for other members of society. This necessary condition for monotonicity implies that for a given coalition size (greater than or equal to three) the allocation method is monotonic for at most one coalition. For example, if a given allocation method is monotonic for a specified coali-
tion with four agents, then it cannot be monotonic for any other four-agent coalition.

The second result is a characterization of allocation methods that are monotonic with respect to a given group of agents: For any given coalition \( S \) with at least three members, which does not include all members of society, a welfare maximizing allocation method is monotonic with respect to coalition \( S \) if and only if the allocation method always maximizes the aggregate income of coalition \( S \). In other words, the members of a group have fully compatible interests in terms of promoting the worth of the group if and only if society always maximizes the total income of this group given the participation constraints (disregarding societies and coalitions of very small size).

In our context of welfare maximization, the results mentioned above provide much stronger insights concerning the lack of monotonicity compared to what can be obtained from Young’s general result. The results demonstrate that only very few coalitions can in fact be monotonic: Indeed, society cannot maximize the aggregate income of more coalitions than there are agents in society. Hence, coalitional monotonicity turns out to be an extremely strong requirement in connection with welfare maximization and a social planner is forced to accept that there are potential conflicts in many subcoalitions of society.

More crucial though is the fact that a social planner cannot even prevent conflicts in society as a whole. Our third result shows that there is no welfare maximizing allocation method which guarantees that the income of all members of society will be monotonically increasing in society’s total income.
(disregarding societies of very small size). It tells us that no matter which allocation method we use there will always be situations where some agents have incentives to prevent society in becoming richer. As monotonicity in society’s total income is a very reasonable requirement (contrary to monotonicity in all participation constraints) and there are indeed many allocation methods that satisfy such a monotonicity requirement the result highlights the consequences of using welfare maximization as the overall objective of a social planner.

2 Basics

A cooperative game with side payments is a pair \((N,v)\) where \(N\) is a non-empty finite set of agents, and \(v\) is a function that to each subset \(S\) of \(N\) associates a real-valued worth \(v(S)\), with the convention \(v(\emptyset) = 0\). As \(N\) is fixed in the following we shall write \(v\) instead of \((N,v)\) for simplicity.

For an income vector \(x \in \mathbb{R}^N\), define \(x(S) = \sum_{i \in S} x_i\). The core of \(v\) is the subset \(C(v)\) of \(\mathbb{R}^N\) defined by

\[
C(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for } S \subset N\}.
\]

In the following we restrict our attention to the class of games \(\mathcal{B}\) for which the core is a non-empty set, i.e. \(\mathcal{B}\) is the class of balanced games, see, e.g., Peleg and Sudhölter [24] for a general treatment. An allocation method is a function \(\phi : \mathcal{B} \to \mathbb{R}^N\), which for each game \(v \in \mathcal{B}\) allocates the total income \(v(N)\) among all agents, that is \(\phi(v) \in \{x \in \mathbb{R}^N \mid x(N) = v(N)\}\). A core allocation method is an allocation method where \(\phi(v) \in C(v)\) for all \(v \in \mathcal{B}\).
We shall be interested in the particular class of allocation methods maximizing welfare subject to the core constraints. Suppose that \( W : \mathbb{R}^N \to \mathbb{R} \) is a differentiable strictly concave social welfare function and define the welfare maximizing allocation method (shorthand, the \( W \)-allocation method) \( \phi^W \) by
\[
\phi^W(v) = \arg \max \{ W(x) \mid x \in C(v) \}.
\]

Since \( C(v) \) is a closed and convex set, there is a unique maximizer of \( W \) on \( C(v) \) and hence \( \phi^W \) is well defined.

**Remark 1.** The \( W \)-allocation method satisfies independence of irrelevant core allocations (see, e.g., Arin et al. [2]) in the sense that if an income vector in the core \( x \in C(v) \) is a \( W \)-allocation for some game \( v \) then \( x \) is also the \( W \)-allocation for any game \( v' \) for which \( x \in C(v') \subseteq C(v) \). This is in contrast to, for example, the nucleolus which may result in different allocations for games with identical cores as shown in Maschler et al. [22].

**Remark 2.** For the class of convex games (i.e., games \( v \) for which \( v(S) + v(T) \leq v(S \cap T) + v(S \cup T) \) for all \( S, T \subseteq N \)), the core is non-empty and contains a unique element maximal with respect to the Lorenz partial ordering of income vectors (which coincides with the constrained egalitarian solution of Dutta and Ray [7]). By Theorem 108 of Hardy et al. [10], it can be found by maximizing any separable function \( W(x) = \sum_{i=1}^n f(x_i) \) where \( f \) is strictly concave, see also Fujishige [9]. For balanced games the set of Lorenz-maximal income vectors in the core may be set-valued. By Theorem 2 in Hougaard et al. [13] an income vector in the core of a balanced game is Lorenz-maximal if and only if it is the maximizer of some symmetric additive strictly concave social welfare function. Note that convex games are
balanced while the converse is not necessarily true.

3 (Non-)monotonicity

Let \( S \subseteq N \) be a non-empty coalition. An allocation method \( \phi \) on \( B \) is called \( S \)-monotonic (that is, monotonic with respect to coalition \( S \)) if for all \( v, w \in B \) where \( v(S) < w(S) \) and \( v(T) = w(T) \) otherwise, that \( \phi_i(v) \leq \phi_i(w) \) for all \( i \in S \).

An allocation method \( \phi \) is said to be coalitionally monotonic if \( \phi \) is \( S \)-monotonic for all coalitions \( S \subseteq N \). For \( |N| \geq 4 \), it is well-known (cf. Young [32] and Housman and Clark [15]) that on the class of balanced games no core allocation method satisfies coalitional monotonicity. In other words, all core allocation methods violate \( S \)-monotonicity for at least one coalition \( S \). We shall return to this result (in the context of welfare maximizing allocation methods) below, where it will emerge as a simple corollary of our characterization results.

3.1 Characterization of \( S \)-monotonicity

As a preliminary for the main characterization result we show that if a \( W \)-allocation method is \( S \)-monotonic then the marginal social welfare of members in \( S \) exceeds that of members in the complement \( N \setminus S \).

**Theorem 1.** Let \( |N| \geq 4, |S| \geq 3 \) and \( S \neq N \). If \( \phi^W \) is \( S \)-monotonic then \( W'_i(x) > W'_j(x) \) for all \( x \in \mathbb{R}^N \) and all \( i \in S, j \in N \setminus S \).

All proofs are provided in the Appendix. The proof of Theorem 1 consists
of showing that if there is \( x \) and \( i \in S, j \in N \setminus S \) for which \( W'_i(x) \leq W'_j(x) \), then we can find a game \( v \) in which it is possible to provoke a contradiction to \( S \)-monotonicity. Disregarding here the case \( W'_i(x) = W'_j(x) \) for brevity, we can sketch the construction of \( v \) as follows: We construct \( v \) such that \( x \in C(v) \) and if income is transferred from agents in \( N \setminus S \) to agents in \( S \) then this transfer must take place between two agents \( j \in N \setminus S \) and \( i \in S \) (if the core constraints should be upheld). Further, we construct the game such that the transfer from agent \( j \) to agent \( i \) must be accompanied by an additional transfer from one agent to another within \( S \setminus \{i\} \), such that this transfer also reduces welfare. In effect, when we consider the game \( v \) and increase \( v(S) \) slightly, a violation of \( S \)-monotonicity is obtained.

For a given social welfare function \( W \), we say that an allocation method \( \phi^W \) is \( S \)-maximal if, for each \( v \in \mathcal{B} \), it maximizes \( x(S) \) over all \( x \in C(v) \). In other words, \( S \)-maximal allocation methods always allocate income such that the aggregate income of coalition \( S \) is as large as possible given the core constraints.

Since the \( W \)-allocation method \( \phi^W \) satisfies independence of irrelevant core allocations it follows that if the allocation method \( \phi^W \) is \( S \)-maximal, \( S \neq N \), it is also \( S \)-monotonic because the solution is unchanged when the coalitional worth of \( S \) is increased. Remarkably, (for \( |N| \geq 4 \)) the reverse implication is also true.

**Theorem 2.** Let \( |N| \geq 4, |S| \geq 3 \) and \( S \neq N \). Then \( \phi^W \) is \( S \)-monotonic if and only if \( \phi^W \) is \( S \)-maximal.

The proof of Theorem 2 demonstrates that if there is an allocation method
φ^W, and a proper subcoalition S for which ∑_{i∈S} φ^W_i(v) is not maximal for at least one game v, then it is always possible to find another game ũ in which a violation of S-monotonicity can be provoked. Constructing such a game ũ is generally non-trivial however, since for a given S, one could imagine that an allocation method φ^W only violates S-maximality in rare cases, in which a reallocation of income from agents in N \ S to agents in S would require a highly complex pattern of accompanying reallocations within N \ S in order to restore the core constraints (we can imagine such cases when |N \ S| is large) and we need to make sure that it is possible to provoke a violation of S-monotonicity even in such cases. The main task in the proof (which is given in the Appendix) is therefore to show that for any given game v as described above, it is possible to construct such a game ũ in a specific way which guarantees that it resembles some key properties of the game v, in particular that of non-S-maximality.

As shown in Young [32] no core allocation method can be coalitionally monotonic (i.e. S-monotonic for all S ⊆ N) on games with five or more agents, a result that was extended by Housman and Clark [15], showing that this result holds also for games with four agents. In the particular case of W-allocation methods, this follows already from Theorem 1, since the conditions of Theorem 1 on marginal social welfare clearly cannot be satisfied for all S ⊆ N. It is easy to show that if φ^W is both S-maximal and S'-maximal, then either S ⊆ S' or S' ⊆ S. Hence, for W-allocation methods we get the following strengthening of the impossibility results by Young and Hausman-Clark, indicating that coalitional monotonicity is, indeed, a very strong requirement.
Corollary 1. Let $|N| \geq 4$. Then for any $3 \leq k < |N|$ there is at most one $S$-monotonic coalition with $k$ agents.

Remark 3. On the domain of convex games, coalitional monotonicity and core stability are compatible requirements. For example the Shapley value is both a core allocation method and coalitionally monotonic (Sprumont [30], Rosenthal [26], see also Shapley [29]), and so are various generalizations of the Fujishige-Dutta-Ray solution (Hokari [12], Hougaard et al. [14]), whereas the nucleolus is not (Hokari [11]). In particular, it was shown in [14] that if an allocation method maximizes an additive (but not necessarily symmetric) social welfare function, i.e. $W(x) = \sum_{i\in N} W_i(x_i)$ where the functions $W_i$ are strictly concave, then $\phi^W$ (called a Generalized Lorenz solution) is coalitionally monotonic.

Theorem 2 does not characterize $S$-monotonicity for the special cases $|S| = 1$ and $|S| = 2$. Since it follows directly from the property of independence of irrelevant core allocations that all $W$-allocation methods are monotonic with respect to single agent coalitions, it remains only to consider the case $|S| = 2$.

Finding violations of monotonicity for coalitions with two agents is not easy (and one may wonder whether all $W$-allocation methods are $\{i, j\}$-monotonic). The following example, which involves a game with seven agents, demonstrates that $\{i, j\}$-monotonicity may, in fact, fail.

Example 1. Let $N = \{1, \ldots, 7\}$, $S = \{1, 2\}$, and $W(x) = \sum_{i=1}^{7} (\alpha_i x_i + f(x_i))$, where $\alpha_1 = 10$, $\alpha_2 = 1$, $\alpha_3 = \ldots = \alpha_7 = 100$ and $f(x_i)$ is a differentiable strictly concave function with derivative bounded between 0 and
Moreover, let the game \( v \) be defined as follows: \( v(N) = 3, v({1, 2}) = v({1, 3}) = v({1, 4}) = v({2, 5}) = v({2, 6}) = v({2, 7}) = 1 \), and \( v(T) = 0 \) otherwise. We observe that \( \phi^W(v) = (0, 1, 1, 0, 0) \). Now, define a new game \( w \) as \( w({1, 2}) = 2 \) and \( w(T) = v(T) \) otherwise. It is readily verified that the solution to \( w \) is \( \phi^W(w) = (\frac{4}{3}, \frac{2}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) where we notice that agent 2 is worse off, i.e. the allocation method \( \phi^W \) is not \( S \)-monotonic.

It is an open question whether a counterpart to Theorem 2 can be established for \( |S| = 2 \) and some lower bound on \( |N| \). (Example 1 suggests \( |N| \geq 7 \)).

### 3.2 \( N \)-monotonicity: an impossibility

The grand coalition has its own interest since all agents of society ought to have incentives to increase the total income to be shared. It is well-known that there are core allocation methods that are \( N \)-monotonic (aggregate monotonic, resource monotonic) as for example the per capita nucleolus, see, e.g., Young [32]. In fact, the class of \( N \)-monotonic core allocation methods seems rather large. For example, it includes all allocation methods constructed as follows: Let \( \phi^0 \) be an arbitrary core allocation method and let \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+^N \) be a function which is nondecreasing in all coordinates such that \( \sum_{i \in N} f_i(t) = t \) for all \( t \). For each balanced game \( v \), find the smallest number \( \alpha(v) \) (which in fact only depends on the worths of the proper subcoalitions) such that the game \( v^0 \) defined by \( v^0(N) = \alpha(v) \) and \( v^0(S) = v(S) \) for all \( S \neq N \) has non-empty core. Then define \( \phi(v) = \phi^0(v^0) + f(v(N) - v^0(N)) \). Note that this class contains continuous allocation methods satisfying inde-
dependence of irrelevant core allocations.

In many ways it would be reasonable to prefer core allocation methods which are $N$-monotonic, and a central question is therefore whether such allocation methods are consistent with maximization of a (differentiable strictly concave) social welfare function. Theorem 3 shows that they are not.

**Theorem 3.** Let $|N| \geq 3$. Then there exists no $N$-monotonic welfare maximizing allocation method $\phi^W$.

The result has somewhat striking implications. If society wants to maximize welfare when it distributes the benefits of social interaction, there will always be some situations (that is, constellations of coalitional worth) for which at least one member of the society will have the incentive to reduce the total income of society, or, dually, will have no incentive to participate in actions that increase the total income of society.

In light of Theorem 2, one might have expected that $N$-monotonicity should be satisfied since any $W$-allocation method is $N$-maximal by definition. However, monotonicity in the worth of $N$ is different since increasing the worth of $N$ displaces the set of core compatible income allocations whereas increasing the worth of any subset of $N$ reduces the set of core allocations. $S$-maximality (for $S \neq N$) implies that the allocation of income does not change when the worth of coalition $S$ is increased. But for any allocation method, the income allocation will change when the worth of $N$ is increased.
3.3 Final remarks and open questions

The proofs of Theorems 1-3 involve games that are balanced but not necessarily superadditive (a game $v$ is superadditive if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$, $S \cap T = \emptyset$). Proofs could have been formulated using only superadditive games, simply by replacing the games with their associated superadditive cover games (note that the core of a balanced game is identical with the core of the associated superadditive cover game). Hence, restricting attention to the class of superadditive games does not alter the results.

Most likely, the differentiability assumption (of the social welfare function $W$) can be dispensed with in Theorems 2 and 3, by limit arguments inspired by those given in relation to the results in Hougaard et al. [14]. A more interesting open question is whether Theorems 2 and 3 can actually be generalized to cover all allocation methods consistent with an underlying social welfare ordering over income allocations that will always select a unique element in the core.

Finally, we notice that it seems clear that $S$-maximal $W$-allocation methods exist for any given $N$ and $S$, since we can, for example, define $W$ such that marginal social welfare of agents in $N \setminus S$ lie within the band $[0, 1]$ and marginal social welfare of agents in $S$ lie within the band $[\lambda, \lambda + 1]$. The allocation method $\phi^W$ would then be $S$-maximal when $\lambda$ is ‘large enough’. We shall refrain from an attempt to specify a bound (which would depend on the cardinality of $S$ and $N$).
A Appendix: Proofs

In the following proofs we make use of the concept of a bilateral transfer, which intuitively can be seen as a reallocation of income from one agent to another. To make this precise, suppose that \( x, y \in \mathbb{R}^N \), and for some \( \gamma \geq 0 \) and some \( i, j \in N \) we have \( y_i + \gamma = x_i \), \( y_j - \gamma = x_j \) and \( x_k = y_k \) for \( k \neq i, j \). We then say that \( y \) is reached from \( x \) after a bilateral transfer of \( \gamma \) from agent \( i \) to \( j \), and that \( \gamma \) is a bilateral transfer leading from \( x \) to \( y \).

**Proof of Theorem 1:** First, we show that if \( \phi^W \) is \( S \)-monotonic then \( W'_i(x) \geq W'_j(x) \) for all \( x \in \mathbb{R}^N, i \in S, j \in N \setminus S \).

Assume that \( \phi^W \) is \( S \)-monotonic and that there exist \( x \in \mathbb{R}^N, i \in S, j \in N \setminus S \) where \( W'_i(x) < W'_j(x) \). Let \( i_1, ..., i_{|S|} \) denote the agents in \( S \) arranged by weakly increasing marginal welfare at \( x \), and similarly let \( j_1, ..., j_{|N\setminus S|} \) denote the agents in \( N \setminus S \) arranged by weakly increasing marginal welfare at \( x \). We therefore have \( W'_{j_{|N\setminus S|}}(x) > W'_{i_1}(x) \).

For \( \theta > 0 \), define the game \( v \) as follows: \( v(S) = x(S), v(\{i_1, j_{|N\setminus S|}\}) = x_{i_1} + x_{j_{|N\setminus S|}}, v(\{i_2, j_{|N\setminus S|}\}) = x_{i_2} + x_{j_{|N\setminus S|}}, v(\{i_1, i_{|S|}\}) = x_{i_1} + x_{i_{|S|}}, v(\{i_2, i_{|S|}\}) = x_{i_2} + x_{i_{|S|}}, v(\{k\}) = x_k \) for \( k \in N \setminus \{i_{|S|}, j_{|N\setminus S|}\}, v(\{k\}) = x_k - \theta \) for \( k \in \{i_{|S|}, j_{|N\setminus S|}\}, v(N) = x(N), \) and \( v(T) = x(T) - 2\theta \) otherwise.

Note that \( x \in C(v) \). In fact, \( C(v) \) is the convex hull of \( x \) and the allocation \( y \) where \( y_k = x_k - \theta \) for \( k \in \{i_{|S|}, j_{|N\setminus S|}\}, y_k = x_k + \theta \) for \( k \in \{i_1, i_2\} \) and \( y_k = x_k \) otherwise: Only agents \( i_{|S|} \) and \( j_{|N\setminus S|} \) are allowed to transfer income given the core constraints, and since the following four coalitions \( \{i_1, j_{|N\setminus S|}\}, \{i_2, j_{|N\setminus S|}\}, \{i_1, i_{|S|}\}, \{i_2, i_{|S|}\} \) have zero excess at \( x \), any transfer of income from \( i_{|S|} \) to \( i_1 \) (\( i_2 \)) must be offset by an equally large transfer from \( j_{|N\setminus S|} \) to
Assume otherwise, i.e.
\[ W \] is sufficient to show that if there exists an allocation \( S \) contradicting \( \phi \); \( \Gamma = [i] \) for all \( i \); \( f \) is neither a payer nor a receiver.

\[ \Gamma = [i] \text{ for } i \in N \text{.} \]

\[ \text{is neither a payer nor a receiver.} \]

\[ \text{agent } f \text{ is both a payer and a receiver.} \]

Define \( \gamma \) following notation and definitions: Let \( \gamma_{ij} \) denote a bilateral transfer from agent \( i \) to agent \( j \). A transfer matrix is a non-negative \( |N| \times |N| \) matrix \( \Gamma = [\gamma_{ij}]_{i,j \in N} \) where if \( \gamma_{ij} > 0 \) there is no \( j' \) such that \( \gamma_{j'i} > 0 \) (no agent is both a payer and a receiver). Hence, if \( \gamma_{ij} > 0 \) then \( \gamma_{ji} = 0 \) (if agent \( i \) transfers income to agent \( j \) then agent \( j \) does not transfer income to agent \( i \)) and \( \gamma_{ii} = 0 \) for all \( i \) (no agent transfers income to himself). An agent that is neither a payer nor a receiver is called unaffected.
For any \( x, y \in \mathbb{R}^N \), with \( x(N) = y(N) \), there exists a transfer matrix leading from \( x \) to \( y \) as can be verified, and we will use the notation \( \Gamma^{xy} \) to denote such a transfer matrix. Let \( \varepsilon_i(\Gamma) \) be the net gain to agent \( i \) induced by the transfer matrix \( \Gamma = [\gamma_{ij}]_{i,j \in N} \), i.e. \( \varepsilon_i(\Gamma) = \sum_{j \neq i} (\gamma_{ji} - \gamma_{ij}) \). We say that two transfer matrices \( \Gamma \) and \( \Gamma' \) are equivalent if the net gains are identical, written \( \Gamma \sim \Gamma' \) in the following.

**Proof of Theorem 2:** As noted in the main text ‘\( \phi^W \) is \( S \)-maximal’ implies ‘\( \phi^W \) is \( S \)-monotonic’. Thus, we focus on the reverse implication: ‘\( \phi^W \) is \( S \)-monotonic’ implies ‘\( \phi^W \) is \( S \)-maximal’.

Our proof strategy is to show that if \( \phi^W(v) \) is not \( S \)-maximal for some game \( v \), then we can construct another game \( \tilde{v} \), which can be used to provoke a contradiction with \( S \)-monotonicity. In order to provoke such a contradiction, we will assume that there exists \( W \) where \( \phi^W \) is \( S \)-monotonic but not \( S \)-maximal. Hence, we assume that there is \( v \) such that \( \phi^W(v) \) does not maximize aggregate income to coalition \( S \) given the core constraints (and we will assume that \( S \) is a zero excess coalition at \( \phi^W(v) \)). A game \( w \) is defined by adding \( \varepsilon \) to \( v(S) \) (i.e. \( w(S) = v(S) + \varepsilon \) and \( w(T) = v(T) \) otherwise), where \( \varepsilon \) will be chosen such that the core of \( w \) is non-empty, and we have \( \sum_{i \in S} \phi_i^W(w) - \sum_{i \in S} \phi_i^W(v) = \varepsilon \). We then construct a game \( \tilde{v} \) with the properties that \( \phi^W(\tilde{v}) \in C(\tilde{v}) \) and, if \( \tilde{w} \) is defined by adding \( \varepsilon \) to \( \tilde{v}(S) \), \( C(\tilde{w}) \) is a singleton (such that the change in social welfare when going from \( \phi^W(\tilde{v}) \) to \( \phi^W(\tilde{w}) \) can easily be assessed). In particular, the games \( \tilde{v} \) and \( \tilde{w} \) will be specified such that one of the agents in \( S \) has strictly higher income in \( \phi^W(\tilde{v}) \) than in \( \phi^W(\tilde{w}) \).
**Step 1.** (Hypothesis). Suppose that for some strictly concave and differentiable social welfare function $W$ that $\phi^W$ is $S$-monotonic but not $S$-maximal, i.e. there exists $v \in B$ such that $\sum_{i \in S} \phi_i^W(v) < \max_{x' \in C(v)} x'(S)$.

Let $x = \phi^W(v)$. Without loss of generality we will assume that $x(S) = v(S)$. Indeed, if $x(S) > v(S)$ we could define the game $v'$ by $v'(S) = x(S)$ and $v'(T) = v(T)$ otherwise, and replace $v$ with $v'$. (Note that since $\phi^W$ satisfies independence of irrelevant core allocations, $\phi^W(v) = \phi^W(v')$, and thus we have $\sum_{i \in S} \phi_i^W(v') < \max_{x' \in C(v')} x'(S)$.)

Furthermore, we notice that by independence of irrelevant core allocations and continuity, for any $\varepsilon$ where $0 < \varepsilon < \max_{x' \in C(v)} x'(S) - \sum_{i \in S} \phi_i^W(v)$ and $w$ defined by $w(S) = v(S) + \varepsilon$ and $w(T) = v(T)$ otherwise, we have $\sum_{i \in S} \phi_i^W(w) = w(S)$. To see this, assume that there is an $\varepsilon$, $0 < \varepsilon < \max_{x' \in C(v)} x'(S) - \sum_{i \in S} \phi_i^W(v)$, for which $\sum_{i \in S} \phi_i^W(w) > w(S)$. For a core allocation method $\phi$ satisfying independence of irrelevant core allocations and continuity, Arin et al. [4, Lemma 3] has established that if $\phi(w) \in C(v)$ and if $w(T) = v(T)$ for all $T$ for which $w(T) = \sum_{i \in T} \phi_i(w)$, then $\phi(v) = \phi(w)$. In our case, those $T$ for which $w(T) = \sum_{i \in T} \phi_i^W(w)$ cannot be equal to $S$ by the preceding hypothesis, so for such $T$, $w(T) = v(T)$ by definition of $w$. Therefore, $\phi^W(v) = \phi^W(w)$, which contradicts $\sum_{i \in S} \phi_i^W(v) = v(S) < w(S) < \sum_{i \in S} \phi_i^W(w)$.

**Step 2.** (Transfers). Let $y = \phi^W(w)$, and let $\Gamma^{xy} = [\gamma_{ij}^{xy}]_{i,j \in N}$ be a transfer matrix leading from $x$ to $y$. Thus, $y_i = x_i + \sum_{j \neq i} (\gamma_{ji}^{xy} - \gamma_{ij}^{xy})$.

Since $\phi^W(v)$ is a continuous function of $v$ (see Hougaard et al. [14]), for any $\lambda > 0$, in Step 1 the amount $\varepsilon > 0$ can be chosen sufficiently small such that any bilateral transfer in $\Gamma^{xy}$ is smaller than $\lambda$. In particular, let
\[ \mu = \min\{x(T) - v(T)|T \subseteq N \text{ and } x(T) - v(T) > 0\}, \text{ and let } \lambda = \mu / \frac{|N|!}{2(|N| - 2)!} \]

then we can choose \( \varepsilon \) sufficiently small, such that any bilateral transfer in \( \Gamma^{xy} \) is smaller than \( \lambda \). Since \( \frac{|N|!}{2(|N| - 2)!} \) is an upper limit on the number of possible bilateral transfers among \( |N| \) agents then for an arbitrary coalition \( T \) with positive excess at \( x \) given \( v \) the change in excess following a transfer matrix \( \Gamma \leq \Gamma^{xy} \) is less than \( \lambda \frac{|N|!}{2(|N| - 2)!} = \mu \).

This means that we can choose \( \varepsilon \) such that if a coalition \( T \) has nonpositive excess after imposing a transfer matrix \( \Gamma \leq \Gamma^{xy} \), then \( v(T) = x(T) \). Indeed, if a coalition \( T \) has nonpositive excess in \( v \) after applying such a transfer matrix \( \Gamma \) to \( x \), then \( x(T) - v(T) < \mu \). This means by definition of \( \mu \) that \( x(T) - v(T) \leq 0 \). But \( x \in C(v) \), so \( x(T) - v(T) = 0 \). This property will be used in Step 8. In the following \( \varepsilon \) is fixed and chosen as above.

**Step 3.** (Categorizing receivers and payers). Since \( \phi^W \) is \( S \)-monotonic, any agent in \( S \) is either a receiver or unaffected. The complement \( N \setminus S \) may consist of both payers, receivers and unaffected agents. Since \( y(S) > x(S) \) there is at least one payer in \( N \setminus S \). If there are no receivers in \( N \setminus S \) (in which case all receivers are in \( S \) and all payers are in \( N \setminus S \)) we have by Theorem 1 that \( W(y) > W(x) \), contradicting \( \phi^W(v) = x \) (since \( y \in C(v) \)). Thus, there is at least one receiver in \( N \setminus S \). (Note that this proves the theorem for the special case \( |N \setminus S| = 1 \), since the single agent in \( N \setminus S \) cannot be both payer and receiver).

Given the transfer matrix \( \Gamma^{xy} \), let \( L \subseteq S \) be the set of receivers in \( S \), let \( H \subseteq N \setminus S \) be the set of receivers in \( N \setminus S \), and let \( K \subseteq N \setminus S \) be the set of payers in \( N \setminus S \). Further, let \( K' \subseteq K \) be the set of agents in \( K \) who transfer income to \( L \), and let \( K'' \subseteq K' \) be the set of agents who transfer income to \( H \).
It is readily verified that the transfer matrix $\Gamma^{xy}$ can be specified such that at most one agent in $K$ transfers income to both agents in $L$ and agents in $H$. In the following, we assume that $|K' \cap K''| \in \{0, 1\}$.

Let $\varepsilon_i = y_i - x_i = \sum_{j \neq i} (\gamma_{ji}^{xy} - \gamma_{ij}^{xy})$ be the net gain to agent $i$, induced by the transfer matrix $\Gamma^{xy}$. Thus, $\varepsilon_i < 0$ for $i \in K$, $\varepsilon_i > 0$ for $i \in L \cup H$, and $\varepsilon_i = 0$ otherwise.

If $K' \cap K'' \neq \emptyset$, let $j^*$ denote the agent in $K' \cap K''$, let $-\varepsilon_{j^*}^L > 0$ be the total amount transferred from $j^*$ to $L$ and let $-\varepsilon_{j^*}^H > 0$ be the total amount transferred from $j^*$ to $H$ specified by the transfer matrix $\Gamma^{xy}$ (so that $\varepsilon_{j^*}^L + \varepsilon_{j^*}^H = \varepsilon_{j^*} < 0$). If $K' \cap K'' = \emptyset$ we define $\varepsilon_{j^*}^H = \varepsilon_{j^*}^L = \varepsilon_{j^*} = 0$. Thus, $\varepsilon = \sum_{i \in L} \varepsilon_i = -\sum_{i \in K'} \varepsilon_i + \varepsilon_{j^*}^H$.

**Step 4.** (Decomposition of the transfer leading from $x$ to $y$). We consider the following decomposition of the transfer matrix $\Gamma^{xy}$: First, a transfer matrix $\Gamma^{x0}$ which involves all the bilateral transfers from $K'$ to $L$ leading from allocation $x$ to an allocation called $x^0$, and, second, a transfer $\Gamma^{x0y}$ which involves all the remaining bilateral transfers from $K''$ to $H$ leading from $x^0$ to $y$. Thus, $\Gamma^{xy} = \Gamma^{x0} + \Gamma^{x0y}$ (see Figure 1).

**FIGURE 1** (Transfers leading from $x$ to $y$) HERE

**Step 5.** (Minimality and uniqueness of the transfer matrix $\Gamma^{x0y}$). We say that $\Gamma^{x0y}$ is *minimal* if, given $x^0$, $\Gamma^{x0y}$ cannot be replaced by some other transfer matrix $\Gamma$, transferring income from $K''$ to $H$, such that $\varepsilon_i(\Gamma) \leq \varepsilon_i(\Gamma^{x0y})$ for all $i \in H$ and $\varepsilon_i(\Gamma) < \varepsilon_i(\Gamma^{x0y})$ for at least one $i \in H$, without some core constraint in $C(w)$ being violated at the allocation obtained.
Consider a minimal transfer matrix $\Gamma_{\text{min}}$, which transfers income from $K''$ to $H$, and for which $\varepsilon_i (\Gamma_{\text{min}}) \leq \varepsilon_i (\Gamma x^0 y)$ for all $i \in H$ (there exists such transfer matrix since the core is a closed set). Moreover, let $\overline{y}$ denote the allocation obtained after applying $\Gamma_{\text{min}}$ to $x^0$. That is, $\overline{y}_i = x^0_i + \varepsilon_i (\Gamma_{\text{min}})$ for each $i$. Note that $\overline{y}_i = y_i$ if $i \notin K'' \cup H$. Now, define a game $v$ by (a) $v(\{i\}) = y_i$ for $i \in K$, (b) $v(\{i\}) = x_i$ for $i \in N \setminus K$, (c) $v(N \setminus \{i\}) = v(N) - \overline{y}_i$ for $i \in S \cup H$, (d) $v(H \cup K'') = x(H \cup K'') + \varepsilon_j (\Gamma x^0)$, (e) $v(N \setminus \{i\}) = v(N) - x_i$ for $i \in K''$, (f) $v((H \cup K'') \setminus K') = x((H \cup K'') \setminus K')$ and (g) $v(T) = v(T)$ otherwise.

We have defined $v$ such that $v(T) \geq v(T)$ for all $T \subseteq N$, and $x \in C(v)$. Indeed, note that $\overline{y}_i \leq x_i$ for $i \in K$, $\overline{y}_i \geq x_i$ for $i \in S \cup H$, and $\varepsilon_j (\Gamma x^0) < 0$. Thus $x = \phi(v)$ by independence of irrelevant core allocations. Note also that we have defined $v$ such that $y \in C(v)$. Let $w$ be the game defined by adding $\varepsilon$ to $v(S)$. Since $\overline{y}(S) = w(S)$ we have $\overline{y} \in C(w)$. We will now show that $C(w) = \{\overline{y}\}$, which in this case corresponds to showing that, given $x^0$, there is a uniquely determined transfer matrix (up to equivalent transfer matrices) leading to an allocation in $C(w)$. Indeed, for this we make the following observations. Let $\overline{x}$ be an arbitrary allocation in $C(w)$. Since by (a) and (b) each agent in $N \setminus (L \cup K'')$ individually has zero excess at $x^0$, we know that any transfer matrix $\Gamma x^0 \overline{x}$ is such that each payer belongs to the set $L \cup K''$. Moreover, since by (d) coalition $H \cup K''$ has zero excess at $x^0$ we can choose $\Gamma x^0 \overline{x}$ such that no agent in $K''$ pays to an agent outside $H \cup K''$. This implies that no agent in $L$ is a payer: if some agent in $L$ is a payer, since $x^0(S) = w(S)$, some other agent in $S$ must be a receiver — but this contradicts (c). Thus each payer is in $K''$. 


By (e), no agent in $K'' \setminus \{j^*\}$ is a receiver. Since $\pi((H \cup K'') \setminus K') \geq \pi((H \cup K'') \setminus K')$, condition (f) implies that the total amount $H$ receives is at least as much as the total amount $K'' \setminus \{j^*\}$ pays. So, as the only remaining possible payer to $H$, $j^*$ cannot be a receiver. Thus the transfer matrix $\Gamma^x\tilde{z}$ consists only of bilateral transfers from agents in $K''$ to agents in $H$. By (c), given $x^0$, no agent in $H$ can receive more from $K''$ than according to $\Gamma^{\min}$. Hence, since $\Gamma^{\min}$ was minimal among transfer matrices transferring income from $K''$ to $H$, $\Gamma^{\min}$ is the only transfer matrix (up to equivalent transfer matrices) transferring income from $K''$ to $H$ which leads to an allocation in $C(\bar{w})$. Therefore, $\Gamma^{\min} \sim \Gamma^{x^0\tilde{z}}$ and we are done.

We assume in the following (replacing $v$ with $\bar{v}$ if necessary) that $v$ is a game where, given $\Gamma^{xx^0}$, the transfer matrix $\Gamma^{x^0y}$ is minimal and uniquely determined up to equivalent transfer matrices. This property will be used in Step 8. Note however that this means that the property of Step 2 (i.e. a coalition $T$ that has negative or zero excess after $\Gamma \leq \Gamma^{xy}$ applied to $x$ has zero excess at $x$) does no longer hold in general, but holds for any $T$ that is an instance of (g).

**Step 6.** (Definition of the game $\tilde{v}$). We define a new game $\tilde{v}$, which we shall use to contradict $S$-monotonicity. Following Step 4, given $x$, a transfer matrix $\Gamma^{xz\tilde{z}}$ that gives coalition $S$ an amount $\varepsilon$ resulting in an allocation $\tilde{z}$ can be decomposed as $\Gamma^{xz\tilde{z}} = \Gamma^{xz\tilde{z}^0} + \Gamma^{x^0\tilde{z}}$ where $\Gamma^{xz\tilde{z}^0}$ is involving bilateral transfers from $N \setminus S$ to $S$ as well as bilateral transfers internally in $S$, and $\Gamma^{x^0\tilde{z}}$ involves bilateral transfers internally in $N \setminus S$. We will construct the game $\tilde{v}$, such that $x \in C(\tilde{v})$, and such that $\Gamma^{xz\tilde{z}}$ will only contain bilateral transfers from $N \setminus S$ to one particular agent in $S$ ($i_1$) plus an internal bilateral transfer.
between two other agents in $S$ (from $i_a$ to $i_b$). Moreover, all the remaining bilateral transfers given by $\Gamma^{\alpha\beta}$ will be equivalent to the transfer matrix $\Gamma_{xy}$.

Let $i_1 \in L$ and let $i_a, i_b \in S \setminus \{i_1\}$. We now define the game $\tilde{v}$ by the conditions (i)-(viii) below. The role of each of the conditions will be observed in the subsequent steps.

(i). $\tilde{v}(\{j\}) = x_j$ for $j \in N \setminus (K \cup \{i_a\})$ and $\tilde{v}(\{i_1\} \cup K \cup H) = x(\{i_1\} \cup K \cup H)$.

(ii). $\tilde{v}(\{i_1\} \cup K') = x(\{i_1\} \cup K') + \varepsilon_{j^*}$ and $\tilde{v}(H \cup K'') = x(H \cup K'') + \varepsilon_{j^*}$.

(iii). $\tilde{v}(\{i_a\}) = x_{i_a} - \varepsilon$, $\tilde{v}(\{i_a, i_b\}) = x(\{i_a, i_b\})$, $\tilde{v}(\{i_a, i_1\}) = x(\{i_a, i_1\})$ and $\tilde{v}(\{i_b\} \cup K \cup H) = x(\{i_b\} \cup K \cup H)$.

(iv). $\tilde{v}(\{j\}) = y_j$ for $j \in K$.

(v). $\tilde{v}(N \setminus \{i\}) = v(N) - y_i$ for $i \in H$, $\tilde{v}(N \setminus \{i\}) = x(N \setminus \{i\})$ for $i \in N \setminus (H \cup \{i_b\} \cup \{i_1\})$, $\tilde{v}(N \setminus \{i\}) = v(N) - x_i - \varepsilon$ for $i \in \{i_1, i_b\}$, and $\tilde{v}((H \cup K'') \setminus K') = x((H \cup K'') \setminus K')$.

(vi). $\tilde{v}(N) = v(N)$ and $\tilde{v}(S) = v(S)$.

(vii). For all other $U \subseteq N \setminus S$ if there is $T'$ such that $T' \setminus S = U$ and $x(T') = v(T')$ define $\tilde{v}(U) = x(U) - \min\{y(T \cap S) - x(T \cap S) \mid v(T) = x(T) \text{ and } T \setminus S = U\}$. See also Figure 2.

(viii). $\tilde{v}(T) = v(T) - 2\varepsilon$ for all other $T$.

**FIGURE 2** (*Defining a new game $\tilde{v}$*) HERE

**STEP 7.** (Properties of allocations in $C(\tilde{v})$). It follows from the construction (i)-(viii) of $\tilde{v}$ that $x \in C(\tilde{v})$ (as one can check). Let $\tilde{w}$ be the game defined
by $\tilde{w}(S) = \tilde{v}(S) + \varepsilon$ and $\tilde{w}(T) = \tilde{v}(T)$ otherwise.

Now, consider any $\tilde{z} \in C(\tilde{w})$, and a transfer matrix $\Gamma^{xz}$. By (vi), the total payment from $N \setminus S$ to $S$ associated with $\Gamma^{xz}$ must be at least $\varepsilon$. Further, notice that if $T$ is a coalition for which $v(T) = x(T)$ and $T \setminus S = N \setminus S$ we have $T \supseteq L$ since $y \in C(w)$. By (vii) it follows, when $U = N \setminus S$ and $T' = N$, that $\tilde{v}(N \setminus S) = x(N \setminus S) - (y(L) - x(L)) = x(N \setminus S) - \varepsilon$. Thus, the total payment from $N \setminus S$ to $S$ associated with $\Gamma^{xz}$ must be $\varepsilon$ exactly, i.e., $\tilde{z}(S) = w(S)$. In particular, we can choose $\Gamma^{xz}$ such that there is a transfer of $\varepsilon$ from $N \setminus S$ to $S$ and possibly some transfers internally in $S$ and $N \setminus S$ respectively (but no transfers from $S$ to $N \setminus S$). In fact, by (i) we are allowed to choose $\Gamma^{xz}$ such that agent $i_1$ is the only receiver in $S$ from agents in $N \setminus S$ (with a net gain of $\varepsilon$ to $i_1$) and furthermore, by (ii), such that all bilateral transfers from $N \setminus S$ to $S$ are taken from $K'$ only (as in $\Gamma^{xy}$).

By (iii), a transfer of $\varepsilon$ from $N \setminus S$ to $i_1$ requires another transfer of $\varepsilon$ internally in $S \setminus \{i_1\}$ from $i_a$ to $i_b$. The incomes to the remaining agents in $S$ cannot be affected because each of them individually has zero excess at $x$ by (i).

Let $\tilde{x}^0$ denote the allocation resulting from implementing the bilateral transfers in $\Gamma^{xz}$ except for the bilateral transfers internally in $N \setminus S$, and denote by $\Gamma^{x0} \tilde{x}$ the associated transfer matrix. Note that in $\Gamma^{x0} \tilde{x}$ every agent in $K$ pays the same amount as in $\Gamma^{x0}$, but there may be a change in who receives income within $S$. Let $\Gamma^{0} \tilde{x}$ be a transfer matrix consisting of the remaining bilateral transfers (involving bilateral transfers internally in $N \setminus S$). Thus, $\Gamma^{xz} = \Gamma^{x0} + \Gamma^{0} \tilde{x}$ and $\tilde{x}^0$ is given by
\[
\tilde{x}_i^0 = \begin{cases} 
    x_i, & \text{if } i \in N \setminus (K' \cup \{i_1, i_a, i_b\}), \\
    x_i + \varepsilon, & \text{if } i = i_1, i_b, \\
    x_i - \varepsilon, & \text{if } i = i_a, \\
    y_i, & \text{if } i \in K' \setminus \{j^*\}, \\
    y_i - \varepsilon_{\mathcal{H}}^j, & \text{if } i = j^*. 
\end{cases}
\]

**Step 8.** \(C(\tilde{w})\) is a singleton. Let \(\tilde{y}\) be the allocation resulting from transfer matrix \(\Gamma x_0^0 y\) applied to \(\tilde{x}_0\). Thus, \(\Gamma x_0^0 \tilde{y} \sim \Gamma x_0^0 y\). We claim that \(\tilde{y} \in C(\tilde{w})\). To verify this claim, let \(Q\) be an arbitrary coalition.

Suppose that \(Q\) is an instance of one of the conditions (i)-(vi) for the definition of \(\tilde{v}\). Then it follows directly from the specification of \(\tilde{x}_0\) (see Step 7) and the definition of \(\Gamma x_0^0 \tilde{y}(\sim \Gamma x_0^0 y)\) that \(\tilde{y}(Q) \geq \tilde{v}(Q)\). We therefore focus on cases where \(Q\) is an instance of (vii) or (viii).

Suppose that \(Q\) is an instance of (vii). Then \(Q \subseteq N \setminus S\) and thus \(\tilde{y}(Q) = y(Q)\). In particular, there is some \(T\), where \(T \setminus S = Q\) and \(v(T) = x(T)\) for which \(\tilde{v}(Q) = x(Q) - (y(T \cap S) - x(T \cap S))\). Thus, \(\tilde{y}(Q) - \tilde{v}(Q) = y(Q) - x(Q) + y(T \setminus Q) - x(T \setminus Q) = y(T) - x(T) = y(T) - v(T) \geq 0\).

Suppose finally that \(Q\) is an instance of (viii). By the construction of \(\tilde{y}\), we have \(\tilde{y}(Q) \geq y(Q) - 2\varepsilon\). Since \(y(Q) \geq v(Q)\) and \(\tilde{v}(Q) = v(Q) - 2\varepsilon\), we have \(\tilde{y}(Q) \geq y(Q) - 2\varepsilon \geq v(Q) - 2\varepsilon = \tilde{v}(Q)\). This verifies our claim; i.e. \(\tilde{y} \in C(\tilde{w})\).

Next, we aim to show that \(\tilde{z} = \tilde{y}\), and therefore that \(C(\tilde{w}) = \{\tilde{y}\}\). By Step 7, it is sufficient to verify that, given \(\tilde{x}_0\), any two transfer matrices that lead from \(\tilde{x}_0\) to an allocation in \(C(\tilde{w})\) are equivalent. It therefore remains to show that \(\Gamma \tilde{x}_0^0 \tilde{z} \sim \Gamma x_0^0 y\) \((\sim \Gamma x_0^0 \tilde{y})\).
It follows from conditions (i) and (iv) that all payers in $\Gamma^{\tilde{x}_0\tilde{z}}$ are from $K''$ since by (i) all members of $N \setminus \{i_1, i_b\} \cup K$ individually have zero excess at $\tilde{x}_0$ and by (iv) all agents in $K \setminus K''$ have zero excess at $\tilde{x}_0$. As in step 5 in which we proved that no agent in $K''$ is a receiver in $\Gamma^{x_0z}$, condition (v) here implies that no agent in $K''$ is a receiver in $\Gamma^{\tilde{x}_0\tilde{z}}$. Moreover, it follows from condition (ii) that all receivers are from $H$ since at $\tilde{x}_0$, we have that $H \cup K''$ is zero excess.

By (iv), $\tilde{z}_j \geq y_j$ for all $j \in K$ and since $x_0^j = \tilde{x}_0^j$ for all $j \in K$ we have $x_0^j - y_j \geq \tilde{x}_0^j - \tilde{z}_j$ for all $j \in K$. By (v), $\tilde{z}_j \leq y_j$ for all $j \in H$ and hence $y_j - x_0^j \geq \tilde{z}_j - \tilde{x}_0^j$ for all $j \in H$. Thus, we can assume $\Gamma^{\tilde{x}_0\tilde{z}} \leq \Gamma^{x_0y}$.

We now claim that a transfer matrix $\Gamma^{\tilde{x}_0\tilde{z}}$ applied to $\tilde{x}_0$ leads to an allocation in $C(\tilde{w})$ only if $\Gamma^{\tilde{x}_0\tilde{z}}$ applied to $x_0$ leads to an allocation in $C(w)$. For this, let the allocation obtained after applying $\Gamma^{\tilde{x}_0\tilde{z}}$ to $x_0$ be denoted by $q$, and suppose that $q \notin C(w)$. Then, since $\Gamma^{\tilde{x}_0\tilde{z}} \leq \Gamma^{x_0y}$, there is a coalition $T$ where $T \cap H \neq \emptyset$ and $q(T) < v(T) = w(T)$. It is clear from inspection of conditions (a)-(g) in Step 5, that $T$ cannot be an instance of (a)-(f). Since $\Gamma^{\tilde{x}_0\tilde{z}} \leq \Gamma^{x_0y}$, we have $\Gamma^{x_0x} + \Gamma^{x_0\tilde{z}} \leq \Gamma^{x_0x} + \Gamma^{x_0y} = \Gamma^{x_0y}$. Thus, by Step 2 and the observation at the end of Step 5, we have $x(T) = v(T)$. By condition (vii) of $\tilde{v}$, for $U = T \setminus S$, we have $\tilde{x}_0(U) - \tilde{v}(U) \leq \tilde{x}_0(U) - (x(U) - (y(T \setminus U) - x(T \setminus U))) = \tilde{x}_0(U) + y(T \setminus U) - x(T) = x_0(T) - v(T)$. Since $\tilde{x}_0(U) - \tilde{z}(U) = x_0(T) - q(T)$ we therefore have $\tilde{z}(U) - \tilde{v}(U) \leq q(T) - v(T) < 0$, i.e. $\tilde{z} \notin C(\tilde{w})$.

Thus, $\Gamma^{\tilde{x}_0\tilde{z}} \sim \Gamma^{x_0y}$ since $\Gamma^{x_0y}$ was uniquely determined up to equivalent transfer matrices (see Step 5), and we have verified that $C(\tilde{w})$ is a singleton. Hence, $\tilde{y} = \varphi^W(\tilde{w})$. See also Figure 3.
FIGURE 3 (Violation of $S$-monotonicity in $\tilde{v}$) HERE

STEP 9. (Assessment of the transfers leading from $\tilde{x}$ to $\tilde{y}$). Let $\tilde{x} = \varphi^W(\tilde{v})$. We will show that for a suitable choice of $i_1, i_a$ and $i_b$ in Step 6, $W(\tilde{y}) < W(\tilde{x})$.

Since $\tilde{x}$ is not specified we utilize the fact that $\Gamma_{\tilde{x}\tilde{y}} \sim \Gamma_{\tilde{xx}} + \Gamma_{\tilde{xy}} + \Gamma_{\tilde{x}y}$

By construction of $\tilde{v}$, $x \in C(\tilde{v})$ and hence $W(\tilde{x}) \geq W(x)$.

Given $i \in L$, let $\Gamma^{xx_i}$ denote a transfer matrix which is similar to $\Gamma^{xy}$ except that all the bilateral transfers from $K'$ to $L$ are redirected such that $i$ receives everything from $K'$ ($\varepsilon$ in total). Let $x^i$ denote the allocation obtained from applying $\Gamma^{xx_i}$ to $x$ (i.e., $x^i_j = y_j$ for $j \notin L$, $x^i_j = x_j + \varepsilon$ for $j = i$ and $x^i_j = x_j$ for $j \in L \setminus \{i\}$). Since $W$ is strictly concave, we can choose $i \in L$ such that $W(x^i) \leq W(y)$. (Indeed, if for any $i \in L$ we had $W(x^i) > W(y)$ this would contradict that $W$ is strictly concave since $y$ is a convex combination of $\{x^i\}_{i \in L}$.) Let $i_1$ be such an $i \in L$. Now, consider the allocation $x^{i_1}$. Since $W$ is strictly concave, we can choose elements $i_a$ and $i_b$ in $S \setminus \{i_1\}$ such that when $\tilde{y}$ is the allocation obtained from $x^{i_1}$ by transferring an amount $\varepsilon$ from $i_a$ to $i_b$ we get $W(\tilde{y}) \leq W(x^{i_1})$. Thus $W(\tilde{y}) \leq W(y)$ and thereby $W(\tilde{y}) - W(\tilde{x}) \leq W(y) - W(x) < 0$. It follows that $\tilde{x}(S) < \tilde{y}(S)$.

STEP 10. (Contradicting $S$-monotonicity). We have $\tilde{x}_{i_a} \geq \tilde{y}_{i_a}$, since $\tilde{v}(\{i_a\}) = x_{i_a} - \varepsilon = y_{i_a}$. Further, we cannot have $\tilde{x}_{i_a} = \tilde{y}_{i_a}$, since by (iii) this would imply that $\tilde{x}_{i_b} \geq \tilde{y}_{i_b}$ and $\tilde{x}_{i_1} \geq \tilde{y}_{i_1}$ and hence by (i) we would have $\tilde{x}(S) \geq \tilde{y}(S)$ which is a contradiction (cf. Step 9). Thus, $\tilde{x}_{i_a} > \tilde{y}_{i_a}$, violating $S$-monotonicity and hence contradicting the hypothesis of Step 1. □

PROOF OF COROLLARY 1: Suppose that $S$ and $S'$ are respectively $S$- and $S'$-
maximal coalitions. We claim that either $S \subseteq S'$ or $S' \subseteq S$. Indeed, suppose that we have $i \in S \setminus S'$ and $j \in S' \setminus S$. Define $v$ as $v(\{i, j\}) = v(N) = 1$ and $v(T) = 0$ otherwise. Then, clearly $x(S)$ and $x(S')$ cannot both be maximized over $C(v)$, proving the claim. Thus, if $|S| = |S'|$ then $S = S'$. 

**Proof of Theorem 3:** Since $W$ is strictly concave the set of gradients is an open set in $\mathbb{R}^N$ (see, e.g., Rockafellar [25], p. 258). In particular, there is $x \in \mathbb{R}^N$ such that the coordinates $W'_i(x)$, for $i \in N$, possibly after a renumbering, satisfy

$$W'_i(x) < W'_{n-2}(x) < W'_{n-1}(x) < W'_n(x),$$

for $i = 1, \ldots, n - 3$. Now, for $\lambda > 0$, define the game $v$ as follows: $v(\{i\}) = x_i$ for all $i \neq n - 2$, $v(\{n - 2, n - 1\}) = x_{n-2} + x_{n-1}$, $v(\{n - 2, n\}) = x_{n-2} + x_n$, $v(N) = x_1 + \ldots + x_n$, and $v(T) = x(T) - \lambda$, otherwise. We then have $C(v) = x = \phi^W(v)$: Indeed, only the income of agent $n - 2$ can be reduced without violating the constraints of the singleton coalitions, but any such reduction must be matched by an equal increase in the incomes of both $n - 1$ and $n$, which is impossible.

Consider a game $w$ where $w(N) = v(N) + \varepsilon$ for $\varepsilon > 0$, and $w(T) = v(T)$ for $T \neq N$. Since $W$ is a differentiable and concave, the functions $W'_i(x)$ are continuous (see, e.g., Rockafellar, p. 246). Hence there is $\varepsilon$ sufficiently small such that, if we write $y = \phi^W(w)$, we have

$$W'_i(y) < W'_{n-2}(y) < W'_{n-1}(y) < W'_n(y),$$

for $i = 1, \ldots, n - 3$. First, suppose that $y_{n-2} > x_{n-2}$. Since $W'_{n-2}(y) < W'_n(y)$ there is a bilateral transfer (possibly smaller than $y_{n-2} - x_{n-2} \leq \varepsilon$) from
agent $n - 2$ which strictly increases total welfare $W$. Further, given $y$, any bilateral transfer up to an amount $y_{n-2} - x_{n-2}$ is feasible in the core of $w$, and we have obtained a contradiction. Second, suppose that $y_{n-2} = x_{n-2}$. Then, since agent $n$ has the highest marginal welfare and all agents different from $n - 2$ have zero excess at $x$, welfare is maximized by allocating the additional amount $\varepsilon$ to the agent with highest marginal welfare, i.e.

$$\phi^W(w) = (x_1, \ldots, x_{n-2}, x_{n-1}, x_n + \varepsilon).$$

But then we could further increase total welfare by a reallocation of $\varepsilon$ from agent $n - 2$ to agent $n - 1$, a contradiction. Thus, we conclude that $y_{n-2} < x_{n-2}$.

\[\square\]

**References**


\[ S \quad N \setminus S \]

- \( S \)
  - \( L \)
    - \( i_a \)
    - \( i_b \)
    - \( i_1 \)
  - \( T \)
  - \( U \)
  - \( H \)

- \( N \setminus S \)
  - \( K' \)
  - \( K'' \)

Dotted lines indicate connections or relationships between the sets.
\[ S, \quad N \setminus S, \quad L, \quad K', \quad K'', \quad H \]

Symbols:
- \( i_a, i_b \)
- \( i_1 \)
- \( \varepsilon \)
- \( \Gamma^{xx^0} \)
- \( j^* \)
- \( \Gamma^{\bar{x}^0 \bar{y}} \sim \Gamma^{x^0 y} \)