Solutions to Exercises in

Game Theory
Chapter 1

1. The strategy space of each player is of the two-elements set {Head, Tail} (the two sides of the coin), and the payoff matrix takes the form

	Head	Tail
Head	$(1, -1)$	$(-1, 1)$
Tail	$(-1, 1)$	$(1, -1)$

Since the payoff of each player depends on what the other player chooses, and the gains or losses at each choice have the same magnitude, there is no obvious choice of strategy for any of the players. We shall see in Chapter 2 that in order to describe solutions to games like this one, we need to extend the notion of a strategy.

2. The situation can be described as a game with two players, the two firms. Each player has as strategy space the set of all nonnegative prices, so that $S_1 = S_2 = \mathbb{R}_+$. The payoff of each player at the strategy pair (p_1, p_2), is found as the value of the resulting sale minus the cost. Here the market is assigned to the player having chosen the smallest price (known as Bertrand competition), so that

$$\pi_1(p_1, p_2) = \begin{cases} (p_1 - 10)(100 - p_1) & p_1 < p_2, \\ \frac{1}{2}(p_1 - 10)(100 - p_1) & p_1 = p_2, \\ 0 & p_1 > p_2. \end{cases}$$

The payoff of is defined similarly as

$$\pi_2(p_1, p_2) = \begin{cases} 0 & p_1 < p_2, \\ \frac{1}{2}(p_1 - 10)(100 - p_2) & p_1 = p_2, \\ (p_1 - 10)(100 - p_2) & p_1 > p_2. \end{cases}$$

3 Assume that the amount of money used by firm i is r_i, $i = 1, \dots, n$. If firm i obtains the patent, it gets W, so that the payoff or net income is $W - r_i$ otherwise net income is $-r_i$. We assume that if several firms reach the patent stage simultaneously, they share the patent income.

The assignment of patent to firms can be expressed as a function I from $\mathbb{R}^n_+$ to subsets of $\{1, \dots, n\}$ given by

$$I(r_1, \dots, r_n) = \begin{cases} \{i \mid T(r_i) \le T(r_j),\ j = 1, \dots, n\} & \text{if } r_i > 0 \text{ for some } i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then the patent race is a game, where strategy sets are $\mathbb{R}_+$, and payoffs are

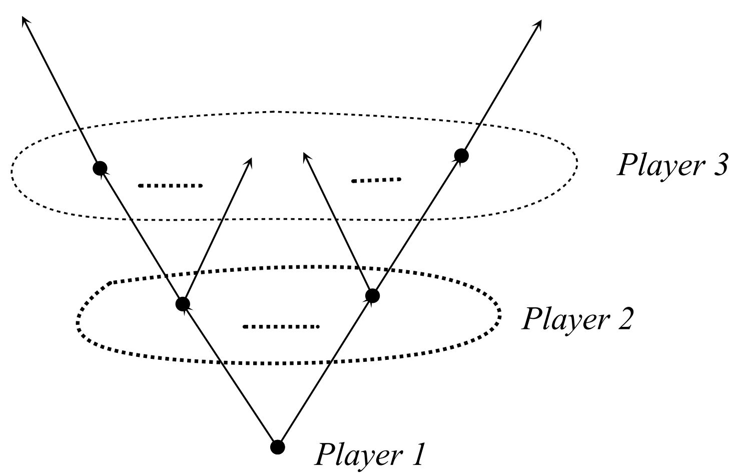
$$\pi_i(r_1, \dots, r_n) = \begin{cases} \dfrac{W}{|I(r_1, \dots, r_n)|} - r_i & \text{if } i \in I(r_1, \dots, r_n) \\ -r_i & \text{otherwise,} \end{cases}$$

for $i = 1, \dots, n$.

4. If the projects are denoted x, y, z, then individuals submit rankings represented as column vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \\ x \end{pmatrix}, \text{ etc.}$$

The initial part of the game tree is shown below.



For each individual, there are $3! = 6$ possible choices, so that there are 6 arrows pointing upwards from each node. Assuming that the players are unaware of the choices made by the others, all nodes at which a particular individual must choose are in a single information set. Thus there are 6 nodes in the information set of player 2, 36 in that of player 3, and 6^4 in the information set of player 5.

At the terminal nodes, a project is chosen according to the maximal number of top positions of the rankings on the path from the root to the terminal node. There may be cases where two different projects have the same maximal number of top positions, so that a rule for solving such cases is necessary.

5. The expected gain given that entry has been paid is infinitely large, since it has the form of a series

$$\frac{1}{2}\,2+\frac{1}{4}\,4+\cdots+\frac{1}{2^n}\,2^n+\cdots$$

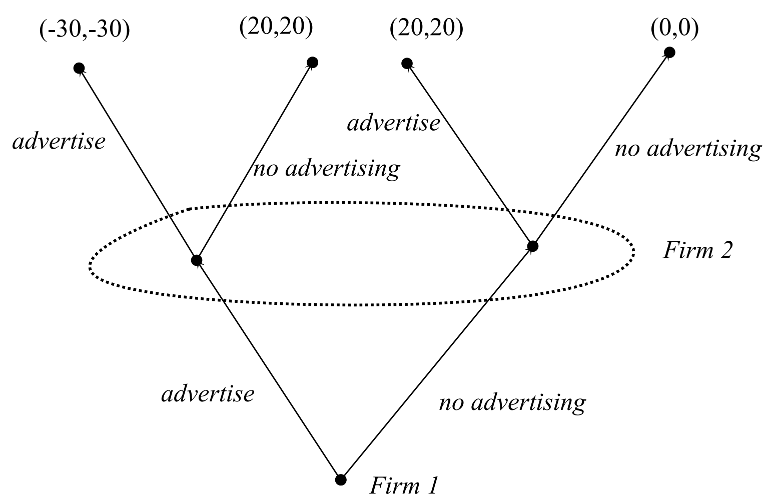
which does not converge. If people decide upon participation using expected values, then the entry fee could be arbitrarily high and still attract costumers, conflicting with an intuitive willingness to pay lower than 10.

The solution to the paradox (suggested by Bernouilli) is that people do not decide according to expected values of gains but to expected value of *utility* of gains. If the utility of a gain w is for example $\ln w$, then the expression

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\ln 2^n=\sum_{n=1}^{\infty}n\frac{1}{2^n}\ln 2\le\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n\frac{1}{2^n}\ln 2=\frac{3}{4}\frac{1}{1-\frac{3}{4}}\ln 2=3\ln 2,$$

where we have used that $n\le\left(\frac{3}{2}\right)^n$ for $n\ge 1$. The main point here is that the expectation is finite, and adjusting the functional form it is easy to get a result which matches intuition.

6. The extensive form is shown below,



The payoffs are net gains from advertising expressed in 1000$.