

Capabilities and Nontrivial Indices of Health-Related Quality of Life:

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Abstract

Health economic assessment has become a major field of application of Health Economics, and the use of utility representation of indices of health-related quality of life, typically in the form of QALYs, is an important ingredient in such assessments. In spite of its widespread application, the foundations of indexes representing preferences over health states may still need some clarifications. A possible approach towards a theoretical background for measuring quality of life might be based on the notion of characteristics considered as the totality of functionings available to an individual. For the concept of capabilities to be useful, an index representing preferences over capabilities should be more just than a weighted average of the assessments over its constituent functionings. In the present paper we investigate whether such an index is possible, and the answer is affirmative although with some qualifications and limitations.

1. Introduction

Since its appearance, the capability approach introduced by Sen (1980, 1985) has been applied in several different fields of economics, such as poverty and inequality and health. To introduce capabilities, one has to start with Sen's more fundamental notion of functionings. Sen distinguishes several forms of relationship between an individual and a good (such as for example a car): a good is an item (the car), utility is the benefit derived from using this item (pleasure from driving), characteristics are qualities of the goods (transport), and the functioning relates to use of the car (moving around). Functionings are important, but what matters more is the capability, the extent to which the individual can function in a particular way, or whether or not she chooses a particular functioning. An approach to the measurement

of standard of living using functionings as a basic concept can be found in Gaertner and Xu (2006).

The literature on capabilities and their possible application in different branches of economics is considerable. For a discussion of characteristics in the context of health economics, see e.g. Anand (2005, 2005a), Cookson (2005), Coast, Smith and Lorgelly (2008), Coast (2009). A recurrent theme in most contributions is the assertion that the capability approach extends the way of dealing with allocative justice and equity beyond the limits posed by traditional welfare theory insisting on individual utility assessments of social states as the basic notion. Although this so-called extra-welfarism has been criticised as being essentially a restructuring of the welfarism approach which it strives to avoid (see Birch and Donaldson (2003)), the focus on sets of functionings rather than on allocations and their utility assessments by individuals suggest that the extra-welfarists may have a point.

Even though the major part of this literature is discussing alternative ways of interpreting and understanding characteristics, there have also been attempts at a more formal treatment. Several studies (Herrero (1996),(1997), Herrero, Iturbe-Ormaetxe and Nieto (1998)) have applied the capability approach to problems of equality, whereby capabilities are taken as depending on the initial state, formalized as an allocation of goods. We shall expand slightly on this formulation, using the concept of characteristics introduced by Lancaster (see, e.g. Lancaster (1971)). We assume that individuals have access to a (household) technology for transforming initial bundles of goods into characteristics. Each such transformation would then correspond to a functioning, but the set of functionings now depend in a rather simple way on the initial bundle, since they will be all the processes which are feasible and use the available endowment as inputs. This approach to capabilities is admittedly simplistic, in particular if we add standard assumptions on the overall technology, but this very simplicity has the advantage that it makes it possible to concentrate on what may or may not be the essential nature of capabilities and whether a theory of capabilities can achieve something which was not possible in the classical approach.

In the present paper, we consider the possibility of assigning an index to individual capabilities, so that one capability set is considered as better than another capability set by the individual considered if and only if it has a higher index value. Assigning utilities to preference relations over a given set, in our case a set of capabilities, is possible if the preference relation is a complete preorder, but not any utility will be of interest when dealing with capabilities. In order to capture the essence of this approach, the ranking of capabilities should not be *trivial* in the sense that it depends only on an underlying ranking of functionings, so that one capability is better than another if the best functioning in the first is better than the best functioning in the other one. For the capability approach to add something new to a traditional analysis of choice, the preferences must involve something more than just utility

maximization over functionings.

The paper is structured as follows. In Section 2, we consider individual preferences over capabilities, with a special view to application to health related quality of life, and in section 3, we discuss problems of allocation in society. Having found that the formalizations so far miss essential parts of the concept of capabilities, we proceed in Section 4 to consider the interpretation of capabilities as options which should be assessed according to the choices that may be made according to preferences on functionings that are not yet known. We conclude with a brief summary in Section 5.

2. Capability indices: The axiomatic approach

As is well-known, measuring individual health is both conceptually and technically complicated; indeed the very nature of health is open to some debate, and it has been suggested that the capability approach outlined above and the measurement of health as QALYs are closely related (see for example Cookson, 2005).

In the following we consider a given family \mathfrak{C} of capabilities, which technically are compact, convex, and comprehensive (meaning that if $x \in C$ and $y \in \mathbb{R}_+^L$ satisfies $y_h \leq x_h$ for $h = 1, \dots, L$, then $y \in C$) sets of functionings (interpreted as representing the activities open to an individual in a given health state). For our subsequent reasoning, it is convenient to assume that \mathfrak{C} is rich enough to contain some distinguished sets: First of all, for each $x \in \mathbb{R}_+^L$, C contains the set

$$x^{\leq} = \left[\{x\} - \mathbb{R}_+^L \right] \cap \mathbb{R}_+^L = \{x' \in \mathbb{R}_+^L \mid x'_h \leq x_h, h = 1, \dots, L\}.$$

Secondly, \mathfrak{C} contains $\{0\}$ and permits the operation of (Minkowski) weighted averages, i.e. if $C, C' \in \mathfrak{C}$ and $\lambda \in [0, 1]$, then the set

$$\lambda C + (1 - \lambda)C' = \left\{ y \in \mathbb{R}_+^L \mid y = \lambda x + (1 - \lambda)x', x \in C, x' \in C' \right\}$$

belongs to \mathfrak{C} as well. We shall say that a family \mathfrak{C} with these properties is *rich*.

This possibility of taking weighted averages of capability sets may not be too restrictive at this point, where we are only defining the set of all conceivable sets of functionings which in some situation might occur as a capability. However, since we shall use the averaging operation repeatedly in what follows, we add some comments: Taking a weighted average of two capabilities, which amounts to taking the set of all weighted averages (with the given weights) of functionings in the two sets, might be interpreted as sampling the functionings of the two sets with given probabilities, or alternatively, if we assume that a fraction λ of

the population has the capability C and the rest has C' , then $\lambda C + (1 - \lambda)C'$ is the average capability set available to this population.

The other assumptions on \mathfrak{C} are more standard, but here again some properties are assumed implicitly. Considering functionings of elements of \mathbb{R}^L already assumes that the vector space operations make sense, and the comprehensiveness assumption on capabilities implies that once an array of functionings is available, then all arrays using less of each functioning is available as well, something which may seem reasonable, but which will again play a certain role in the sequel.

We consider the situation where an individual ranking of capabilities has a utility representation. This implies that there is complete preorder \succeq on the sets $C \in \mathfrak{C}$; we let $>$ and \sim denote the associated strict order and indifference, respectively. For completeness of exposition, this is stated as a first axiom.

AXIOM 1 *The preference relation \succeq on the family \mathfrak{C} is a complete preorder, and it is continuous in the sense that $\{C' \in \mathfrak{C} \mid C \succeq C'\}$ and $\{C'' \in \mathfrak{C} \mid C'' \succeq C\}$ are closed¹ subsets of \mathfrak{C} .*

We shall consider in some more details the properties of this preorder which seem reasonable if it is to be represented by an index with QALY-like properties. Consider two alternative health states s_1 and s_2 for which the index I assigns the same value $I(s_1) = I(s_2)$, and let s'_1, s'_2 be some other states with the same value $I(s'_1) = I(s'_2)$, but not necessarily the same as $I(s_1)$ or $I(s_2)$. If a share λ of the population is in health state s_1 and $1 - \lambda$ is in health state s'_1 , then the index yields the same average value $\lambda I(s_1) + (1 - \lambda)I(s'_1)$ as when a share λ is in state s_2 and $1 - \lambda$ in state s'_2 . We assume that a similar result will emerge when using our operation of weighted averages on capabilities:

AXIOM 2 *Let (C_1, C_2) and (C'_1, C'_2) be pairs of elements of \mathfrak{C} with $C_1 \sim C_2$, $C'_1 \sim C'_2$, and let $\lambda \in [0, 1]$. Then*

$$\lambda C_1 + (1 - \lambda)C'_1 \sim \lambda C_2 + (1 - \lambda)C'_2.$$

The property stated as Axiom 2 is a strong one, inducing some linearity into the preferences (which indeed is what comes out of the present characterization). On the other hand it seems no more restrictive than what is usually assumed when considering preferences over health states, where indifference between suitable lotteries are instrumental for assessing the values of the utility indices.

The next axiom is a monotonicity property of the ranking of capabilities – larger sets are better than smaller sets. Here and in the sequel, $\text{int } C$ is the interior of the set C relative to \mathbb{R}_+^L , and $\text{cl } A$ for $A \subset \mathbb{R}_+^L$ denotes the closure of A

¹Here closedness is considered w.r.t. the topology on \mathfrak{C} induced by the Hausdorff distance on the compact subsets of \mathbb{R}_+^L .

AXIOM 3 *If $C_1 \subset \text{int } C_2$, then $C_2 \succ C_1$.*

In its present form, this axiom can hardly be controversial, stating that if there are strictly less functionings available, then the resulting smaller capability set is less desired than the large one. We shall later have to consider modifications of this axiom which are perhaps less immediately acceptable.

Our fourth axiom is much more open to criticism, indeed it may be seen as a violation of the very principles behind the use of capabilities. The basic idea here is that every capability set can be matched (in terms of individual preferences) by a capability set which is focussed on a particular combination of functionings already available.

AXIOM 4 *For each $C \in \mathfrak{C}$, there exists $x \in C$ such that $\{x\} - \mathbb{R}_+^L \sim C$.*

This axiom can be recognized as a version of Independence of Irrelevant Alternatives (IIA). If preferences over capabilities can be rationalized by a utility function, then the utility-maximizing element of the availability set (extended by free disposal to satisfy comprehensiveness) should be exactly as good as the larger choice set containing options that will not be chosen anyway. Thus, an IIA axiom of some type (and we shall consider another type of IIA axiom later) seems to be a necessary ingredient in any system of axioms for preferences on availability sets which can be rationalized by utility maximization. We note that if Axiom 3 holds, then the vector x of Axiom 4 must belong to the boundary of C .

As mentioned above, the third axiom is too strong if we want to keep some of the intuitive content of the capability approach. This is formally confirmed by the following result.

THEOREM 1 *Let \mathfrak{C} be a rich family of subsets of \mathbb{R}_+^L , and let \succsim be a relation on \mathfrak{C} . Then the following are equivalent:*

- (i) (\mathfrak{C}, \succsim) satisfies Axioms 1 – 4,
- (ii) there is a linear map $u : \mathbb{R}_+^L \rightarrow \mathbb{R}$ such that

$$C \succsim C' \Leftrightarrow \max_{x \in C} u(x) \geq \max_{x \in C'} u(x).$$

Thus, a preference relation on capabilities satisfying all four axioms is trivial in the sense that it has an index representation derived from a utility on functionings, so that the index value of a capability C is the maximal utility attained on a functioning in C .

PROOF OF THEOREM 1: The proof of the implication (ii) \Rightarrow (i) is straightforward and left to the reader. Define the set $L^1 = \{x \in \mathbb{R}_+^L \mid x^\leq \in \mathfrak{C}, x^\leq \succsim C', \text{ all } C' \in \mathfrak{C}\}$. By Axiom 1 there are maximal elements for \succsim on \mathfrak{C} , and by Axiom 4, we get that L^1 is nonempty.

Next, choose any $x^* \in L^1$ and define for each $\lambda \in [0, 1]$ the set

$$L^\lambda = \left\{ x \in \mathbb{R}_+^L \mid x^{\leq} \sim (\lambda x^*)^{\leq} \right\}. \quad (1)$$

By Axiom 2, each set L^λ is convex, and we have that $\lambda' \lambda^{-1} L^\lambda \subseteq L^{\lambda'}$ whenever $\lambda' \leq \lambda$. Letting $\hat{L}^1 = \{x \mid \lambda x \in L^1 \text{ for some } \lambda \in [0, 1]\}$ we get that \hat{L}^1 is convex and that for each $x \in \hat{L}^1$, the sets $\{x' \mid x' < x\}$ and \hat{L}^1 are disjoint. Consequently, by separation of convex sets there is $c \in \mathbb{R}_+^L$, $c \neq 0$, such that $L^1 \subset \{x' \mid c \cdot x' = 1\}$. It follows that $L^\lambda \subset \{x' \mid c \cdot x' = \lambda\}$ for each $\lambda \in [0, 1]$.

Define the map $u : \mathbb{R}_+^L \rightarrow \mathbb{R}$ by $u(x) = c \cdot x$ for each x . We show that u satisfies the conditions in (ii). Let C be arbitrary, and assume that $C \sim x^{\leq}$ for some $x \in L^1$. Then there is $y \in C$ with $y^{\leq} \sim x^{\leq}$, and since $u(y) = u(x) = \lambda$, we have that $\max_{z \in C} u(z) \geq \lambda$. Suppose that $\max_{z \in C} u(z) = \lambda' > \lambda$; then C contains some vector $z \in L^{\lambda'}$, meaning that $z' = \lambda(\lambda')^{-1}z$ must belong to L^1 . But since $(z')^{\leq}$ is contained in the interior of C by Axiom 3, we have a contradiction. \square

It can be checked that the linear function u is uniquely determined (up to a positive multiple) in the case where $\dim L^1 = L - 1$ (where L^1 was defined in (1) above) for some $\lambda \in [0, 1]$. Also, if we do not insist on u being linear, we still need that $u(x) = \lambda$ for all $x \in L^1$, each λ , meaning that the level sets of u will have large flat segments, possibly of dimension $L - 1$.

Since the assumption that \mathfrak{C} contains a sufficient large supply of sets of the form x^{\leq} may be difficult to justify in the applications at hand, we consider below another axiom system which is tailored for our purpose. As before, we assume \mathfrak{C} to be compact (in topology induced by the Hausdorff metric), convex and to contain $\{0\}$. In our present setup we assume moreover that \mathfrak{C} is closed under the operation of taking (arbitrary) unions followed by convexifying. Thus, if $\mathcal{D} \subset \mathfrak{C}$, then the set

$$\text{conv}^*(\cup \mathcal{D}) = \text{cl} \left\{ x \mid x = \sum_{i=1}^r \mu_i x_i, \mu_i \in [0, 1], \sum_{i=1}^r \mu_i = 1, x_i \in C_i \in \mathcal{D}, i = 1, \dots, r \right\}$$

belongs to \mathfrak{C} . For ease of reference, a family satisfying all these conditions shall be called *regular*.

For the following result, we need to modify some of the axioms stated above; clearly Axiom 4 must be replaced by another one, but also Axiom 3 must be sharpened slightly.

AXIOM 5 *Let $C, C' \in \mathfrak{C}$. If $C \subseteq C'$ then $C' \succsim C$, and if $C \sim C'$, then the set $\text{bd } C' \cap C$ is convex.*

In the context of regular families of convex sets, we replace Axiom 4 above by the following axiom which exploits the new structure.

AXIOM 6 Let \mathfrak{C}' be a family of sets from \mathfrak{C} such that $C \sim C'$ for all $C, C' \in \mathfrak{C}'$. Then $\text{conv}(\cup \mathfrak{C}') \sim C$ for each $C \in \mathfrak{C}'$.

On the face of it, Axiom 6 is very different from 4. Instead of postulating an equivalent subset of a capability C dominated by a single array of functionings, we assume some regularity of the superset consisting of all equivalent sets. Taken together with the new version of Axiom 3 we however end up with the same situation, that is triviality of the representing index on capabilities.

THEOREM 2 Let \mathfrak{C} be a rich family of closed, convex, and comprehensive subsets of \mathbb{R}_+^L , and let \succeq be a complete and continuous preorder on \mathfrak{C} . Then the following are equivalent:

- (i) (\mathfrak{C}, \succeq) satisfies Axioms 1,2,5 and 6,
- (ii) there is an affine function $u : \mathbb{R}_+^L \rightarrow \mathbb{R}$ such that for all $C, C' \in \mathfrak{C}$,

$$C \succeq C' \Leftrightarrow \max_{x \in C} u(x) \geq \max_{x \in C'} u(x).$$

PROOF: As before we leave the implication (i) \Rightarrow (ii) to the reader. For the converse implication, we follow the steps in the proof of Theorem 1. Let \mathcal{D}^1 be the set of elements of \mathfrak{C} which are maximal for \succeq , and let $C^1 = \text{conv}^*(\cup \mathcal{D}^1)$. Then C^1 belongs to \mathcal{D}^1 by Axiom 4'. For each $\lambda \in [0, 1]$, we define \mathcal{L}^λ as the set of all C such that $C \sim \lambda C^1$, and let $C^\lambda = \text{conv}^*(\cup \mathcal{D}^\lambda)$; again $C^\lambda \in \mathcal{D}^\lambda$ by Axiom 4'. If for all λ , the set \mathcal{D}^λ consists of the single element C^λ , then $C^\lambda = \lambda C^1$ for each λ . Choosing any $x \in \text{bd } C^1$ and any p which supports C^1 at x (i.e. $p \cdot c \leq p \cdot x$ for all $c \in C^1$), we have that the function u defined by $u(x) = p \cdot x$ satisfies the conditions stated in (ii).

If there is some λ such that \mathcal{D}^λ contains more than one element, then we define the set

$$E = \left\{ x \in \mathbb{R}_+^L \mid \exists \lambda \in [0, 1], C \in \mathcal{D}^\lambda, C \neq C^\lambda : \lambda x \in \text{bd } C^\lambda \cap C \right\}.$$

We claim that E is convex. Indeed, let $x_1, x_2 \in E$ and let $\mu \in [0, 1]$ be arbitrary. Then there is some λ (chosen small enough), together with sets $C_1, C_2 \in \mathcal{D}^\lambda$, $C_1, C_2 \neq C^\lambda$, such that $\lambda x_i \in \text{bd } C^\lambda \cap C_i$, $i = 1, 2$. It follows that $\lambda x_1, \lambda x_2 \in \text{bd } C^\lambda \cap \text{conv}(C_1 \cup C_2)$, and by Axiom 3', we have that

$$\mu \lambda x_1 + (1 - \mu) \lambda x_2 \in \text{bd } C^\lambda \cap \text{conv}(C_1 \cup C_2)$$

for any $\mu \in [0, 1]$. If $\text{conv}(C_1 \cup C_2) \neq C^\lambda$, then $\mu \lambda x_1 + (1 - \mu) \lambda x_2 \in E$; if not, we repeat the argument above with C_1 replaced by $\nu C_1 + (1 - \nu) C_2$ for ν small enough so that $\mu \lambda x_1 + (1 - \mu) \lambda x_2$ can be written as a convex combination of $\nu \lambda x_1 + (1 - \nu) \lambda x_2$ and λx_2 .

Next, consider the family of closed convex sets $(\lambda^{-1} C^\lambda)_{0 < \lambda \leq 1}$. We have that $\lambda^{-1} C^\lambda \subseteq \hat{\lambda}^{-1} C^\lambda$ for $\hat{\lambda} \leq \lambda$, so the set $F = \cup_{0 < \lambda \leq 1} \lambda^{-1} C^\lambda$ is convex. Moreover, E does not intersect the interior

of E , since in that case E would intersect the interior of C^λ for some $\lambda > 0$, a contradiction. It follows that E can be separated from the set $\text{int } F$, so that there is a linear form p such that $p \cdot x = 1$ for $x \in E$ and $p \cdot x \leq 1$ for $x \in F$.

Define u by $u(x) = p \cdot x$; we check that u has the desired properties by showing that $\max_{x \in C} u(x) = \lambda$ for $C \in \mathcal{D}^\lambda$, $\lambda \in [0, 1]$. Thus, let $C \in \mathcal{D}^\lambda$. Then $\text{bd } C^\lambda \cap C \neq \emptyset$, since otherwise $C < C^\lambda$ by Axiom 5. Since $u(\hat{x}) = \lambda$ for $\hat{x} \in \text{bd } C^\lambda \cap C$, we have that $\max_{x \in C} u(x) \geq \lambda$. The fact that $\max_{x \in C} u(x) \leq \lambda$ follows from the separation property, since $u(x) \leq 1$ on F implies $u(x) \leq \lambda$ for $x \in C^\lambda$. \square

We conclude this section with some considerations of the conditions for a family \mathfrak{C} to be regular. In the application that we have had in mind, sets $C \in \mathfrak{C}$ arise as sets of feasible (characteristics) outputs in a technology with commodity bundles as inputs. Let $T \subset \mathbb{R}_+^l \times \mathbb{R}_+^L$ be a convex set, interpreted as a production set for the household transforming commodity bundles $x \in \mathbb{R}_+^l$ to characteristics bundles $\xi \in \mathbb{R}_+^L$. Then any set of the form

$$T(x) = \{\xi \mid (x, \xi) \in T\}$$

would be a feasible availability set. The set $\{0\}$ will appear as $T(0)$ provided that T satisfies the standard assumption that no output is obtainable without input. The remaining properties of regular or rich families do not however follow if we restrict ourselves to sets of the type $T(x)$ or even sets $\cup_{x \in A} T(x)$ for suitable subsets A of \mathbb{R}_+^l , the standard example of A being bundles obtainable in a market from a given bundle of resources. Thus, for our results above to make sense in this context we might have to assume that individuals can order also availability sets that do not arise in any natural way.

Needless to say, any rich family \mathfrak{C} may arise from *some* technology T , at least if we allow for infinite-dimensional commodity spaces; as the input bundle giving rise to C , we may then choose the support function of C , giving us a suitable subspace of C^0 as the commodity space; the operations of averaging and of convex combinations carry over to support functions. Since this construction has only limited interest unless combined with some method of reduction to finite dimensions, we shall not pursue this matter any further.

3. Nontrivial capability indices

In the previous section, we have seen that an index representation of a complete preorder on capabilities becomes trivial when we pose too strong conditions on the way in which particular sub- or supersets of a capability inherit the ranking of the capability itself. It might seem that nontriviality will not occur very often. That is can indeed occur, so that index representations are not all trivial, can be seen by the example in Figure 1, where the family \mathfrak{C} of capabilities

contain the triangles marked by A and C and the square B ; we assume that the sides of B have unit length while the longest side of the triangles has length $a > 1$. Assume now that $A \sim C \succ B$. If this ordering had a trivial index representation, then there should be a linear form on \mathbb{R}^2 which attains the same maximum on the sets A and C , but such a linear form would necessarily attain a larger value on B .

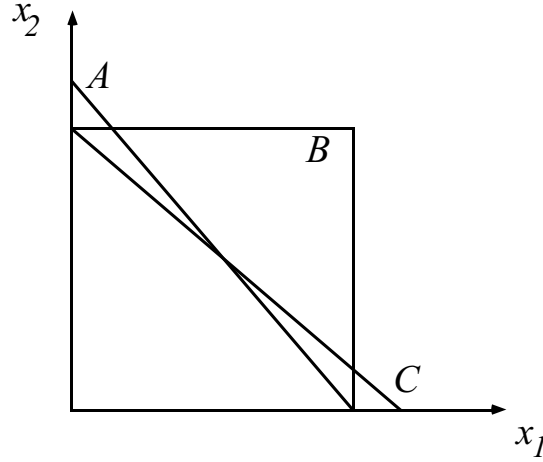


Figure 1: The ordering $A \sim C > B$ admits no trivial index representation

On the other hand, it is not too complicated to exhibit a function on subsets of \mathbb{R}_+^2 which takes the same value on A and C and a smaller value on B , indeed, we may define this index as

$$I(D) = \frac{1}{2} \max_{x \in D} (1, 0) \cdot x + \frac{1}{2} \max_{x \in D} (0, 1) \cdot x$$

for D a subset of \mathbb{R}_+^2 . Then

$$I(A) = I(C) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot a > \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = I(B).$$

We may think of this representation as arising from two trivial representations after a random choice, in our case the random utility on functionings is

$$\tilde{u}(x) = \begin{cases} (1, 0) \cdot x & \text{with probability } 1/2 \\ (0, 1) \cdot x & \text{with probability } 1/2 \end{cases}$$

and the index representing \succsim appears as

$$I(D) = \mathbf{E} [\max_{x \in D} \tilde{u}(x)].$$

It is seen that although the realizations of \tilde{u} give rise to trivial representations, using the expectation gives a nontrivial one, taking into consideration more than a single utility-maximizing functioning in the capability set.

We now propose a characterization of complete preorderings of capabilities which have an index representation of this form. For this we use support functions of convex sets: For $C \in \mathfrak{C}$ a capability, the support function $\delta^*(\cdot | C) : \Delta \rightarrow \mathbb{R}_+$ defined by

$$\delta^*(\cdot | C) = \max_{u \in \Delta_L} \{u \cdot x \mid x \in C\}, \quad (2)$$

where Δ_L is the unit simplex in \mathbb{R}^L ,

$$\Delta_L = \left\{ u \in \mathbb{R}_+^L \mid \sum_{h=1}^L u_h = 1 \right\}.$$

The support function is convex, and it characterizes the capability C fully, in the sense that if $g : \Delta \rightarrow \mathbb{R}_+$ is a convex function, then $g(u) = \delta^*(u | C^g)$, all $u \in \Delta$, where

$$C^g = \bigcap_{u \in \Delta} \{x \in \mathbb{R} \mid u \cdot x \leq g(u)\}.$$

Using the notion of support functions, we may write the generalized version of the index considered as

$$I(C) = \int_{\Delta_L} \delta^*(u | C) dP(u) \quad (3)$$

for some probability measure P on Δ_L .

The orderings admitting a representation of the form given by (3) In this section, we present a characterization of dynamically rationalizable orderings of families of capabilities. We begin with the following straightforward result:

THEOREM 3 *Let \mathfrak{C} be a regular family of capabilities ordered by \succeq , and suppose that \succeq admits a representation (3). Then \succeq satisfies Axioms 1 – 3.*

PROOF OF THEOREM 3: To show that Axiom 1 holds, we need only to check the continuity part, since it is clear that \succeq must be a complete preorder. Let $(C_n)_{n \in \mathbb{N}}$ be a sequence of capabilities with $C_n \succeq C$ converging to C_0 in the Hausdorff topology. Using (3) we have that there is P such that

$$\int_{\Delta} \delta^*(u | C_n) dP(u) \geq \int_{\Delta} \delta^*(u | C_n) dP(u)$$

for all $n \in \mathbb{N}$. Since the map $C' \mapsto \delta^*(\cdot | C')$ is continuous, we get that

$$\int_{\Delta} \delta^*(u | C_0) dP(u) \geq \int_{\Delta} \delta^*(u | C_0) dP(u)$$

or $C_0 \succeq C$, and we conclude that the set $\{C' \mid C' \succeq C\}$ is closed for every $C \in \mathfrak{C}$. Closedness of $\{C' \mid C \succeq C'\}$ is established in similar way.

For Axiom 2, let (C_1, C_2) and (C'_1, C'_2) be pairs of elements of \mathfrak{C} with $C_1 \sim C_2$, $C'_1 \sim C'_2$, and let $\lambda \in [0, 1]$. Using the definition of support functions, we have that

$$\delta^*(u \mid \lambda C_j + (1 - \lambda) \mid C'_j) = \lambda \delta^*(u \mid C_j) + (1 - \lambda) \delta^*(u \mid C'_j),$$

$j = 1, 2$, for each $u \in \Delta$, so that

$$\begin{aligned} \int_{\Delta} \delta^*(u \mid \lambda C_j + (1 - \lambda) \mid C'_j) dP(u) \\ = \lambda \int_{\Delta} \delta^*(u \mid C_1) dP(u) + (1 - \lambda) \int_{\Delta} \delta^*(u \mid C'_1) dP(u), \end{aligned}$$

$j = 1, 2$, we get from repeated use of Theorem 1 that $\lambda C_1 + (1 - \lambda)C'_1 \sim \lambda C_2 + (1 - \lambda)C'_2$.

Axiom 3 follows easily from the fact that $\delta^*(u, C_1) < \delta^*(u, C_2)$ for all $u \in \Delta$. \square

To proceed we shall need a general result about ordered topological vector spaces.

THEOREM 4 *Let (V, \succeq) be an ordered topological vector space with positive cone V_+ having nonempty interior, let A be a convex cone contained in V_+ , and let \succeq be a continuous total preorder on A . Then the following are equivalent:*

(i) \succeq is monotonic (in the sense that $x, x' \in A$, $x \geq x'$ implies $x \succeq x'$) and respects nonnegative linear combinations in the following sense: If $x, x', y, y' \in A$ with $x \succeq x'$ and $y \succeq y'$, and $\lambda, \mu \in \mathbb{R}_+$, then

$$[x \succeq x', y \succeq y', \lambda x + \mu x' \in A, \lambda y + \mu y' \in A] \Rightarrow \lambda x + \mu x' \succeq \lambda y + \mu y'. \quad (4)$$

(ii) \succeq has a utility representation u , where $u = v|_A : A \rightarrow \mathbb{R}_+$ is the restriction to A of a positive linear form v on V .

PROOF : (ii) \Rightarrow (i): Since for $x, x' \in A$, $x \succeq x'$ if and only if $u(x) \geq u(x')$, and u is the restriction of a positive linear form on V , we have immediately that \succeq is monotonic. The second condition in (ii) follows similarly: If $u(x) \geq u(y)$ and $u(x') \geq u(y')$, then by linearity of u we get that

$$u(\lambda x + \mu x') = \lambda u(x) + \mu u(x'), \quad u(\lambda y + \mu y') = \lambda u(y) + \mu u(y'),$$

and the conclusion follows.

(i) \Rightarrow (ii): Choose an element x of A with $x > 0$ (such an element exists since 0 is minimal for \succeq by monotonicity). Clearly, $\lambda x > x$ for $\lambda > 1$ and $x > \lambda x$ for $\lambda < 1$. Let $I(x) = \{x' \in A \mid x' \sim x\}$ be the set of elements of A which are equivalent to x in \succeq (that is, such that $x' \succeq x$ and $x \succeq x'$). Using (4) with $y = y' = x$ we get that $I(x)$ is the intersection of A with an affine subset in V ; we write this set as $K + \{x\}$, where K is a closed subspace of V .

By monotonicity, the set K does not intersect $\text{int } V_+$, and by the Hahn-Banach theorem (see, e.g., Rudin (1973), Theorem 3.4), there is a continuous linear form v on V such that $v(x) = 0$ for $x \in K$ and $v(x) > 0$ for $x \in \text{int } V_+$. It is easily seen that $u = v|_A$ satisfies the conditions in (i). \square

Now we may prove a converse of Theorem 3, giving a characterization of index representations of rankings of capabilities.

THEOREM 5 *Let \mathfrak{C} be a regular family of closed, convex, and comprehensive subsets of \mathbb{R}^L , and assume that if \mathfrak{C} contains sets C and C' , then it contains also $\lambda C + \mu C'$ for all $\lambda, \mu \in \mathbb{R}_+$.*

If \succsim is an ordering of \mathfrak{C} which satisfies Axioms 1 – 3, then there exists a probability measure P on Δ_L such that \succsim has a representation of the form (3).

PROOF OF THEOREM 5: Let $V = C(\Delta)$ be the set of continuous real functions on Δ . Endowed with the topology of uniform convergence, V is a topological vector space, and its positive cone (the set of all nonnegative functions on Δ) has nonempty interior (consisting of the strictly positive functions).

We define the set $A = \{\delta^*(\cdot | C) | C \in \mathfrak{C}\}$. Since $\lambda C + \mu C' \in \mathfrak{C}$ for all $C, C' \in \mathfrak{C}$ and $\lambda, \mu \in \mathbb{R}_+$, and $\delta^*(\cdot | \lambda C) = \lambda \delta^*(\cdot | C)$, $\delta^*(\cdot | \mu C') = \mu \delta^*(\cdot | C')$ we have that A is a convex cone.

Define the preorder \succsim on A in the obvious way, that is by

$$\delta^*(\cdot | C) \succsim \delta^*(\cdot | C') \Leftrightarrow C \succsim C'$$

(since this is basically the “same” ordering, only transferred from sets to their support functions, we have kept the same notation). We leave it to the reader to check that \succsim is a continuous preorder on A (the continuity part was shown in the proof of Theorem 2).

Monotonicity of \succsim defined on A is a straightforward consequence of Axiom 3. Similarly, it is easily seen that (1) follows from Axiom 2. We thus have that all the conditions in (i) of Theorem 3 are fulfilled.

Using now Theorem 4, we get that \succsim has a utility representation U , which is the restriction to A of a positive linear form on $V = C(\Delta)$. By the Riesz representation theorem (see e.g., Rudin (1966), Thm.6.19), a positive linear form on V can be identified with a probability measure P on Δ , so that

$$U(\delta^*(\cdot | C)) = \int_{\Delta} \delta^*(u | C) dP(u),$$

which by Theorem 1 is exactly the expression of dynamic rationalizability. \square

It is seen that the randomized trivial representation described in (3) appears as the only functional form of an index representing a preorder which satisfies the Axioms 1 – 3, and if we

consider these as the least controversial of the axioms, then the result is remarkably general, showing that the idea of maximizing utility over functionings remain fundamental, since the index emerges as the expected value of all the possible trivial representations, weighted in a suitable way.

To interpret the results, we must return to the starting point for the axiomatic approach to ordering capabilities: Why are we at all interested in such orderings? The point here is that in order to use capabilities for measuring inequality, we would like capabilities to be ordered in a more or less objective way, making it possible to decide whether or not an individual is better or worse off getting one capability instead of another; in the absence of such a common scale of preferences we would have to be satisfied with weak measures of inequality such as non-*envy* or *egalitarian-equivalence*, which clearly are much less appealing than standard equality measures. Our result shows that the members of a reasonably broad class of orderings on capabilities are uniquely determined by a probability distribution over utility assignments. Assuming this probability distribution given in society, a unique ranking of capabilities will emerge. This is a much better result than what we obtained in the first paper, where orderings related directly to utilities, that is to subjective properties of the individual, and therefore could not reasonably be taken as valid for all individuals in society. With the options approach to capabilities, it does make sense to speak of objective ranking of capabilities as better or worse. In this sense, the results of the paper may be seen as a possibility result opening up for measuring inequality through capabilities. Needless to say, this measurement faces many other problems than the conceptual one considered here, but at least we have moved one step towards the goal.

4. Capability indices and choice of allocation

In the present section, we return to the model considered section, we considered briefly at the end of Section 2, where we treated capabilities as sets of possible outputs in the form of functionings from a given input of commodities using a common technology. Since we are now treating allocations of commodities as well as of functionings, we distinguish in notation using lower case greek letters ξ, η etc. for arrays of functionings and keeping standard notation x, y, z for allocations of commodities.

In our simple model, individuals $i \in N = \{1, \dots, n\}$ receive commodity bundles to be transformed according to a known and common technology. If individual i inserts the bundle x_i into the common technology T , she obtains the option of choosing an array of functionings ξ_i from $T(x_i)$. Thus, a given input may give rise to many different outputs, which seems to be in accordance with both our intuition pertaining to individual choice of life style and health profile and with the properties of technologies in general. As in the previous section,

we consider orderings of the possible capabilities which may or may not be trivial, that is defined by maximizing a given utility function on functionings under the constraints given by the capability.

An allocation in this model is an array $x = (x_1, \dots, x_n)$ of commodity bundles in \mathbb{R}_+^l , and we work with a given set X of allocations, which might be all allocations obtained by distributing a given initial endowment, or it might arise in some other way, we shall need only that X is a convex set, and that it allows redistribution: if $x \in X$ is an allocation, and $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ satisfies $\sum_{i=1}^n z_i = 0$, then there is some $\lambda > 0$ such that the array $x + \lambda z$ belongs to X .

Each allocation $x = (x_1, \dots, x_n)$ gives rise to an array of capabilities $(T(x_1), \dots, T(x_n))$. If individual i has preferences over capabilities, formalized as a complete preorder \succeq_i on the family $\{T(x_i) \mid x_i \in \mathbb{R}_+^l\}$, then allocations $x = (x_1, \dots, x_n)$ and (x'_1, \dots, x'_n) may be assessed by the individual according to whether or not $T(x_i) \succeq_i T(x'_i)$.

We shall be interested in the case where an allocations $x \in X$ with associated capabilities $(T(x_1), \dots, T(x_n))$ is Pareto optimal in the sense that there is no allocation $x' \in X$ with associated capabilities $(T(x'_1), \dots, T(x'_n))$ such that $T(x'_i) \succeq_i T(x_i)$ for all $i \in N$ and $T(x'_j) \succ_j T(x_j)$ for some $j \in N$.

THEOREM 6 *Assume that*

(a) *for each $i \in N$, the preference relation \succeq_i over capabilities is monotonic in the sense that if*

$$C \subseteq C' \Rightarrow C' \succeq_i C, \quad C \subset \text{int } C' \Rightarrow C' \succ_i C$$

for any two capability sets C and C' .

(b) *T is monotonic in inputs: if $x_h > x'_h$ for $h = 1, \dots, l$, then $T(x') \subset \text{int } T(x)$.*

Let $x \in X$ be an allocation which is not Pareto optimal. Then there is there is a profile (u_1, \dots, u_n) of linear utility functions on \mathbb{R}_+^l an an allocation $x' \in X$ such that

$$\max_{\xi_i \in T(x'_i)} u_i(\xi_i) > \max_{\xi_i \in T(x_i)} u_i(\xi_i), \quad \text{all } i \in N. \quad (5)$$

PROOF: Since x is not Pareto optimal, there is an allocation $\hat{x}'' \in X$ such that $T(\hat{x}''_i) \succeq_i T(x_i)$ for $i \in N$ and $T(\hat{x}''_j) \succ_j T(x_j)$ for at least one $j \in N$. Choose j^0 such that $T(\hat{x}''_{j^0}) \succ_{j^0} T(x_{j^0})$. By continuity of \succeq_{j^0} we may assume that x''_{j^0} belongs to the interior of \mathbb{R}_+^l . Choose λ such that x' defined by

$$x'_{j^0} = x''_{j^0} - \lambda e, \quad x'_i = x''_i + \frac{\lambda}{n-1} e,$$

where $e = (1, 1, \dots, 1)$ is the diagonal vector in \mathbb{R}^l , and $x'' \in X$ (this is possible by our assumptions on X). If λ is chosen small enough, then $T(x'_{j^0}) \succ_{j^0} T(x_{j^0})$ and $T(x'_i) \succ_i T(x_i)$ by

monotonicity of \succeq_i , $i \neq j$.

For each $i \in N$, since $T(x'_i) \succ_j T(x_i)$, there is some $u_i \in \Delta_L$ such that

$$\max_{\xi_i \in T(x'_i)} u_i(\xi_i) > \max_{\xi_i \in T(x_i)} u_i(\xi_i),$$

since otherwise we would have $T(x'_i) \subseteq T(x_i)$, a contradiction. It follows that (5) is satisfied at the profile (u_1, \dots, u_n) . \square

The result of Theorem 6 seems to vindicate that linear utilities on functionings suffice for tracing inefficiency in an economy of the type modelled here. However it should be noticed that the profile of utility function which is used in (5) depends on the particular allocation x' which Pareto dominates x , so that it cannot be used as a method for detecting Pareto improving allocations.

It may be noticed that if $T(x_i) \subset T(x'_i)$ then any linear utility in Δ_L may be used in the profile (u_1, \dots, u_n) , so that in some cases the profile may consist of identical utility functions. As mentioned above, the practical advantages of this are however limited.

The above result can be combined with the standard welfare theorem to show that if there is no instance where a profile of linear utilities on functionings detects that all individuals can be better off, then the allocation must be an price equilibrium in the sense that for each individual, a bundle which induces a preferred capability must be more expensive than the actual bundle. Not surprisingly we need a convexity assumption on the preferences over capabilities.

THEOREM 7 *Assume the assumptions (a) and (b) of Theorem 6 are fulfilled, and that in addition*

(c) *for each $i \in N$ and $x_i \in \mathbb{R}_+^l$, the set $\{x'_i \in \mathbb{R}_+^l \mid T(x'_i) \succeq_i T(x_i)\}$ is convex.*

Let $x = (x_1, \dots, x_n)$ be a feasible allocation. Then one of the following holds:

(i) *there is a profile of linear utility functions on \mathbb{R}_+^l $u = (u_1, \dots, u_n)$ and a feasible allocation $x' = (x'_1, \dots, x'_n)$ such that*

$$\max_{\xi_i \in T(x'_i + z_i)} u_i(\xi_i) > \max_{\xi_i \in T(x_i^*)} u_i(\xi_i) \tag{6}$$

for each $i \in N$,

(ii) *there is a price vector $p \in \mathbb{R}_+^l$ such that (x_1^*, \dots, x_n^*, p) is an equilibrium in the sense that for each $i \in N$, if $x'_i \in \mathbb{R}_+^l$ is such that $T(x'_i) \succ_i T(x_i^*)$, then $p \cdot x'_i > p \cdot x_i^*$.*

PROOF: If there is a Pareto improvement of x , then (i) holds by Theorem 6, so there is a profile (u_1, \dots, u_n) and an allocation $x' \in X$ such that (6) is satisfied. Suppose now that the allocation

x is Pareto optimal. Define for each $i \in N$ the set

$$Z_i = \{z_i \in \mathbb{R}^l \mid T(x_i + z_i) \succeq_i T(x_i)\},$$

and let Z be the set of all vectors $\sum_{i=1}^n z_i$ with $z_i \in Z_i$. Then each Z_i and consequently also Z are convex by assumption (c).

We claim that Z does not intersect the set $\{y \in \mathbb{R}^l \mid y_h < 0, h = 1, \dots, l\}$. Indeed, if $\sum_{i=1}^n z_i \cdot h < 0$, all h , for some $(z_i)_{i \in N}$ with $T(x_i + z_i) \succeq_i T(x_i)$, all i , then there would be a feasible allocation $x' = (x'_1, \dots, x'_n)$ with $x'_{ih} > x_{ih}$ for all i and h , so that $T(x'_i) \succ_i T(x_i)$ for all i , contradicting Pareto optimality of x .

By separation of convex sets, there is $p \in \mathbb{R}_+^l$, $p_h > 0$, $h = 1, \dots, l$, such that $p \cdot z \geq 0$ for $z \in Z$. It follows that $p \cdot x'_i \geq p \cdot x_i$ whenever $T(x'_i) \succeq_i T(x_i)$, all $i \in N$, so that (x, p) is indeed a price equilibrium. \square

5. Interpretation of results

In the present paper, we considered utility representations of preferences on capabilities, considered as subsets of functionings, with the specific aim of obtaining a representation which is non-trivial in the sense that it does not reduce to maximization on of a utility function defined directly on the functionings, the interpretation being that capabilities represent something over and above a constraint on functionings available.

Looking first at a single individual, we found that an assignment of index values to capabilities which satisfied some rather reasonable properties turned out to be trivial, so that the properties had to be revised. Retaining what might be considered a minimum of structure we found that the only possible representation has the form of a suitably weighted average of trivial indices. Such an index does not correspond to maximizing a utility over the functionings contained in the capability, although it retains some resemblance to the trivial representations.

In the context of preferences over capability profiles in a society, where capabilities are derived from using a common individual (health-promoting) technology whose inputs are commodity bundles obtained in the market, we saw although trivial utility representations are enough for detecting cases of inefficient allocation, the particular choice of such utilities will depend on both the allocation considered and the Pareto improvement. Consequently, we cannot rely of the trivial index representation of preferences over capabilities, and the more general version considered in Section 3 is appropriate also in this model. It should be stressed that the results obtained are not to be considered as in any way reducing the potential usefulness of the capability approach. Rather, they should be seen as an additional argument for intensifying research in the nature of capabilities; since preferences on capabilities cannot be

trivially deduced from standard utility maximization on functionings, we need an explanation of how preferences on sets of functionings are formed and whether they can at all be reduced to simpler structures.

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