Solutions to Exercises in

Game Theory
Chapter 16

1. If $R_N \in \mathcal{L}(A)^N$ and a is a Condorcet winner at R^N, then $|\{i \mid a\,R_i\,b\} > \frac{n}{2}$ for all $b \neq a$, and since $a\,R_i\,b$ means that we cannot have $b\,R_i\,a$, it follows that a must be unique. However, there are profiles such as

$$\begin{matrix} x & y & z \\ y & z & x \\ z & x & y \end{matrix}$$

for which there are no Condorcet winners.

For $R_N \in Q(A)^N$, there can be more than one Condorcet winner: Suppose that $x \sim z$ (in the sense that $x\,R_i\,z$ *and* $z\,R_i\,z$) for the second individual in the profile above, then both x and z are Condorcet winners.

Suppose that there is a unique Condorcet winner a at R_N, but that there is a manipulation at R_N. This means that there is some profile Q_N with a Condorcet winner b such that b is strictly preferred to a in R_N (that is $b\,R_i\,a$ and not $a\,R_i\,b$) for all i such that $Q_i \neq R_i$ (taking account also for coalitional manipulation). Since b is not a Condorcet winner at R_N, the set of individuals i such that $b\,R_i\,a$ cannot be a majority, and it follows that b cannot be a Condorcet winner at Q_N, a contradiction.

2. Assume that $h : Q(A)^N \to A$ can be implemented (understood here as full implementation) in Nash equilibria, let $R_N \in Q(A)^N$ be a profile with $h(R_N) = a$, and consider a profile $R'_N \in Q(A)^N$ such that the set $\{a' \in A \mid a'\,R'_i\,a\}$ of alternatives at least as good as a in R'_i is contained in the set $\{a' \in A \mid a'\,R_i\,a\}$ of alternatives at least as good as a in R_i, for all $i \in N$.

Suppose first that R'_i differs from R_i only for one individual. Since h is implemented in Nash equilibrium via some game form $G = (N, (S_i)_{i\in N}, \pi)$, there is a strategy array $s' = (s'_1, \dots, s'_n)$ which is a Nash equilibrium at R'_N and satisfies $\pi(s') = b$. If $b\,R_i\,a$, then b is also a Nash equilibrium outcome at R_N (with strategy array s'), contradicting that $h(R_N) = a$. It follows that a is strictly preferred to b at R_i and consequently at R'_i, and if $s = (s_1, \dots, s_n)$ is a Nash equilibrium of $G(R_N)$ with $\pi(s) = a$, then s is also a Nash equilibrium in $G(R'_N)$, and we conclude that $b = a$, which gives the desired result. The case where R'_i differs from R_i for more than one individual follows by successive application of the above result.

[For the converse, one needs an additional assumption of no veto power.]

3. Let E be the given effectivity function, and choose $i \in N$ arbitrarily. Then one can define an effectivity function $E^{)i(} : \mathcal{P}(N\backslash\{i\}) \to \mathcal{P}^2(A)$ by

$$E^{)i(}(S) = E(S) \cup E(S \cup \{i\}), S \subseteq N\backslash\{i\}.$$

We check that $E^{)i(}$ is convex: Let T_1, T_2 be in $\mathcal{P}(N\backslash\{i\})$; the

$$E^{)i(}(T_1) \cap E^{)i(}(T_2) = [E(T_1) \cup E(T_1 \cup \{i\})]\,[E(T_2) \cup E(T_2 \cup \{i\})]$$
$$\subseteq E(T_1 \cap T_2) \cup E(T_1 \cup T_2 \cup \{i\}) \subseteq E^{)i(}(T_1 \cap T_2) \cup E^{)i(}(T_1 \cup T_2),$$

so that $E^{)i(}$ is indeed convex.

Next, we show that if the core of E is empty, then so is the core of $E^{)i(}$: Choose an arbitrary profile R_N and an alternative a. Then there is $S \subset N$ such that $E(S)$ contains an alternative b with $b\,R_i\,a$ for all $i \in S$. It follows that $E^{)i(}(S\backslash\{i\})$ contains an alternative b with $b\,R_i\,a$ for all $i \in S\backslash\{i\}$, so that a is not in the core of $E^{)i(}$.

Stability of convex effectivity functions now follows by successive reduction of E to a convex effectivity function on a two-individual set, which is trivially stable.

4. Let $E : \mathcal{P}(A) \to \mathcal{P}^2(A)$ be a representable effectivity function, and let $G(N, (S_i)_{i\in N}, \pi)$ a representation of E, so that $E = E^G_\alpha$.

If S, T are two disjoint coalitions and $B \in E(S)$, $C \in E(T)$, then there is an S-strategy $(s_i)_{i\in S}$ with $s_i \in S_i$, each $i \in S$, such that $\pi(s_{i\in S}, t_{i\in N\backslash S}) \in B$ for all $(t_i)_{i\in N\backslash S} \in \prod_{i\in N\backslash S} S_i$, and T-strategy $(t_i)_{i\in T}$ with $t_i \in S_i$, each $i \in T$, such that $\pi(t_{i\in T}, s_{i\in N\backslash T}) \in C$ for all $(s_i)_{i\in N\backslash T} \in \prod_{i\in N\backslash S} T_i$. It follows that $\pi((s_i)_{i\in S}, (t_i)_{i\in T}, (w_j)_{j\in N\backslash S\cup T}) \in B \cap C$, so that E is indeed superadditive. Monotonicity is straightforward: If S has a strategy which guarantees that outcome is in B, than any T with $S \subseteq T$ has a strategt which guarantees that outcome is in B and consequently in any superset C of B.

To show the converse, assume that E is superadditive and monotonic. Let G be the game form defined as follows: Choose a fixed linear order R on A, and for each i, let S_i consist of all pairs $(S, B) \in \mathcal{P}(N) \times \mathcal{P}(A)$ such that $i \in S$ and $B \in E(S)$. Then π is defined by

$$\pi((S_i, B_i)_{i\in N}) = \max_R \cap\{B \mid \exists S : (S_i, B_i) = (S, B), i \in S\}$$

(with $\pi((S_i, B_i)_{i\in N}) = \max_R A$ if there are no sets of the above type). The outcome function is well-defined by superadditivity of E, and for every S and $B \in E(S)$ the S-strategy array where all individuals in S choose (S, B) will result in an element of B independent of the choices of the remaining individuals.

5. We choose the version of the deferred acceptance algorithm, where at the first step each college C send out proposals to the most preferred students to a number which is the smallest of q_C and the total number of students, and each student accepts the best college (preliminary) having proposed and rejects the rest. In the kth step, the college sends out the maximal possible number of proposals to students which have not rejected it so far, and the student preliminary accepts the best of all previous proposals and rejects the rest. The algorithm stops when there are no rejections.

The algorithm must stop after finitely many steps: For each college C, let $S^{(1)}_C$ be the set of students at least as good as C itself (interpreted as the level below which admission of a student is worse than no admission), and for $k > 1$, let $S^{(k)}_C$ similarly be the set of students at least as good as C who have not yet rejected an admission from C. Then $W^{(k)}_C \subseteq W^{(k-1)}_C$ for $k > 1$, and it the algorithm does not stop, then some student must have rejected an admission from some college, so that $\sum_C |W_C^{(k)}| < \sum_C |W_C^{(k-1)}|$, which means that the sum be 0 after finitely many steps.

We then check that the matching achieved by the algorithm is stable: First of all no singleton can improve: Colleges send proposals only to students which are better than C, and since at any step the student can choose among all proposals achieved, she will never have accepted a college worse than no admission. If pairs (s, C) can improve, then the student s is better for C then some $s' \in \mu(C)$, and C is better for s than $\mu(s)$. However, since s is better than $s' \in \mu(C)$, s must have been among the proposals sent out by C at some step, and since at each step s chooses the best from all proposals yet received, s would not have rejected C and chosen $\mu(s)$. It follows that the matching obtained is stable.

6. An example of a room-mates problem which has no stable matching is the following

x	y	z	w
y	z	x	x
z	x	y	y
w	w	w	z
x	y	z	w

(there are four persons x, y, z, w, and the table displays the preferences of the individuals as columns). The result of a matching will be a partition of $\{x, y, z, w\}$ into either 2-sets or singletons. If there are singletons, then the matching cannot be stable, so there must be two pairs, and one of these pairs must contain w. The other individual in this pair has been ranked at top by another person different from w, and these two individuals can improve, so the matching cannot be stable.

7. Suppose that μ is a stable matching in the marriage problem, and let m be such that $\mu(m) = m$. By stability, there is no w in W such that

$$u_m(w) > u_m(m), u_w(m) > u_w(\mu(w)).$$

This means that all w better for m than m himself either are matched, and then to somebody preferred to m, or prefer to be single. If in some other matching w would be paired to m' with $u_w(m') < u_w(m)$, then this matching could not be stable

If there is a stable matching μ' where m is matched to some $w^1 = \mu'(m) \in W$, then $u_m(w^1) > u_m(m)$ by stability of μ', and then $u_{w^1}(m^1) > u_{w^1}(m)$, where $m^1 = \mu(w^1)$, by stability of μ. Since μ' is stable, we cannot have that $u_{m^1}(w^1) > u_{m^1}(w^2)$, where $w^2 = \mu'(m^1)$, so $u_{m^1}(w^2) > u_{m^1}(w^1)$. Proceeding in this way and using the finiteness of $M \cup W$, we must reach a situation where either $\mu'(m^k) = w^r$ or $\mu(w^k) = m^r$ for some $r < k$, a contradiction, so that m cannot be matched to some $w \in W$. The same reasoning shows that if w is not matched in some stable matching, then w is unmatched in all stable matchings.