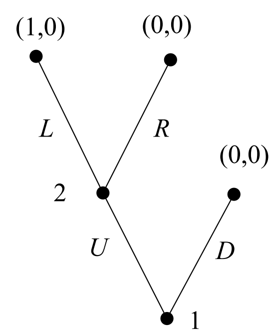
Solutions to Exercises in

Game Theory
Chapter 7

1. In the extensive form game below, the strategy pair (D,R) (player 1 chooses D, and player 2 chooses the strategy which selects R if choice becomes necessary) is a subgame perfect Nash equilibrium (no player has incentive to change given the choice of the other player in any subgame).



The normal form of the game is

	L	R
U	$(1,0)$	$(0,0)$
D	$(0,0)$	$(0,0)$

and (D,R) is not a perfect Nash equilibrium, since player 1 will have U as best reply to any completely mixed strategy of player 2.

2. In the game below, (T,L) is a perfect equilibrium. The row T is weakly dominated by C, and the column L is weakly dominated by R

	L	C	R
T	$(1,1)$	$(0,0)$	$(0,1)$
M	$(1,0)$	$(0,0)$	$(0,0)$
B	$(0,1)$	$(0,0)$	$(0,1)$

The row C contains a Nash equilibrium, namely (C, M), which is not perfect (for every mixed strategy of player 1, the best reply will be either T or B. Similarly, R contains the Nash equilibrium (B, R) which is not perfect, since best reply of player 1 to any mixed strategy of player 2 will be T or C.

3. The check for Nash equilibrium is straighforward, since each of the pairs are such that the strategy of player 1(2) is best reply given the strategy of player 2(1).

Perfectness: The pair (T, L) is clearly perfect, since T is best reply to any mixed strategy of player 2 with small weight on R, and L is best replay to any mixed strategt with small weight on B. The pair (M, C) is perfect since is best replay on any mixed strategy with small weight on L and equal weight on C and R. Similarly, C is best reply to mixed strategies with small weight on T, equal weight on M and B. But (B, R) is not perfect, since any completely mixed strategy of player 2 will have best reply either T or B, and similarly by symmetry for player 1.

Properness: For (T, L) it is seen that for any $\varepsilon > 0$, there are mixed strategies of player such 2 (1) that payoff at T (L) is sufficiently larger than payoff at any other pure strategy. The same argument applies to (M, C) (the mixed strategies should have sufficiently large weight on M and C, respectively. It is seen immediately that no such mixed strategies can be designed for (B, R).

Since (T, L) payoff dominates (M, C), the first one would seem more reasonable, but this of course depends on what is put into the notion of reasonableness.

4. An equilibrium mixed strategy of player 1 must satisfy

$$2\sigma_2(L) + 4\sigma_2(R) = 4\sigma_2(L) + 3\sigma_2(R)$$

with solution $\sigma_2(L) = \dfrac{1}{3}$. Similarly, the mixed strategy σ_1 of player 1 is found from

$$2\sigma_1(T) + \sigma_1(B) = \sigma_1(T) + 3\sigma_1(B)$$

with solution $\sigma_1(T) = \dfrac{2}{3}$. Expected payoff to player 1 is $\frac{1}{3}2 + \frac{2}{3}4 = \frac{10}{3}$. To find player 1's maximin, let p be the weight of any mixed strategy of player 1 on T and q the weight of any mixed strategy on L. Then expected payoff to player 1 is

$$2pq + 4p(1 - q) + 4(1 - p)(1 - q) + 3(1 - p)(1 - q) = q(1 - 3p) + p + 3,$$

and the minimum over q is attained at $q = 1$ for $p > \frac{1}{3}$ and at $q = 0$ for $p < \frac{1}{3}$. For $p = \frac{1}{3}$ expected payoff is $\frac{10}{3}$, which is seen to be the maximum attainable.

Player 1 may be considered as better off choosing $\sigma_1' = \left(\frac{1}{3}, \frac{2}{3}\right)$, since the payoff can never be below the Nash equilibrium payoff and may well be above, depending on the choice of the other player.

5. [There is a typo in the graph, the payoffs in the two last nodes corresponding to the choice of B should be interchanged] The pure strategies in the normal form are T, M and B for player 1 and L and R (choosing L or R whenever there is a choice) for player 2. This gives the payoffs in the table. The path along Y which is a mixture of T and M with weight β on T will give the same output as a mixture of row 1 and 2 (where the weights are computed after correcting by β.

Clearly, choosing T will be at least as good as choosing M, since an equilibrium strategy of player 2 will put some weight on R. Therefore, the mixture Y is at least as good as M, so that equilibrium strategies will be such that payoff given the mixed strategy with weight q on L gives the same expected payoff to player 1 in the two strategies Y and B, meaning that

$$\beta + (1-\beta)[2p - (1-q)2] = -2q + 2(1-q)$$

which gives that $p = \dfrac{4\beta - 8}{3\beta - 4}$, so that the strategy depends on β which did not show up in the normal form. This means that another extensive form game (for example with another value of β) with the same normal form must have other sequential equilibria, so that this concept cannot be used as equilibrium selection for normal form games.

6. Eliminating the row B which is weakly dominated gives the game

	L	R
T	(3, 2)	(2, 3)
M	(1, 1)	(0, 0)

and by (vi), the solution should be in this game. By admissibility (iii), it must be (T,L). However, we could just as well have deleted M, and then (iii) gives that the solution must be (T, R), and we conclude that the solution to the original game cannot be single-valued.

7. Without information, player 2 must maximize

$$-\frac{1}{2}(-w-a)^2 - \frac{1}{2}(w-a)^2.$$

First order conditions are $2(-w-a) + 2(w-a) = 0$, which has solution $a = 0$, corresponding to a maximum since the objective function is concave.

Open rule: Let t be the message sent by player 1 and a the decision chosen by player 2. Since optimal choice for player 2 is $a = t$, we get that the payoff to player 1 is $-(\theta + b - t)^2$, which equals $-b^2$ if $t = \theta$ (truthful reporting) and $-(-2\theta + b)^2$ if $t = -\theta$ (false reporting). Truthful reporting is better than false reporting if $b < 2\theta - b$ which amounts to $b \leq w$.

Closed rule: The recommendation $a = \theta + b$ is truthful if when $\theta = b - a$, and it will be accepted if $-(\theta - (\theta + b))^2 \geq -(\theta)^2$ which is equivalent to $b \leq w$.

In cases where $b > w$, it may be advantageous to player 1 to recommend $a = -w + b$ in cases where $\theta = w$, since the recommendation $w + b$ would be rejected by player 2, giving instead $a = 0$, and $-w + b > 0$.