Solutions to Exercises in

Game Theory
Chapter 12

1. Let the price of the output commodity be 1 and the endowment of input commodities be $\omega_i \in \mathbb{R}^l_+$.If the production function is $g : \mathbb{R}^l_+ \to \mathbb{R}_+$, then the coalition S can produce $g\left(\sum_{i\in S} \omega_i\right)$. The game (N, v) with $v(S) = g\left(\sum_{i\in S} \omega_i\right)$ is a cooperative game, and it is superadditive since

$$v(S \cup T) = g\left(\textstyle\sum_{i\in S\cup T} \omega_i\right) \geq g\left(\textstyle\sum_{i\in S} \omega_i\right) + g\left(\textstyle\sum_{i\in S} \omega_i\right) = v(S) + v(T)$$

for $S \cup T = \emptyset$ (where we have assumed that the technology is additive, which it will be if the set $\{(z, y) \mid y \leq g(z)\}$ is convex and satisfies constant returns to scale).

Let $(c_S)_{S\in C}$ be a balanced family of coalitions in (N, v), so that $\sum_{S\in\mathcal{S}:i\in S} c_S = 1$ for each i. Then

$$\begin{aligned}\textstyle\sum_{S\in\mathcal{S}} c_S v(S) = \textstyle\sum_{S\in\mathcal{S}} c_S g\left(\sum_{i\in S} \omega_i\right) &\leq g\left(\sum_{S\in\mathcal{S}} c_S \sum_{i\in S} \omega_i\right) \\ &= g\left(\sum_{i\in N} \sum_{S\in\mathcal{S}} c_S \omega_i\right) = g\left(\sum_{i\in N} \omega_i\right) = v(N),\end{aligned}$$

where we have used convexity and constant returns to scale of g. Thus, (N, v) is balanced.

The restriction of (N, v) to any coalition $S \subset N$ is again a production game, and consequently each subgame (S, v) is balanced.

2. The game $(\{1, 2, 3\}, v)$ with

$$v(\{i\}) = 0,\ i = 1, 2, 3, v(\{1, 2\}) = v(\{2, 3\}) = v(\{1, 3\}) = v(\{1, 2, 3\}) = 1$$

is superadditive, but and its core is empty: Indeed, if x is an imputation with $x_1 + x_2 + x_3 = 1$, then $x \in \text{Core}(\{1, 2, 3\}, v)$ would imply that $x_i + x_j = 1$ for $i, j \in \{1, 2, 3\}, i \neq j$, since otherwise x could be improved by $\{i, j\}$. But then $x_i = 0$ for each i, a contradiction.

We define $(N, \hat{v})$ from (N, v) by

$$\hat{v}(S) = \max\{x \mid \exists T_1, \ldots, T_k \subseteq S, T_i \cap T_j = \emptyset, i, j \leq k, i \neq j, \textstyle\sum_{i=1}^k v(T_i) = x\}.$$

Then $(N, \hat{v})$ is superadditive: If $S_1, S_2 \subseteq N$ with $S \cap S' = \emptyset$, then for $i = 1, 2$ there is a partition $T^i_1, \ldots T^i_{k_i}$ of S_i with

$$\hat{v}(S_i) = \sum_{j=1}^{k_i} v(T^i_j), i = 1, 2.$$

Then $T_1^1, \dots T_{k_1}^1, T_1^2, \dots T_{k_2}^2$ is a partition of $S_1 \cup S_2$, and consequently $\hat{v}(S_1 \cup S_2) \geq \hat{v}(S_1) + \hat{v}(S_2)$. We conclude that $(N, \hat{v})$ is superadditive.

Since $\hat{v}(S) \geq v(S)$ for each coalition S, we have that $\text{Core}(N, \hat{v}) \subset \text{Core}(N, v)$. Next, suppose that $x \in \text{Core}(N, v)$. For each coalition S, if $T_1, \dots T_k$ is a partition of S, then $v(T_j) \leq \sum_{i \in T_j} x_i$ by the core property, so that $\sum_{j=1}^k v(T_j) \leq \sum_{i \in S} x_i$, and since the partition was chosen arbitrarily, we have that $\hat{v}(S) \leq \sum_{i \in S} x_i$, from which we get that $x \in \text{Core}(N, \hat{v})$.

3. [Typos: The function $\mathcal{U}$ should be defined as $\mathcal{U}(z) = \sum_{i \in S} z_i$ (for $S = N$, the selection is trivial since any element in the core maximizes the function). Unfortunately, also the statement of S-monotonicity is imprecise: A core selection ϕ is S-monotonic at v if for all w with $w(S) > v(S)$ and $W(T) = v(T)$ for $T \neq S$, one has that for all $x \in \phi(v)$, there is $y \in \phi(w)$ with $y_i \geq x_i$, all $i \in S$.] Once the formulation has been set right, the S-monotonicity of any $\phi_{\mathcal{U}}$ for $\mathcal{U}(z) = \sum_{i \in S} z_i$ at any v is straightforward: If w satisfies the assumptions, then $x \in \phi)(v) \subset \text{Core}(v)$ implies that $x \in \phi(w)$, so that S-monotonicity is fulfilled trivially.

The selection $\phi_{\mathcal{U}'}$ with $\mathcal{U}_i = \sum_{i \in T} z_i$ where $T \cap S \neq \emptyset$, $T \backslash S \neq \emptyset$ ia not S-monotonic, since the game v can be selected such that an isolated increase in the worth of S will lead to smaller core payoff for individuals in $S \backslash T$.

4. The pair (N, v), where $v(S) = \sum_{i \in S} c(\{i\}) - c(S)$ is the cost saving of the coalition $S \subseteq N$, is a TU game: Indeed, v has the properties of a characteristic function. Moreover, v is superadditive: Let $S_1, S_2 \in \mathcal{S}$ be coalitions with $S_1 \cap S_2 = \emptyset$. Then

$$v(S_1) + v(S_2) = \sum_{i \in S_1} + \sum_{i \in S_2} - [c(S_1) + c(S_2)] \leq \sum_{i \in S_1 \cup S_2} + c(S_1 \cup S_2) = v(S_1 \cup S_2),$$

where we have used that $c(S_1 \cup S_2) \leq c(S_1) + c(S_2)$ (subadditivity if cost) according to the definition of c as the cost of providing the projects in $S_1 \cup S_2$.

Let x be the cost allocation with

$$x_i = s_i + \frac{c(\{i\}) - s_i}{\sum_{j \in N}(c(\{j\}) - s_j)} \left(c(N) - \sum_{j \in N} s_j \right)$$

for all $i \in N$ (the separable cost plus a share in the cost savings from cooperation determined by alternate cost avoided. Since $c(N) \leq \sum_{j \in N} c(\{j\})$, we get that

$$\frac{c(N) - \sum_{j \in N} s_j}{\sum_{j \in N}(c(\{j\}) - s_j)} \leq 1,$$

so that

$$x_i = s_i + \frac{c(\{i\}) - s_i}{\sum_{j \in N}(c(\{j\}) - s_j)} \left(c(N) - \sum_{j \in N} s_j \right) \leq s_i + (c(\{i\}) - s_i) = c(\{i\}).$$

By subadditivity, we have that $s_i - c(\{i\}) = c(N) - c(N \backslash \{i\}) - c(\{i\}) \leq 0$ for all i. If the semicore is non-empty, then there is some x with $\sum_{i \in N} x_i = c(N)$ such that $\sum_{j \neq i} x_j \leq c(N \backslash \{i\})$ for all i, so that $s_i = c(N) - c(N \backslash \{i\} \leq x_i$, all i, and therefore $\sum_{i \in N} s_i \leq c(N)$.

Now, let x be determined by the alternative cost method and assume that the semicore is nonempty, then

$$\sum_{j\neq i} x_j = c(N) - x_i = c(N) - [c(N) - c(N\backslash\{i\})] - \frac{c(\{i\}) - s_i}{\sum_{j\in N}(c(\{j\}) - s_j)}\left(c(N) - \sum_{j\in N} s_j\right)$$
$$= c(N\backslash\{i\}) - \frac{c(\{i\}) - s_i}{\sum_{j\in N}(c(\{j\}) - s_j)}\left(c(N) - \sum_{j\in N} s_j\right) \leq c(N\backslash\{i\})$$

since $c(N) \geq \sum_{j\in N} s_j$.

5. In the game $(\{1,2,3,4\}, v)$, where $v(S) = 1/8$ if $1 \in S$ and $S \neq \{1,2,3,4\}$, and $v(S) = \frac{\|S|}{4}$ for all other coalitions, the marginal vector x^{id} (where id is the identical permutation) given by

$$x^{\text{id}} = \left(\frac{1}{8}, 0, 0, \frac{9}{10}\right)$$

is in the Weber set (the convex hull of all the marginal vectors) but not in the core, since it can be improved by $\{2,3\}$ with $v(\{2,3\}) = 1/2$.

6. Let α be a given choice function, selecting a member of any coalition, and let $m^\alpha(v)$ be the selector value, assigning to each player i the payoff

$$m_i^\alpha(v) = \sum_{S:\alpha(S)=i} \Delta_v(S),$$

where $\Delta_v(S)$ is the Harsanyi dividend of the coalition S at the game v. Then from $\Delta_v(N) = v(N) - \sum_{S\subset N} \Delta_v(S)$ we get that

$$\sum_{i\in N} m_i^\alpha(v) = \sum_{i\in N}\sum_{S:\alpha(S)=i} \Delta_v(S) = \sum_{S\subseteq N} \Delta_v(S) = v(N),$$

so that $m^\alpha(v)$ is Pareto optimal and therefore a preimputation.

The game (N, v) with $N = \{1,2,3,4\}$ and $v(\{i\} = 0$ for $i \in N$, $v(S) = 1$ for $S \subseteq N$, $|S| \geq 2$, has empty core, and for α the choice function selecting the smallest index of the players in S, we trivially obtain that $m^\alpha(v)$ does not belong to the core.

7. The Shapley-Shubik power index

$$\phi_i(v) = \sum_{S\subseteq N: i\in S}\left(\frac{(s-1)!(n-s)!}{n!}\right)[v(S) - v(S\backslash\{i\})$$

has the desired form with $\lambda_s = \left(\frac{(s-1)!(n-s)!}{n!}\right)$, since

$$\sum_{s=1}^{n}\binom{n-1}{s-1}\left(\frac{(s-1)!(n-s)!}{n!}\right) = \sum_{s=1}^{n}\frac{(s-1)!(n-s)!(n-1)!}{n!(s-1)!(n-s)!} = \sum_{s=1}^{n}\frac{1}{n} = 1.$$

The Banzhaf-Coleman index

$$\psi_i(v) = \sum_{S \subseteq N : i \in S} \frac{1}{2^{n-1}} [v(S) - v(S \setminus \{i\})]$$

also has this form with $\lambda_s = \dfrac{1}{2^{n-1}}$ for all s, since

$$\sum_{s=1}^{n} \binom{n-1}{s-1} \frac{1}{2^{n-1}} = \sum_{t=0}^{n-1} \binom{n-1}{t} \frac{1}{2^{n-1}} = 1.$$