Solutions to Exercises in

Game Theory
Chapter 8

1. The minimax payoff vector in the repeated game is $(v_1, v_2) = (-9, -9)$, and the set of payoffs attainable in Nash equilibria in the repeated game is

$$\{(v_1, v_2) \in \mathbb{R}^2 \,|v_i \geq -9, i = 1, 2\} \\ \cap \{(v_1, v_2) \in \mathbb{R}^2 \mid \exists v' \in \mathrm{conv}(\{(0, -10), (-10, 0), (-1, -1)\}), v_i \leq v'_i, i = 1, 2\}.$$

when the game terminates with probability $\frac{1}{2}$ at every stage, the payoffs must be modified accordingly. For the minimax payoffs, we get

$$\frac{\lambda}{1-\lambda}\sum_{t=1}^{\infty}(1-\lambda)^t 2^{-t}(-9) = \frac{\lambda}{1-\lambda}\frac{2}{1+\lambda}(-9)$$

which tends to 0 as $\lambda \to 0$. By similar reasoning, one sees that all the payoffs if the repeated game approach $(0, 0)$ as $\lambda \to 0$, so this payoff is trivially a Nash equilibrium payoff.

2. The trigger strategy for the repeated prisoners' dilemma game prescribe that the row (column) player should choose B (R) at each stage unless either row player has chosen T or column player has chosen L at some earlier state, in which case the choice should be T (L).

The trigger strategy as described above is a subgame perfect Nash equilibrium, since it is Nash and in any stage where some player has deviated from (B, R), the strategy of the other player is such that each player will be better off following the prescription of the strategy rather than deviating. This property does not depend on λ but is of course a specific feature of the game.

3. If the duopolists coordinate their strategy, they would choose the monopoly price $p^{mon} = \frac{b+c}{2}$ [typo in formula p.75] and share the monopoly profit from the sale of $q^{mon} = \frac{b-c}{2a}$, giving each a payoff of

$$\Pi = \frac{1}{2}\left[\frac{b+c}{2} - c\right]\frac{b-c}{2a} = \frac{(b-c)^2}{8a}.$$

We assume that the duopolists are compete by choosing prices. In particular, any firm may undercut the other firm and receive the whole market.

In the repeated game, the minimax payoffs are 0, and the equal division of the monopoly profit may be sustained by trigger strategies, whereby the duopolists choose the monopoly price (if price-setting) if both have being doing so previously, and charge the price $p = c$

otherwise. Deviating from the cooperative strategy at any stage the firm may gain Π (the other half of the monopoly profit) at the given stage but will lose all future profits, the discounted value of which amount to

$$\sum_{t=1}^{\infty} \beta^t \Pi = \frac{\beta}{1-\beta}\Pi,$$

so that no deviation is optimal if $\Pi \le \frac{\beta}{1-\beta}\Pi$ or $\beta \ge \frac{1}{2}$.

4. Complications arise in punishing deviations, since the exact identity of the deviating player cannot be established. Consequently, the attainment of payoff in Nash equilibria of the repeated game can be established only for payoffs $(v_1, \dots, v_n)$ such that there is a strategy array σ in the one-shot game such that for each i,

$$\max_{\sigma_i'} u_i(\sigma_i', \sigma_{-i}) \le v_i.$$

If this is satisfied, a repeated game strategy leading to $(v_1, \dots, v_n)$, which turns to σ in all future after a deviation will be a Nash equilibrium. The set of the Nash equilibrium payoffs thus has a more complicated structure.

5. "Only if" follows from the definition. To show the "if"-part, assume that a player has a deviation beginning at t^0 and running over more than one period. If there is a last period t^* for which the deviation yields a gain to the player, then the subgame starting at t^* has a profitable one-stage deviation. If the deviation has no final stage, then for every $\varepsilon > 0$ there is a period T such that the payoff of a strategy equals that of the deviation for $t \le T$ and has no deviations for $t > T$ differs with less than ε from the deviation, and the previous reasoning applies.

In the repeated Prisoners' dilemma game (Problem 2), the strategies prescribing L to player 1 unless player 2 has chosen L at some stage, in which case T is chosen, whereas player 2 chooses R unless player 1 has chosen T at some stage, in which case the choice is L, is a Nash equilibrium, but it is not subgame perfect: If player 1 deviates and chooses T, then player 2 should choose L in the next period, but given that the equilibrium strategies prescribe that player 1 will choose B from now on, it is better to choose R at this stage.

A similar case can be made in the repeated Cournot duopoly of Problem 3.