Fluctuations in overlapping generations economies*

Mich Tvede†

Abstract

In the present paper stationary pure-exchange overlapping generations economies with ℓ goods per date and m consumers per generation are considered. It is shown that for an open and dense set of utility functions there exist endowment vectors such that n-cycles exist for $n \leq \ell + 1$ and $\ell \leq m$. The approach to existence of endogenous fluctuations is basic in the sense that the prime ingredients are the implicit function theorem and linear algebra. Moreover the approach is applied to show that for an open and dense set of utility functions there exist endowment vectors such that sunspot equilibria, where prices at every date only depends on the state at that date, exist.

Keywords: cycles, overlapping generations economies, sunspot equilibria.

JEL-classification: D51, E32.

*I would like to thank an anonymous referee for useful suggestions including pointing out an error and Enrique Covarrubias for a careful reading.

†Department of Economics, University of Copenhagen, Studiestraede 6, DK-1455 Copenhagen K, Denmark; Tel: +45 35 32 30 92; Fax: +45 35 32 30 85; email: mich.tvede@econ.ku.dk.

1 Introduction

For dynamic economies, endogenous fluctuations are equilibria, where endogenous variables vary with time even though no exogenous shocks are hitting fundamentals. An early example of a cycle (prices alternate deterministically between finitely many price vectors) in a overlapping generations economy (or OG economies for short) is provided in Gale (1973) where it is noted that "*everyone has perfect foresight but cycling nevertheless occurs as a consequence of the equilibrium price mechanism*". An effect of endogenous fluctuations is that possibilities of otherwise identical consumers vary with the date of birth. In the present paper the existence of endogenous fluctuations in stationary pure-exchange OG economies is studied.

Cycles in pure-exchange OG economies have been an object of interest in several papers. For simple economies with one good per date and one consumer per generation, the existence of cycles has been studied in Gale (1973), Benhabib & Day (1983) and Grandmont (1985) among others. For economies with many goods per date and many consumers per generation, the existence of cycles has been studied in Ghiglino & Tvede (1995, 2004). In Ghiglino & Tvede (1995) it is shown that for almost all n and almost all lists of individual utility functions, if the number of consumers per generation is at least twice the number of goods per date, then there exist lists of endowment vectors such that there exist n-cycles. In general, cycles in pure-exchange OG economies are caused by the interaction of wealth effects and sustitution effects in demand.

Stationary n-state sunspot equilibria (prices jump stochastically between n price vectors and there is a transition matrix of probabilities for jumps between all pairs of price vectors) in pure-exchange OG economies have been an object of interest in some papers. For simple economies the existence of sunspot equilibria has been studied in Shell (1977), Azariadis (1981) and Azariadis & Guesnerie (1986) among others. In Azariadis & Guesnerie (1986) it is shown that there exists stationary 2-state sunspot equilibria if and only if there exists a 2-cycle which is robust to small changes in fundamentals. However for economies with many goods per date and many consumers per generation, stationary sunspot equilibria typically do not exist as indicated in Davila (1997) and Citanna & Siconolfi (2007).

To understand the importance of the number of goods per date for the existence/non-existence of stationary n-state sunspot equilibria it should be noted that: for economies with one good per date, the net demand of the old consumers is just equal to the real value of the stock of money, so past prices have no impact on aggregate net demand of the old consumers, and; for economies with more than one good per date, past prices do have an impact on aggregate net demand of old consumers as aggregate net demand of old consumers depends on their consumption at the previous date as well as the distribution of the stock of money. Put mathematically, for economies with more than one good per date there are more equations than unknowns.

The question of existence of cycles has been addressed for productive OG economies in Reichlin (1986), Julian (1988) and Benhabib & Laroque (1988) among others. Moreover cycles also have been studied in optimal growth economies in Benhabib & Nishimura (1979), Boldrin & Montrucchio (1986) and Sorger (1994) among others.

In the present paper stationary pure-exchange OG economies with ℓ goods per date and m consumers per generation are considered and a basic approach for establishing existence of endogenous fluctuations is provided. Indeed in order to establish existence of n-cycles (price vectors alternate deterministically between n price vectors), where $n \leq \ell + 1$ and $\ell \leq m$, two steps are needed: 1. the implicit function theorem is applied to the market clearing condition for n subsequent dates at a stationary equilibrium to show that price vectors for the n subsequent dates can be changed in any direction by changing incomes across dates and consumers, and; 2. linear algebra is applied to budget constraints and equilibrium conditions to show that there exist endowment vectors such that both budget constraints and equilibrium conditions are satisfied for the changed price vectors and endowment vectors. Moreover n-cycles are shown to be robust.

It is sketched how the approach can be applied to establish existence of stationary n-state sunspot equilibria. Next it is sketched how stationary sunspot equilibria can be shown not to be robust as indicated in previous work. However existence of stationary sunspot equilibria may still be of interest because economies with endowment vectors in a neighborhood of economies with stationary sunspot equilibria should be expected to have sunspot equilibria close to stationary.

The usual approach to existence of endogenous fluctuations is by applications of bifurcation theory to stationary equilibria. In order to apply bifurcation theory it has to be shown that there exist fundamentals such that the matrix of first-order derivatives of the dynamical system has an eigenvalue of -1 for the period doubling bifurcation or a pair of complex eigenvalues of modulus one for the Naimark-Sacker (Hopf) bifurcation. Thus the approach of the present paper is basic compared to the usual approach as it rests on the implicit function theorem and linear algebra rather than bifurcation theory.

The paper is organized follows. In Section 2 the structure of economies, assumptions and the notion of equilibrium is presented. In Section 3 the main result of the present paper and a sketch of the proof are presented. In Section 4 the proof of the main result is presented. Finally in Section 5 it is sketched how the approach can be used to establish the existence of stationary sunspot equilibria.

2 Set-Up

Consider a stationary pure-exchange overlapping generations economy where time extends from $-\infty$ to ∞. At every date there is a finite number ℓ of goods and a finite number m of consumers, who live for two dates, is born.

Let $p_t = (p_t^1, \ldots, p_t^\ell) \in \mathbb{R}^\ell_{++}$ be the price vector at date t and let $(p_t)_t$ be a price system.

Consumers are described by their identical consumption sets $X = \mathbb{R}^{2\ell}$, their endowment vectors $\omega_i = (\omega_i^y, \omega_i^o) \in X$, where $\omega_i^y, \omega_i^o \in \mathbb{R}^\ell$, and their utility functions $u_i : X \to \mathbb{R}$. An economy is a list of consumers $(\omega_i, u_i)_i$.

Consumer i is supposed to satisfy the following assumptions

(A.1) $u_i \in C^2(X, \mathbb{R})$.

(A.2) $Du_i(x) \in \mathbb{R}^{2\ell}_{++}$.

(A.3) $z^T D^2 u_i(x) z < 0$ for all $z \in \mathbb{R}^{2\ell} \setminus \{0\}$.

(A.4) For all $a \in \mathbb{R}$ there exists $y \in X$ such that if $u(x) = a$ then $x \geq y$.

All assumptions are standard. The set of utility function satisfying (A.1)-(A.4) is endowed with the Whitney topology and the set of lists of individual utility functions $(u_i)_i$ is endowed with the product topology.

The problem of consumer i in generation t for a pair of price vectors (p_t, p_{t+1}) and an income w_{it}, where $w_{it} = p_t \cdot \omega_i^y + p_{t+1} \cdot \omega_i^o$, is

$$\begin{aligned} \max_{(x^y, x^o)} \quad & u_i(x^y, x^o) \\ \text{s.t.} \quad & p_t \cdot x^y + p_{t+1} \cdot x^o \leq w_{it} \end{aligned}$$

For every pair of price vectors (p_t, p_{t+1}) and income w_{it} there exists unique solution to the problem of consumer i. Therefore let $f_i : \mathbb{R}^{2\ell}_{++} \times \mathbb{R} \to X$ be the demand function of consumer i.

Let $r = \sum_i \omega_i^y + \sum_i \omega_i^o$ be the vector of available resources, then an equilibrium is a price system and a list of individual endowment vectors such that all markets clear.

Definition 1 *An* **equilibrium** *is a price system and a list of individual endowment vectors* $((p_t)_t, (\omega_i)_i)$ *such that*

$$\sum_i f_i^y(p_t, p_{t+1}, p_t \cdot \omega_i^y + p_{t+1} \cdot \omega_i^o) + \sum_i f_i^o(p_{t-1}, p_t, p_{t-1} \cdot \omega_i^y + p_t \cdot \omega_i^o) \;=\; r$$

for all t.

3 Cycles

A n-cycle is an equilibrium where price vectors alternate deterministically between n price vectors.

Definition 2 *A* n**-cycle** *is an equilibrium* $((p_t)_t, (\omega_i)_i)$ *such that* $p_{t+n} = p_t$ *for all* t. *A* **non-trivial** n**-cycle** *is a* n*-cycle that is not a* k*-cycle for any* $k < n$.

In Kehoe & Levine (1984) it is shown that every economy has a 1-cycle or a steady state. For simplicity n-cycles are denoted $((p_j)_{j=1}^n, (\omega_i)_i)$.

Let $\mathcal{U}(p, (w_i)_i)$ denote the set of lists of individual utility functions for which the $\ell \times \ell$-matrices

$$(D_w f_1^y \dots D_w f_\ell^y)$$

and

$$I + (-1)^{n-1}((D_w f_1^y \dots D_w f_\ell^y)^{-1}(D_w f_1^o \dots D_w f_\ell^o))^n,$$

where $D_w f_i^y = D_w f_i^y(p, p, w_i)$ and $D_w f_i^o = D_w f_i^o(p, p, w_i)$, have rank ℓ.

Lemma 1 *Suppose that* $m \geq \ell$. *Then* $\mathcal{U}(p, (w_i)_i)$ *is open and dense for all* $(p, (w_i)_i)$.

Remark: Since the proof of Lemma 1 is rather long and not too complicated it is delegated to the Appendix.

End of remark

Let $\mathcal{P}_n \subset (\mathbb{R}_{++}^\ell)^n$ denote the set of lists of n price vectors $(p_j)_{j=1}^n$, such that the $\ell \times (n-1)$-matrix $(p_1 - p_2 \dots p_{n-1} - p_n)$ has rank $n - 1$.

Theorem 1 *Suppose that* $n \leq \ell + 1$ *and* $\ell \leq m$. *For every price vector and list of individual incomes* $(p, (w_i)_i)$, *list of individual utility functions* $(u_i)_i \in \mathcal{U}(p, (w_i)_i)$ *and neighborhood* $\mathcal{N}_w$ *of* $(w_i)_i$, *there exists a neighborhood* $\mathcal{N}_p$ *of* p *such that for all* $(p_j)_{j=1}^n \in \mathcal{N}_p^n \cap \mathcal{P}_n$, *there exists* $(\omega_i)_i$ *such that* $((p_j)_j, (\omega_i)_i)$ *is a non-trivial* n*-cycle and* $(w_i^j)_{i,j} \in \mathcal{N}_w^n$, *where* $w_i^j = p_j \cdot \omega_i^y + p_{j+1} \cdot \omega_i^o$ *for all* i *and* j.

Sketch of proof: The proof consists of two major steps. Here the steps are explained and in the next section the steps are done in detail.

In the first step the following system of equations is considered

$$\begin{aligned}
\sum_i f_i^y(p_1, p_2, w_i^1) + \sum_i f_i^o(p_n, p_1, w_i^n) &= r \\
\sum_i f_i^y(p_2, p_3, w_i^2) + \sum_i f_i^o(p_1, p_2, w_i^1) &= r \\
&\vdots \\
\sum_i f_i^y(p_{n-1}, p_n, w_i^{n-1}) + \sum_i f_i^o(p_{n-2}, p_{n-1}, w_i^{n-2}) &= r \\
\sum_i f_i^y(p_n, p_1, w_i^n) + \sum_i f_i^o(p_{n-1}, p_n, w_i^{n-1}) &= r.
\end{aligned} \tag{1}$$

For a price vector and a list of individual incomes $(p, (w_i)_i)$ the implicit function theorem can applied to the system of equations (1) at $((p_j)_j, (w_i^j)_{i,j}, r)$, where $p_j = p$, $w_i^j = w_i$ and $r = \sum_i f_i^y(p, p, w_i) + \sum_i f_i^o(p, p, w_i)$ to obtain some of the individual incomes as functions of lists of price vectors and the rest of the individual incomes.

In the second step for a list of price vectors, a list of individual incomes and a vector of available resources $((p_j)_j, (w_i^j)_{i,j}, r)$ the following systems of equations are considered

$$\begin{aligned}
p_1 \cdot \omega_i^y + p_2 \cdot \omega_i^o &= w_i^1 \\
p_2 \cdot \omega_i^y + p_3 \cdot \omega_i^o &= w_i^2 \\
&\vdots \\
p_{n-1} \cdot \omega_i^y + p_n \cdot \omega_i^o &= w_i^{n-1} \\
p_n \cdot \omega_i^y + p_1 \cdot \omega_i^o &= w_i^n
\end{aligned} \tag{2}$$

for all i and

$$\sum_i \omega_i^y + \sum_i \omega_i^o = r. \tag{3}$$

A list of individual endowment vectors $(\omega_i)_i$, such that the systems of equations (2) and (3) are satisfied, is found.

The first step is used to find a list of price vectors $(p_j)_j$ and the second step is used to find a list of individual endowment vectors $(\omega_i)_i$ such that $((p_j)_j, (\omega_i)_i)$ is a n-cycle.

End of sketch

Remark: As explained in Section 2 consumption sets are unbounded rather than bounded from below. Clearly both steps in the proof of Theorem 1 remain valid for consumption sets that are bounded from below. However if consumption sets are bounded from below and the endowment vectors have to be in the consumption sets, then the second step could fail.

End of remark

For a n-cycle there are $\ell n - 1$ equilibrium conditions because of Walras law and $\ell n - 1$ prices because demand functions are homogenous of degree zero. Therefore n-cycles should be expected to be robust in following sense: if an economy has a n-cycles, then there exists a sequence of economies that converges to the economy such that every economy in the sequence has a n-cycle and for every economy in the sequence there exists a neighborhood such that every economy in the neighborhood has a n-cycle.

Proposition 1 *Suppose that $((p_j)_j, (\omega_i)_i)$ is a non-trivial n-cycle for $(u_i)_i$. Then in every neighborhood of $(u_i)_i$ there exist $(u'_i)_i$ and a neighborhood of $(\omega_i)_i$ such that for every $(\omega'_i)_i$ in the neighborhood of $(\omega_i)_i$ there exists $(p'_j)_j$ such that $((p'_j)_j, (\omega'_i)_i)$ is a non-trivial n-cycle for $(u'_i)_i$.*

Remark: Since the proof of Proposition 1 is based on the proof of Lemma 1 it is delegated to the Appendix.

End of remark

4 Proof of Theorem 1

The proof of Theorem 1 consists of two lemmas.

Definition 3 *A* **price-income** n**-cycle** *for $r \in \mathbb{R}^\ell$ is a list of price vectors and a list of individual incomes $((p_j)_{j=1}^n, ((w_i^j)_i)_{j=1}^n)$ such that the system of equations (1) is satisfied.*

Lemma 2 *Suppose that $m \geq \ell$. For all $(p, (w_i)_i)$ and $(u_i)_i \in \mathcal{U}(p, (w_i)_i)$ if*

$$r = \sum_i f_i^y(p, p, w_i) + \sum_i f_i^o(p, p, w_i)$$

then there exist a neighborhood $\mathcal{N}_p$ of p, a neighborhood $\mathcal{N}_w$ of $(w_i)_i$ and a differentiable map $\Gamma : \mathcal{N}_p^n \times (pr_{\{\ell+1,\ldots,m\}}\mathcal{N}_w)^n \to (pr_{\{1,\ldots,\ell\}}\mathcal{N}_w)^n$ such that for all $(p_j)_{j=1}^n$, where $p_j \in \mathcal{N}_p$ for all j, and $(w_i^j)_{i,j}$, where $(w_i^j)_i \in \mathcal{N}_w$ for all j,

$$(w_i^j)_{i\in\{1,\ldots,\ell\},j} = \Gamma(p_1, \ldots, p_n, (w_i^j)_{i\in\{\ell+1,\ldots,m\},j})$$

if and only if $((p_j)_{j=1}^n, ((w_i^j)_i)_{j=1}^n)$ is a price-income n-cycle for r.

Proof: For $(p,(w_i)_i,r)$, where $r = \sum_i f_i^y(p,p,w_i) + \sum_i f_i^o(p,p,w)$, suppose $((p_j)_j,(w_i^j)_i)_j)$ is defined by $p_1 = \ldots = p_n = p$ and $w_i^1 = \ldots = w_i^n = w_i$, then $((p_j)_j,(w_i^j)_i)_j)$ is a solution to the system of equations (1). Let the $\ell\times\ell$ matrices A and B be defined by $A = (D_w f_1^y \ldots D_w f_\ell^y)$ and $B = (D_w f_1^o \ldots D_w f_\ell^o)$, then the derivatives of the system of equations (1) with respect to $((w_i^j)_{i=1}^\ell)_{j=1}^n$ is

$$\begin{pmatrix} A & & & B \\ B & \ddots & & \\ & \ddots & A & \\ & & B & A \end{pmatrix}.$$

Consider the following operations on matrix of derivatives of the system of equations (1): the first row-block is multiplied by $-BA^{-1}$ from the left and added to the next row-block,... , the second last row-block is multiplied by $-BA^{-1}$ from the left and added to the last row-block. Then the matrix of derivatives of the system of equations (1) becomes

$$\begin{pmatrix} A & & & & B \\ & A & & & -BA^{-1}B \\ & & \ddots & & \vdots \\ & & & A & (-1)^{n-2}B(A^{-1}B)^{n-2} \\ & & & & A+(-1)^{n-1}B(A^{-1}B)^{n-1} \end{pmatrix}.$$

Therefore the matrix of derivatives of the system of equations (1) has rank ℓn, because the $\ell \times \ell$ matrices A and $I + (-1)^{n-1}(A^{-1}B)^n$ have rank ℓ by assumption and $A + (-1)^{n-1}B(A^{-1}B)^{n-1} = A(I + (-1)^{n-1}(A^{-1}B)^n)$.

Hence according to the implicit function theorem there exist a neighborhood $\mathcal{N}_p$ of p, a neighborhood $\mathcal{N}_w$ of $(w_i)_i$ and a differentiable map $\Gamma : \mathcal{N}_p^n \times (\mathrm{pr}_{\{\ell+1,\ldots,m\}}\mathcal{N}_w)^n \to (pr_{\{1,\ldots,\ell\}}\mathcal{N}_w)^n$ such that for all $(p_j)_{j=1}^n$ where $p_j \in \mathcal{N}_p$ for all j, and $(w_i^j)_{i,j}$ where $(w_i^j)_i \in \mathcal{N}_w$ for all j

$$(w_i^j)_{i\in\{1,\ldots,\ell\},j} = \Gamma(p_1,\ldots,p_n,(w_i^j)_{i\in\{\ell+1,\ldots,m\},j})$$

if and only if $((p_j)_{j=1}^n,((w_i^j)_i)_{j=1}^n)$ is a price-income n-cycle.

□

Definition 4 *For a list of price vectors, a list of individual incomes and a vector of total endowment* $((p_j)_{j=1}^n,((w_i^j)_i)_{j=1}^n,r)$, *an* **endowment** n**-cycle** *is a list of individual endowment vectors* $(\omega_i)_i$ *such that the systems of equations (2) and (3) are satisfied.*

Lemma 3 *Suppose that $n \leq \ell + 1$. For all $(p_j)_{j=1}^n \in \mathcal{P}_n$, $((w_i^j)_i)_{j=1}^n$ and r, if $\sum_{i,j} w_i^j = (\sum_j p_j) \cdot r$, then there exists an endowment n-cycle.*

Proof: For all $((p_j)_j, (w_i^j)_{i,j}, r)$ where $\sum_{i,j} w_i^j = (\sum_j p_j) \cdot r$, if $(\omega_i)_i$ satisfies the system of equations (2) except the last equation and the system of equations (3), then the last equation of the system of equations (2) is satisfied too, so $(\omega_i)_i$ is an endowment n-cycle.

For $(p_j)_{j=1}^n$ let the $n \times 2\ell$-matrix P be defined by

$$P = \begin{pmatrix} p_1^T & p_2^T \\ \vdots & \vdots \\ p_n^T & p_1^T \end{pmatrix}$$

and let the $(n-1) \times 2\ell$-matrix $(Q_1 Q_2)$ be the matrix P without the last row. Then $(\omega_i)_i$ is an endowment n-cycles if and only if

$$\begin{pmatrix} P & & & & \\ & \ddots & & & \\ & & P & & \\ & & & Q_1 & Q_2 \\ I\ I & \cdots & I\ I & I & I \end{pmatrix} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix} = \begin{pmatrix} w_1^1 \\ \vdots \\ w_m^{n-1} \\ r \end{pmatrix}.$$

Consider the following operations on the matrix above: the column-block for ω_m is multiplied by -1 and added to the column-block for ω_1; $\ldots$; the column-block for ω_m is multiplied by -1 and added to the column-block for ω_{m-1}; $\ldots$; the row-block for consumer 1 except the last row is multiplied by -1 and added to row-block for consumer m; $\ldots$; the row-block for consumer $m-1$ except the last row is multiplied by -1 and added to the row-block for consumer m, and; the column-block for ω_i^o is multiplied by -1 and added to the column-block for ω_i^y. Then the matrix above becomes

$$\begin{pmatrix} P & & & & \\ & \ddots & & & \\ & & P & & \\ & & & Q_1 - Q_2 & Q_2 \\ & & & & I \end{pmatrix}.$$

If the matrix P has rank n and the matrix $Q_1 - Q_2$ has rank $n-1$, then the matrix above has rank $\ell + mn - 1$.

The matrix $Q_1 - Q_2 = (p_1 - p_2 \dots p_{n-1} - p_n)^T$ has rank $n-1$, because $(p_j)_{j=1}^n \in \mathcal{P}_n$. Consider the following operations on the matrix P: The first $n-1$ rows are added to the last row, and; the last column-block is multiplied by -1 and added to the first column-block. Then the matrix becomes

$$\begin{pmatrix} Q_1 - Q_2 & Q_2 \\ & \sum_{j=1}^n p_j^T \end{pmatrix}.$$

Therefore the matrix P has rank n as the matrix $Q_1 - Q_2$ has rank $n-1$. Hence there exists a endowment n-cycle.

□

5 Sunspot Equilibria

In the present section it is explained how the approach to existence of n-cycles can be applied to existence of stationary n-state sunspot equilibria. In stationary n-state sunspot equilibria there are n price vectors $p_1, \dots, p_n$ and a transition matrix π such that if the price at date t is p_j, then the price at date $t+1$ is p_k with probability $\pi_{jk} > 0$ so $\sum_k \pi_{jk} = 1$.

If the price at date t is p_j, then the problem of consumer i is

$$\max_{(x^y,(x^k)_k)} \quad \sum_k \pi_{jk} u_i(x^y, x^k)$$
$$\text{s.t.} \quad \begin{cases} p_j \cdot x^y + p_1 \cdot x^1 & = & w_i^{j1} \\ & \vdots & \\ p_j \cdot x^y + p_n \cdot x^n & = & w_i^{jn} \end{cases}$$

Let $g_i = (g_i^y, (g_i^j)_j) : \mathbb{R}_{++}^{\ell(1+n)} \times \mathbb{R}^n \times]0,1[^n \to \mathbb{R}^{\ell(1+n)}$ be the demand function of consumer i.

Definition 5 *A* **stationary n-state sunspot equilibrium** *is a list of n price vectors, a list of individual endowment vectors and a transition matrix $((p_j)_j, (\omega_i)_i, \pi)$ such that*

$$\sum_i g_i^y(p_j, (p_{j'})_{j'}, (p_j \cdot \omega_i^y + p_{j'} \cdot \omega_i^o)_{j'}, \pi_j)$$
$$+ \sum_i g_i^j(p_k, (p_{k'})_{k'}, (p_k \cdot \omega_i^y + p_{k'} \cdot \omega_i^o)_{k'}, \pi_k) \quad = \quad r$$

for all j and k. A **non-trivial n-state sunspot equilibrium** *is a n-state sunspot equilibrium where $p_k \neq p_j$ and $\pi_{jk} \neq 0$ for all j and k.*