Solutions to Exercises in

Game Theory
Chapter 2

1. It is easily seen that the game has no equilibria in pure strategies (since maximin = $0 \neq 100$ = minimax). Let $\tau = (\tau_1, \tau_2)$ be a mixed strategy equilibrium strategy of player 2, then

$$\tau_1 \cdot 100 + (1 - \tau_1) \cdot (-50) = \tau_1 \cdot 0 + (1 - \tau_1) \cdot 100$$

(if player 1 chooses mixed strategies, then each of the pure strategies having positive probability must yield the same expected payoff). so that $\tau_1 = \frac{3}{5}$. Similarly, an equilibrium mixed strategy of player is found from

$$\sigma_1 \cdot (-100) + (1 - \sigma_1) \cdot 0 = \sigma_1 \cdot 50 + (1 - \sigma_1) \cdot (-100),$$

so that $\sigma_1 = \frac{2}{5}$. The unique equilibrium strategies are thus $(\sigma, \tau) = \left(\left(\frac{2}{5}, \frac{3}{5}\right), \left(\frac{3}{5}, \frac{2}{5}\right)\right)$.

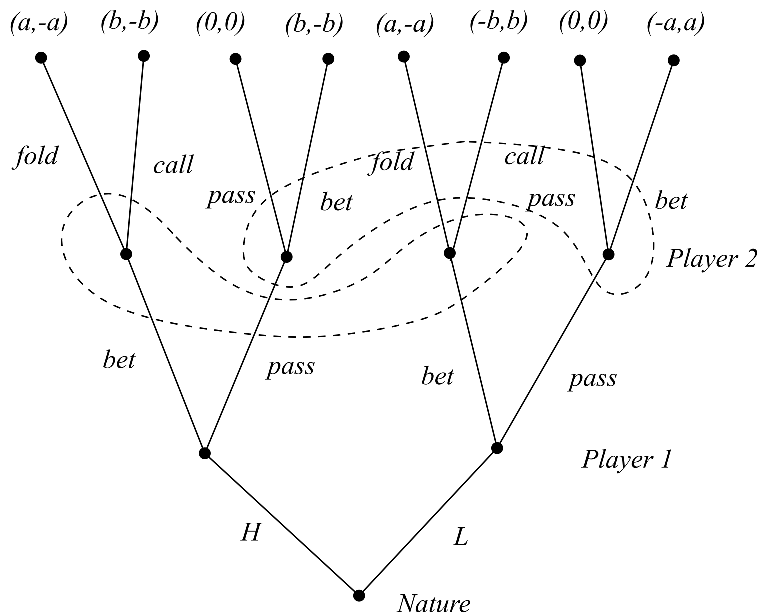
2. Let (σ, τ) be an equilibrium of Γ'. We identify σ with the mixed strategy σ in Γ for which $\sigma(s) = 0$. Since $\pi(\sigma, t) > \pi(s, t)$ for all t, it follows that $\pi(\sigma, \tau) > \pi(\sigma', \tau)$ for all mixed strategies σ' with $\sigma'(s) > 0$, so that (σ, τ) is an equilibrium in Γ.

Conversely, let (σ, τ) be an equilibrium in Γ. Then $\sigma(s) = 0$, since otherwise the mixed strategy σ' defined by

$$\sigma'(s') = \begin{cases} \frac{\sigma(s')}{1-\sigma(s)} & s' \neq s, \\ 0 & s' = s, \end{cases}$$

would result in $\pi(\sigma', \tau) > \pi(\sigma, \tau)$, contradicting the equilibrium property of (σ, τ).

3. The extensive form of the game is shown below:



Player 1 has 4 pure strategies ((1)bet if H, bet if L, (2)bet if H pass if L, (3) pass if H, pass if L, (4) pass if H, bet if L), and player 2 has 4 pure strategies ((1) fold if 1 bets, pass if 1 passes, (1) fold if 1 bets, bet if 1 passes, (3) call if 1 bets, pass if 1 passes, (4) call if 1 bets, bet if 1 passes). Numbering rows and columns from 1 to 4 as above, we get the game matrix

$$\begin{pmatrix} a & a & 0 & 0 \\ \frac{a}{2} & 0 & \frac{b}{2} & \frac{b-a}{2} \\ 0 & \frac{b-a}{2} & 0 & \frac{b-a}{2} \\ \frac{a}{2} & \frac{a+b}{2} & -\frac{b}{2} & 0 \end{pmatrix}$$

4. The game Γ with matrix A has an equilibrium in pure strategies if and only if $\operatorname{maxmin} A = \operatorname{minmax} A$, which in this case means that

$$\max\{\min\{a_{11}, a_{12}\}, \min\{a_{21}, a_{22}\}\} = \min\{\max\{a_{11}, a_{21}\}, \max\{a_{12}, a_{22}\}\}. \qquad (1)$$

If the $a_i j$ are independent and uniformly distributed in the interval $[0, 1]$, equality in (**??**) obtains only with probability 0.

5. [The word 'same' is missing: players have the *same* number of strategies.]

Since there is only one non-zero element ind each row or column, we may reorder the strategies of player 2 so that in the game matrix, $a_{ij} \neq 0$ if and only $i = j$. The game matrix then takes the form

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Equilibrium mixed strategies τ of player 2 must satisfy $\tau(1) = \cdots = \tau(n)$, so that

$$\tau(i) = \frac{\lambda}{\lambda_i},\ \lambda = \left[\sum_{i=1}^{n} \frac{1}{\lambda_i}\right]^{-1}.$$

By symmetry, the equilibrium strategy σ of player 1 has the same form, and λ is the value of the game.

6. Let V be the value of a target, and choose a numbering of targets from left to right as 1,2,...,6. Then the game has the following normal form, where the strategies of player A is to choose defence of target centered in $i \in \{1, \ldots, 6\}$ (rows) and strategies of player B is to attack target i columns):

	1	2	3	4	5	6
1	$(0,0)$	$(0,0)$	$(-V,V)$	$(-V,V)$	$(-V,V)$	$(-V,V)$
2	$(0,0)$	$(0,0)$	$(0,0)$	$(-V,V)$	$(-V,V)$	$(-V,V)$
3	$(-V,V)$	$(0,0)$	$(0,0)$	$(0,0)$	$(-V,V)$	$(-V,V)$
4	$(-V,V)$	$(-V,V)$	$(0,0)$	$(0,0)$	$(0,0)$	$(-V,V)$
5	$(-V,V)$	$(-V,V)$	$(-V,V)$	$(0,0)$	$(0,0)$	$(0,0)$
6	$(-V,V)$	$(-V,V)$	$(-V,V)$	$(-V,V)$	$(0,0)$	$(0,0)$

giving rise to the matrix

$$\begin{pmatrix} 0 & 0 & -V & -V & -V & -V \\ 0 & 0 & 0 & -V & -V & -V \\ -V & 0 & 0 & 0 & -V & -V \\ -V & -V & 0 & 0 & 0 & -V \\ -V & -V & -V & 0 & 0 & 0 \\ -V & -V & -V & -V & 0 & 0 \end{pmatrix}$$

It is seen here that the first and the last row are (weakly undominated), so that we expect the equilibrium mixed strategies to have 0 weight on these. Similarly, the second and the 5th column are (weakly) dominated. Given the symmetry of the payoffs, we expect the weight on the remaining rows and columns to be of equal size, giving equilibrium strategies $\left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right)$ for A and $\left(\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}\right)$ for B. The value of the game is $-\frac{V}{2}$.

7. The game has the following normal form:

	2	3
2	$(4,-4)$	$(-6,6)$
3	$(-6,6)$	$(9,-9)$

with game matrix

$$\begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix}.$$

Equilibrium strategies τ for player 2 are found from

$$4\tau_1 - 6(1 - \tau_1) = -6\tau_1 + 9(1 - \tau_1),$$

which gives $\tau_1 = \frac{3}{5}$. By symmetry, we get that the equilibrium is $\left(\left(\frac{3}{5}, \frac{2}{5}\right), \left(\frac{3}{5}, \frac{2}{5}\right)\right)$. The value of the game is the equilibrium expected payoff of player 1, which is 0.