**Solutions to Exercises in**

# Game Theory
# Chapter 4

**1.** Both players have two strategies 'stay' and 'swerve off', and the normal form is

| | L | R |
|---|---|---|
| M | $(-2,-2)$ | $(2,-1)$ |
| B | $(-1,2)$ | $(1,1)$ |

There are two pure strategy Nash equilibria, namely those where one player stays and the other one swerves off. In addition the this, there is a mixed strategy Nash equilibrium where each player chooses 'stay' and 'swerve off' with probability $1/2$.

**2.** The first row (the strategy T) is dominated by the second row (the strategy which should have been denoted 'M'), and eliminating T, we get the new game

| | Stay | Swerve off |
|---|---|---|
| M | $(7,2)$ | $(9,1)$ |
| B | $(8,4)$ | $(2,2)$ |

Here the strategy R for player 2 is dominated by the strategy L, so R is eliminated, and we get a game where only player 1 has a choice between strategies, and clearly B dominates M, so M is eliminated. The final result is therefore the pair $(B,L)$.

It is seen that $(B,L)$ is als a pure strategy Nash equilibrium.

**3.** (a) Let $\sigma_j$ be an arbitrary mixed strategy of player $j$, $j \neq i$. Since $\pi_i(s_i', s_{-i}) > \pi(s_i, s_{-i}$ for all $s_{-i} \in S_{-i}$, then

$$\sum_{s_{-i} \in S_{-i}} \left[\prod\nolimits_{j \neq i} \sigma_j(s_i)\right] \pi_i(s_i', s_{-i}) > \sum_{s_{-i} \in S_{-i}} \left[\prod\nolimits_{j \neq i} \sigma_j(s_i)\right] \pi_i(s_i', s_{-i}),$$

so that indeed expected payoff to player $i$ at $s_i'$ exceeds that at $s_i'$. So the first statement is true.

(b) This is false, as is seen from the following example:

| | L | R |
|---|---|---|
| T | $(6,2)$ | $(9,1)$ |
| M | $(8,4)$ | $(3,2)$ |
| B | $\left(\frac{13}{2},2\right)$ | $(5,1)$ |

The mixed strategy for player 1 with equal weights $\frac{1}{2}$ on T and M gives 7 if 2 chooses L and 6 if 2 chooses R, so that it dominates B. However, neither T nor M dominates B.

**4.**. Let $n_2$ be the number of pure strategies of player 2, and let $U_1$ be the set of points in $\mathbb{R}^{n_2}$ which are $\leq$ some point in the convex hull of the points $(u_1(s_1,s_2^1),\dots,u_1(s_1,s_2^{n_2}))$, $s_1 \in S_1$. Then $s_1$ is not dominated by a pure or mixed strategy if and only if $s_1$ is a point on the boundary of $U_1$.

By (weak) separation of convex sets, there is a linear form $p \in \mathbb{R}_+^{n_2}$ such that $p$ attains its maximum over $U$ at the point $(u_1(s_1,s_2^1),\dots,u_1(s_1,s_2^{n_2}))$. Normalizing $p$ such that $p\cdot(1,\dots,1) = 1$, we may identify $p$ with a mixed strategy on $S_2$, and clearly $s_1$ gives the maximal expected utility to player 1 given this mixed strategy of player 2, so that $s_1$ is a best response to $p$.

Suppose now that after iterated elimination of dominated strategies, player $i$ is left with the set $S_i'$ of pure strategies. Then by the above, each $s_i \in S_i'$ is a best response to some mixed strategy in $\Delta S_j'$, $j \neq i$, and by the characterization theorem for rationalizable strategies, the $(\Delta S_i')_{i=1}^2$ is the set of rationalizable strategies in $\Gamma$.

**5.** The pair $(T,L)$ where player 1 chooses $T$ and player 2 chooses $L$, is a Nash equilibrium. It is easily seen that there are no other Nash equilibria, neither pure nor mixed, since in that case one it would contain a dominated strategy for at least one of the players, a contradiction.

For an evolutionary stable equilibrium, we must first define the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of fitness coefficients. Now $q = (q_1,q_2) \in \Delta\{1,2\}$ is evolutionary stable if $(q,q)$ is a Nash equilibrium of the game, which means that $q = (1,0)$, and $q$ is stable in the sense that $p\cdot Aq = q\cdot Aq$ and $p \neq q$ implies $q\cdot Ap > p\cdot Ap$. In our case we have that

$$p\cdot A\begin{pmatrix}1\\0\end{pmatrix} = p_1 a + (1-p_1)c = a = \begin{pmatrix}1 & 0\end{pmatrix} A \left(1//0\right)$$

implies that $p_1 = 1$, so the stability condition is fulfilled trivially.

**6.** It is easily checked that no array of pure strategies can be a Nash equilibrium. If there is a Nash equlibrium in mized strategies, then the equations system

$$5q + (1-q) = 3q + 2(1-q)$$
$$p + 4(1-p) = 5p + 3(1-p)$$

should have a solution $(p, q)$. The equations have the unique solution $p = \frac{1}{3}$, $q = \frac{1}{5}$, so the mixed strategies $\left(\left(\frac{1}{5}, \frac{4}{5}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$ gives the unique Nash equilibrium.

For a correlated strategy $(p_{TL}, p_{TR}, p_{BL}, p_{BR})$ in $\Delta(\{T, B\} \times \{L, R\})$ to be a correlated equilibrium, it must satisfy the inequalities

$$\begin{aligned} p_{TL}(5-3) + p_{TR}(1-2) &\geq 0 \\ p_{BL}(3-5) + p_{BR}(2-1) &\geq 0 \\ p_{TL}(1-5)) + p_{BL}(4-3) &\geq 0 \\ p_{TR}(5-1) + p_{BR}(3-4) &\geq 0 \end{aligned}$$

The equations can be reduced to

$$p_{BR} \geq 2p_{BL} \geq 8p_{TL} \geq 4p_{TR} \geq p_{BR},$$

which shows that all $\geq$ must be equalities. It follows that only the Nash equilibrium gives rise to a correlated equilbrium (if there was another one satisfying the equalities, then all convex combinations of this correlated strategy and that Nash equilibrium would be correlated equilibria as well, but they would violate the equalities).

**7.** Consider the following game (a version of Chicken):

| | L | R |
|---|---|---|
| T | $(0, 0)$ | $(7, 2)$ |
| B | $(2, 7)$ | $(6, 6)$ |

The game has no pure strategy Nash equilibria and a unique mixed strategy Nas equilibrium $\left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$. Consider the correlated strategy

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}.$$

We claim that this correlated strategy is a correlated equilibrium. Indeed, if player 1 is ordered to play T and defects playing B, then expected change in payoff is $6-7 = -1$ (player 1 knows that if he ist told to play T, then player 2 is told to play B), and if player 1 should choose B but instead selects T, the expected gain is $\frac{1}{2}(7-6)(0-2) + \frac{1}{2}(0-2) = -\frac{1}{2}$ (player 1 knows that player 2 has equal probability to choose L or B). Thus, player 1 has no incentive to deviate, and by symmetry, neither has player 2.