A Fast Nested Endogenous Grid Method for Solving General Consumption-Saving Models*

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Abstract

This paper proposes a new fast method to solve multi-dimensional consumption-saving models. A model with non-durable and durable consumption subject to adjustment costs can e.g. be solved in a few seconds. The solution method is based on nesting an endogenous grid method (EGM) step inside a standard value function iteration (VFI) algorithm. This is shown to both simplify the optimization problem in the VFI step, and reduce the overall need for time consuming numerical integration. A novel vectorized multi-linear interpolation algorithm is furthermore introduced to speed up the EGM step. The solution method is applicable to a much broader class of models than existing pure EGMs. For a benchmark two-asset consumption-saving model with liquid and illiquid assets, the proposed solution method is more than a 100 times faster than pure value function iteration for a given level of accuracy. Software is provided. (JEL: C6; D91; E21)

Keywords: Endogenous grid method, post-decision states, stochastic dynamic programming, continuous and discrete choices; occasionally binding constraints.

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1 Introduction

The endogenous grid method (EGM) developed by Carroll (2006) let us solve the canonical buffer-stock model much faster than with other methods.[1] The only downside is that the original EGM only applies to one-dimensional models without non-convexities.

The first contribution of this paper is to show that an EGM step can be nested in an outer value function iteration (VFI) algorithm for a broad class of multi-dimensional consumption-saving models even if they contain non-convexities such as discrete choices or adjustment costs. The EGM step only rely on the consumption Euler equation, and unlike existing multi-dimensional pure EGMs (White, 2015; Druedahl and Jørgensen, 2017) the proposed solution method does not require us to keep track of value function derivatives or invert systems of Euler equations.

The second contribution of the paper is the introduction of a novel multi-linear interpolation algorithm for evaluating an ordered vector of points for multiple functions on the same grid. This substantially speeds up the EGM step because it relies heavily on interpolation. The proposed vectorized multi-linear interpolation algorithm can also be incorporated in alternative EGM based solution methods, or in other numerical tasks where interpolation is the bottleneck.

The third and final contribution of the paper is to show that once the EGM step has been precomputed, the remaining optimization problem in the VFI step can both be dimension reduced and solved without any numerical integration. In total, this might allow us to take expectations less than once per grid point in the state space.

I refer to the proposed solution method as the *nested endogenous grid method* (NEGM). Using a two-asset consumption-saving model with a liquid asset and an illiquid asset (similar to Kaplan and Violante, 2014), I show that the NEGM is more than a 100 times faster than pure VFI for a given level of accuracy. Without my novel vectorized interpolation algorithm the speed-up factor is reduced by almost two-thirds.

[1] See Jørgensen (2013) and Low and Meghir (2017).

In sum, the NEGM makes it possible to solve and estimate richer consumption-saving models than previously, and thus makes it computationally feasible to perform policy analysis based on more realistic models. Software is provided as supplemental material at the author's website.

Example To focus ideas consider a standard life-cycle consumption-saving model with non-durable and durable consumption,[2]

$$\begin{aligned}
V_t(M_t, \overline{D}_t, P_t) &= \max_{C_t, D_t} \frac{(C_t^{\alpha} D_t^{1-\alpha})^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t[V_{t+1}(\bullet)] \\
& \text{s.t.} \\
A_t &= M_t + \overline{D}_t - \tau \overline{D}_t \mathbf{1}_{D_t \neq \overline{D}_t} - C_t - D_t \\
M_{t+1} &= R \cdot A_t + Y_{t+1} \\
\overline{D}_{t+1} &= (1-\delta) D_t \\
P_{t+1} &= G_t \psi_{t+1} P_t, \quad \log \psi_{t+1} \sim \mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2) \\
Y_{t+1} &= P_{t+1} \xi_{t+1}, \quad \log \xi_{t+1} \sim \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2) \\
A_t &\geq 0,
\end{aligned} \tag{1.1}$$

where M_t is cash-on-hand, $\overline{D}_t$ is the stock of durables, P_t is permanent income, C_t is non-durable consumption, D_t is durable consumption, A_t is end-of-period assets, Y_t is income and ψ_t and ξ_t are permanent and transitory shocks. For $\tau > 0$ there are adjustment costs proportional to the stock of durables. Households live for T periods, and after period T_R they are retired and $G_t = 1$, $\psi_t = 1$ and $\xi_t = \kappa$. Using the NEGM the model in (1.1) can be solved in a few seconds on a standard desktop computer.[3]
Denote normalized variables by lowercases, $c_t = C_t/P_t$, $d_t = D_t/P_t$ etc.,

[2] Similar to Berger and Vavra (2015) and Harmenberg and Oberg (2016).

[3] The code for this example, provided as supplemental material at the author's website, is simpler than for the model with liquid and illiquid assets considered later.

and define the post-decision value function by

$$\begin{aligned} w_t(a_t, d_t) &\equiv \beta \mathbb{E}_t[(G_t\psi_{t+1})^{1-\rho} v_{t+1}(m_{t+1}, \overline{d}_{t+1})] \\ & m_{t+1} \equiv R \cdot a_t/(G_t\psi_{t+1}) + \xi_{t+1} \\ & \overline{d}_{t+1} \equiv (1-\delta)d_t/(G_t\psi_{t+1}), \end{aligned} \tag{1.2}$$

where $v_{t+1}(\bullet) = V_{t+1}(\bullet)/P_t^{1-\rho}$.

The problem can then be rewritten as

$$\begin{aligned} v_t(m_t, \overline{d}_t) &= \max\{v_t^{keep}(m_t, \overline{d}_t), v_t^{adj.}(x_t)\} \\ & \text{s.t.} \\ x_t &= m_t + (1-\tau)\overline{d}_t, \end{aligned} \tag{1.3}$$

where the value of keeping the current stock of durables is

$$\begin{aligned} v_t^{keep}(m_t, \overline{d}_t) &= \max_{c_t \in [0, m_t]} \frac{(c_t^{\alpha} \overline{d}_t^{1-\alpha})^{1-\rho}}{1-\rho} + w_t(m_t, \overline{d}_t) \\ & \text{s.t.} \\ a_t &= m_t - c_t, \end{aligned} \tag{1.4}$$

and the value of adjusting the stock of durables is only a function of resources after liquidation, x_t,

$$\begin{aligned} v_t^{adj.}(x_t) &= \max_{s_t \in [0,1]} \frac{(c_t^{\alpha} d_t^{1-\alpha})^{1-\rho}}{1-\rho} + w_t(a_t, d_t) \\ & \text{s.t.} \\ d_t &= s_t x_t \\ m_t &= x_t - d_t \\ c_t &= c_t^{\star,keep}(m_t, d_t), \end{aligned} \tag{1.5}$$

where $c_t^{\star,keep}(m_t, \overline{d}_t)$ is the solution to the problem of the keepers. The insight that once we have found $c_t^{\star,keep}(m_t, \overline{d}_t)$ the optimization problem for the adjusters can be dimension reduced is shown below to be valuable for a broad class of consumption-saving models. Note also that no numerical integration is needed when solving the optimization problem for the adjusters in equation (1.5).

The main idea in the NEGM is to solve the problem of the keepers with an EGM. This is possible because all interior consumption choices of the keepers must satisfy the Euler-equation

$$\alpha c_t^{\alpha(1-\rho)-1}\overline{d}_t^{(1-\alpha)(1-\rho)} = q(a_t, \overline{d}_t), \tag{1.6}$$

where

$$q(a_t, d_t) \equiv \beta R \mathbb{E}_t[(G_t\psi_{t+1})^{1-\rho}\alpha c_{t+1}^{\alpha(1-\rho)-1} d_{t+1}^{(1-\alpha)(1-\rho)}]. \tag{1.7}$$

The Euler-equation is, however, only necessary and not sufficient, due to potential jumps in the next period consumption function given the discrete choice whether to adjust or not. The problem can, however, still be solved using EGM combined with an upper envelope algorithm (Fella, 2014; Iskhakov, Jørgensen, Rust and Schjerning, 2017; Druedahl and Jørgensen, 2017). The novel vectorized multi-linear interpolation algorithm, proposed in this paper, is very useful when computing $q(a_t, d_t)$, where we for given d_t, ψ_{t+1} and ξ_{t+1} need to compute, c_{t+1}, d_{t+1} and v_{t+1} for both *keep* and *adjust* in the next period over an ordered grid of a_t values.

Structure. The paper is henceforth structured as follows. Section 2 discuss the related literature. Section 3 presents the NEGM in general terms. Section 4 presents a benchmark model to test the NEGM in practice. Section 5 presents speed and accuracy results comparing the NEGM with VFI. Section 6 concludes.

2 Related Literature

A number of generalizations of EGM have previously been proposed. Closest to this paper is Druedahl and Jørgensen (2017), who develops a generalized EGM algorithm, denoted G^2EGM, for solving multi-dimensional models with both non-convexities and constraints. The main benefit of G^2EGM, relative to the NEGM proposed in this paper, is that G^2EGM avoids any VFI step altogether. The cost is that a much smaller set of models can be solved with G^2EGM than with the NEGM. Specifically, G^2EGM requires that the equation system of stacked discrepancies in the first order conditions and post-decision state equations have a unique solution. This is not required by NEGM, where only the first order condition for consumption is used. As an example, the benchmark model used in this paper cannot be solved by G^2EGM. Additionally, G^2EGM rely on a complex multi-dimensional upper envelope algorithm, while NEGM rely on a simple one-dimensional upper envelope.

Previously, Barillas and Fernández-Villaverde (2007) extended EGM to the case with two control variables, but still one state variable and no non-convexities. Fella (2014) and Iskhakov, Jørgensen, Rust and Schjerning (2017) considered extensions allowing for non-convexities, but both only consider one-dimensional models. White (2015) showed how a pure EGM can be used to solve multi-dimensional models without neither non-convexities nor constraints.

Similar to this paper, Hintermaier and Koeniger (2010) and Ludwig and Schön (2016) also propose a hybrid solution method with both an EGM step and a time iteration step (like the value function iteration step in the NEGM proposed in this paper). They both, however, restrict attention to models without non-convexities, handle constraints using a more model-specific approach, and finally use numerical integration in their time iteration steps, which I in this paper show can be avoided.

The idea of precomputing post-decision functions, which EGM fundamentally relies on, is well-known in the operations research and engineering literature. For *infinite* horizon models is it thus common to iterate on a Bellman equation in the post-decision value function; see in particular Van Roy, Bertsekas, Lee and Tsitsiklis (1997), Powell (2011) and Bertsekas (2012).[4] The idea of using precomputations to limit the costs of numerical integration was also proposed by Judd, Maliar and Maliar (2017) in an algorithm where the value function is approximated by a parametric function.

[4] See Hull (2015) for some extensions and an application in economics.

3 Solution Method

In this section, I present the proposed *nested endogenous grid method*, henceforth NEGM, in general terms. In Section 4, I will turn to a specific example.

Let M_t denote beginning-of-period *cash-on-hand* (i.e. liquid resources), C_t denote *consumption*, and A_t denote end-of-period *liquid assets*. Further let $N_t = (N_t^1, \dots, N_t^{\#_N})$ be the vector of the remaining discrete and continuous states, and $D_t = (D_t^1, \dots, D_t^{\#_N}) \in \mathcal{D}(M_t, N_t)$ the vector of the remaining discrete and continuous choices. Also let $B_t = (B_t^1, \dots, B_t^{\#_B})$ be the vector of the remaining *post-decision states* besides A_t. Assume that B_t for some function $\mathcal{B}$ is given by

$$B_t = \mathcal{B}(N_t, D_t). \tag{3.1}$$

For a vector of stochastic shocks, ψ, let the transition functions for M_t and N_t be denoted $\mathcal{T}_M$ and $\mathcal{T}_N$. Assume that $\mathcal{T}_M$ is continuous, differentiable and strictly monotone wrt. A_t, and that $\mathcal{T}_N$ is independent of A_t.[5]

The fundamental restrictive assumption so far, is that consumption, C_t, only affect the future through its effect on first end-of-period assets, A_t, and then future cash-on-hand, M_{t+1}. This is a mild assumption satisfied in most models.

Further assume that the N_t states and D_t choices only affect end-of-period assets *additively*, i.e. that I for some function $\mathcal{L}$ can write

$$L_t = M_t + \mathcal{L}(N_t, D_t) \tag{3.2}$$

$$A_t = L_t - C_t, \tag{3.3}$$

where L_t are liquid assets just-before-consumption. This is also a rather weak assumption, and fulfilled in most consumption-saving models in the literature. Likewise assume that the minimum level of end-of-period liquid assets is only a function of the other post-decision states, i.e. $A_t \geq \underline{A}(B_t)$. This implies that the maximum level of consumption is given by

[5] I can allow $\mathcal{T}_M$ to be non-differentiable wrt. A_t at a finite number of kinks, see sub-section 3.6 below

$\overline{C}(L_t, B_t) \equiv L_t - \underline{A}(B_t)$.

Finally, assume that the effect of N_t and D_t on utility is *additively separable* such that the period utility function for functions u and h has the form

$$u(C_t, B_t) + h(N_t, D_t), \tag{3.4}$$

where it is required that u'_C exists and (i) $u'_C > 0$, (ii) $u''_{CC} < 0$, and (iii) u'_C is invertible wrt. to C_t (i.e. $u'^{-1}_{C,C}$ exists).[6]

In sum, the Bellman equation for the considered model class can be written on the form

$$\begin{aligned}
V_t(M_t, N_t) &= \max_{C_t, D_t} u(C_t, B_t) + h(N_t, D_t) + \beta \mathbb{E}_t \left[V_{t+1}(\bullet) \right] \\
&\text{s.t.} \\
L_t &= M_t + \mathcal{L}(N_t, D_t) \\
A_t &= L_t - C_t \\
B_t &= \mathcal{B}(N_t, D_t) \\
M_{t+1} &= \mathcal{T}_M(A_t, B_t, \psi) \\
N_{t+1} &= \mathcal{T}_N(B_t, \psi) \\
D_t &\in \mathcal{D}(M_t, N_t) \\
C_t &\in [0, \overline{C}(L_t, B_t)],
\end{aligned} \tag{3.5}$$

where β is the discount factor.[7] Denote the optimal choice function by $C^{\star}_t(M_t, N_t)$ and $D^{\star}_t(M_t, N_t)$, and the implied optimal post-decision functions by $A^{\star}_t(M_t, N_t)$ and $B^{\star}_t(M_t, N_t)$.

VFI. Solving the model in (3.5) by *value function iteration* (VFI) is in general time consuming. For each grid point in the (discretized) state space of M_t and N_t, the value-of-choice needs to be calculated for each guess of C_t and D_t when searching for the optimal choices. And to calculate the value-

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[6] The assumption of additive separability for N_t and D_t can be loosed somewhat (but at a potential cost) as discussed in sub-section 3.6 below.

[7] For simplicity, I only consider additively separable time preferences. The NEGM is, however, also fully applicable to models with e.g. Epstein-Zin preferences.

of-choice it is necessary to compute the continuation value, $\mathbb{E}_t[V_{t+1}(\bullet)]$, using numerical integration. This typically requires multiple interpolations of V_{t+1} for a set of Monte Carlo or quadrature notes of ψ.

Especially in multi-dimensional models, interpolation can be a computationally costly operation. All methods which can simplify the optimization problem implying fewer needed calculations of the value-of-choice, or can limit the use of numerical integration, are thus computationally very beneficial.

In the following sections, I show that the proposed NEGM delivers on both these fronts.

3.1 Nested Structure

In order to reformulate the problem in (3.5), it is beneficial to introduce the *post-decision value function* given by

$$W_t(A_t, B_t) \equiv \mathbb{E}_t\left[V_{t+1}(\mathcal{T}_M(A_t, B_t, \psi), \mathcal{T}_N(B_t, \psi))\right]. \tag{3.6}$$

Next, define the following pure consumption problem

$$\begin{aligned} \mathcal{V}_t(L_t, B_t) &= \max_{C_t} u(C_t, B_t) + \beta W_t(A_t, B_t) \qquad (3.7) \\ &\text{s.t.} \\ A_t &= L_t - C_t \\ C_t &\in [0, \overline{C}(L_t, B_t)], \end{aligned}$$

where the optimal consumption function is denoted $C_t^{\star}(L_t, B_t)$.

Given *precomputed interpolants* of $C_t^{\star}(L_t, B_t)$ and $W_t(A_t, B_t)$ on suitable grids (I return to this below), the original model in (3.5) can be reformulated as

$$
\begin{aligned}
V_t(M_t, N_t) &= \max_{D_t} u(C_t, A_t, B_t) + h(N_t, D_t) + \beta W_t(A_t, B_t) \quad (3.8)\\
&\text{s.t.}\\
B_t &= \mathcal{B}(N_t, D_t)\\
L_t &= M_t + \mathcal{L}(N_t, D_t)\\
C_t &= C_t^{\star}(L_t, B_t)\\
A_t &= L_t - C_t\\
M_{t+1} &= \mathcal{T}_M(A_t, B_t, \psi)\\
N_{t+1} &= \mathcal{T}_N(B_t, \psi)\\
D_t &\in \mathcal{D}(M_t, N_t)\\
C_t &\in [0, \overline{C}(L_t, B_t)].
\end{aligned}
$$

This reformulation is greatly beneficial for two reasons. Firstly, it reduces the dimensionality of the optimization problem (only max over D_t, not also C_t). Secondly, it replaces multiple interpolations of the next period value function required to calculate the continuation value, $\mathbb{E}_t[V_{t+1}(\bullet)]$, with two interpolations in total; one interpolation of the consumption function $C_t^{\star}(L_t, B_t)$ and one interpolation of the post-decision value function $W_t(A_t, B_t)$.[8]

The nail in the coffin is that I in the following sections show that $C_t^{\star}(L_t, B_t)$ can be found by the EGM.

3.2 Euler Equation

To show $C_t^{\star}(L_t, B_t)$ can be found by the EGM, it is first necessary to first show that optimal interior consumption choices, $C_t \in (0, \overline{C}(\bullet))$, must satisfy an Euler equation. This can be done using a variational argument.

Specifically, consider a household assumed to be following the optimal plan. If the current consumption choice is interior, $C_t \in (0, \overline{C}(\bullet))$, it is feasible for

[8] Note what while I propose to solve the problem in (3.5) by value function iteration, a time iteration method could alternatively be used. For details on time iteration see Rendahl (2015).

the household to change its consumption today by a small amount $\Delta \neq 0$. It is then also feasible for it to change consumption tomorrow by a likewise small amount $-\mathcal{T}'_{M,A,t+1}\Delta$, where we use the definition

$$\mathcal{T}'_{M,A,t+1} \equiv \frac{\partial \mathcal{T}_M(A_t, B_t, \psi)}{\partial A_t}.$$

Leaving all other *choices* in $t+1$ unchanged, this implies that A_{t+1} and B_{t+1}, and thus future utility from $t+2$ onward, are unchanged. For $\Delta \to 0$ the benefit of this deviation from the plan consequently is

$$\text{sign}(\Delta)(u'_{C_t} - \beta \mathbb{E}_t[\mathcal{T}'_{M,A,t+1} u'_{C_{t+1}}]) = 0, \tag{3.9}$$

All optimal interior consumption, choices $C_t \in (0, \overline{C}(\bullet))$, must therefore satisfy the Euler equation

$$u'_{C_t} = \beta \mathbb{E}_t[\mathcal{T}'_{M,A,t+1} u'_{C_{t+1}}]. \tag{3.10}$$

If equation (3.10) does not hold it implies that there exists a $\Delta \neq 0$ which can increase the expected discounted utility violating the initial assumption that the household were following the optimal plan.

The Euler equation is thus a *necessary* requirement for optimal interior consumption choices. Due to the presence of discrete choices and other non-convexities, it can, however, in general not be proven that the Euler equation is also *sufficient* as in the canonical buffer-stock model.

3.3 EGM

To implement EGM first define the *post-decision expected discounted marginal value of liquid assets* by

$$Q_t(A_t, B_t) = \beta \mathbb{E}_t[\mathcal{T}'_{M,A,t+1} u'_{C,t+1}]. \tag{3.11}$$

The fundamental idea in EGM, originally developed in Carroll (2006) for one-dimensional models, is that the Euler equation in (3.10) implies

$$C_t = u'^{-1}_{C,C}(Q_t(A_t, B_t)), \tag{3.12}$$

and that the budget constraint can be used to further deduce the endogenous pre-decision state, here $L_t = A_t + C_t$. When the Euler equation is sufficient this immediately maps out the consumption function. When the Euler equation, as here, is not necessarily sufficient, Fella (2014) and Iskhakov, Jørgensen, Rust and Schjerning (2017) showed that an *upper envelope algorithm* can be used to discard solutions to the Euler equation which are not globally optimal.

3.4 Upper Envelope

Specifically, I rely on a variation of the upper envelope algorithm proposed in Druedahl and Jørgensen (2017). This allows me to compute the consumption function $C_t^\star(L_t, B_t)$ on exogenously chosen grids for L_t and B_t such that it afterwards can be evaluated using simple multi-linear interpolation. The upper envelope consists of the following four steps:

- **Step 1. Grids.** Define grids for L_t and B_t denoted $\mathcal{G}_L = \{L_j\}_{j=1}^{\#_L}$ and $\mathcal{G}_B = \{B^k\}_{k=1}^{\#_B}$. Define B_t-specific grids for A_t denoted $\mathcal{G}_A^k = \{A^{ik}\}_{i=1}^{\#_A}$. Let the first grid point be given by $A^{1k} = \underline{A}(B^k) + \epsilon$ where ϵ is a tiny number.

- **Step 2. EGM.** For each grid point in $(A^{ik}, B^k) \in \mathcal{G}_A^k \oplus \mathcal{G}_B$, compute the post-decision variables $W^{ik} = W_t(A^k, B^k)$ and $Q^{ik} = Q_t(A^{ik}, B^k)$. Find the consumption choice from

$$C^{ik} = u_{C,C}^{\prime -1}(Q^{ik}), \tag{3.13}$$

  and deduce the liquid asset level just before consumption by

$$L^{ik} = A^{ik} + C^{ik}. \tag{3.14}$$

- **Step 3. Constraint.** For each grid point $(L_j, B^k) \in \mathcal{G}_L \oplus \mathcal{G}_B$ where $L_j < L^{1k}$ set $C_j^k = L_j$ implying that the constraint is binding. This is a consequence of the monotonicity of the marginal utility function together with the lowest grid point in $\mathcal{G}_A^k$ being just above the constraint (see step 1).

- **Step 4. Upper envelope.** For each grid point $(L_j, B^k) \in \mathcal{G}_L \oplus \mathcal{G}_B$ where $L_j \geq L^{1k}$ find the corresponding consumption level, C_j^k, by interpolation. Note that for any i where $L_j \in [L^{ik}, L^{i+1,k}]$ a good consumption guess can be calculated as

$$C_j^{ik} = C^{ik} + \frac{C^{i+1,k} - C^{ik}}{L^{i+1,k} - L^{ik}}(L_j - L^{ik}). \tag{3.15}$$

Specifically, set $C_j^k = C_j^{i^\star(j,k)k}$ where $i^\star(j,k)$ is the i implying the highest value-of-choice, i.e.

$$\begin{aligned} i^\star(j,k) &= \arg\max_{i\in\{1,\ldots\#_A-1\}} u(C_j^{ik}, B^k) + \beta W_j^{ik} \qquad (3.16) \\ &\text{s.t.} \\ L_j &= [L^{ik}, L^{i+1,k}] \\ A_j^{ik} &= L_j - C_j^{ik} \\ W_j^{ik} &= W^{ik} + \frac{W^{i+1,k} - W^{ik}}{A^{i+1,k} - A^{ik}}(A_j^{ik} - A^{ik}). \end{aligned}$$

This upper envelope is in itself a very cheap operation. A very dense common cash-on-hand grid $\mathcal{G}_M$ is thus feasible. The expensive part of EGM is the computation of $W_t(A_t, B_t)$ and $Q_t(A_t, B_t)$ as this requires numerical integration and multiple interpolations of the next period value and consumption functions. The next section shows how to speed up this part of the algorithm

3.5 Vectorized multi-linear interpolation

For numerical integration with a grid of shocks $\mathcal{G}_\psi = \{\psi^q\}_{q=1}^{\#_q}$, and associated weights $\mathcal{G}_\omega = \{\omega^q\}_{q=1}^{\#_q}$, the typical approach to computing $W_t(A_t, B_t)$ and $Q_t(A_t, B_t)$ can be formulated as a simple weighted sum

$$
\begin{aligned}
W^{ik} &\approx \sum_{q=1}^{\#_q} \omega^q V_+^{ikq} && (3.17)\\
Q^{ik} &\approx \beta \sum_{q=1}^{\#_q} \omega^q \mathcal{T}'_{M,A,t+1}(A^{ik}, B^k, \psi^q) u'_C(C_+^{ikq}, B_+^{ikq}) && (3.18)\\
& M_+^{ikq} \equiv \mathcal{T}_M(A^{ik}, B^k, \psi^q)\\
& N_+^{kq} \equiv \mathcal{T}_N(B^k, \psi^q)\\
& V_+^{ikq} \equiv V_{t+1}(M_+^{ikq}, N_+^{kq})\\
& C_+^{ikq} \equiv C_{t+1}^{\star}(M_+^{ikq}, N_+^{kq})\\
& B_+^{ikq} \equiv B_{t+1}^{\star}(M_+^{ikq}, N_+^{kq})
\end{aligned}
$$

A substantial speed-up can, however, be achieved if some special structure in the considered model class is used. Fixing B^k and ψ^q, and thus N_+^{kq}, note that V_{t+1}, $C_{t+1}^{\star}$, and $B_{t+1}^{\star}$ has to be interpolated for same vector of grid points $(M_+^{1:\#_A,kq}, N_+^{kq})$ where $M_+^{1:\#_A,kq} \equiv \mathcal{T}_M(\mathcal{G}_A^k, B^k, \psi^q)$. Using that the $\mathcal{G}_A^k$-grids are increasing by construction, and the assumption that $\mathcal{T}_M$ is strictly monotonically increasing in A_t, it can be inferred that the vector $M_+^{1:\#_A,kq}$ is increasing. This allow me to apply the novel vectorized multi-linear interpolation algorithm for multiple functions on the same grid presented in Algorithm 1.

Breaking down Algorithm 1 into blocks we have:

1. Line 1-3 find the grids positions of N_+^{kq} (e.g. with binary search).
2. Line 4-11 calculate the baseline weights.
3. Line 12-14 find the grid positions of $M_+^{1:\#_A,kq}$ (e.g. with binary search).
4. Line 15-22 compute the interpolated values.

The first benefit of using Algorithm 1 is that the lines 1-11 (grid positions for N_+^{kq}) is only computed *once* for each element in the x-vector (the grid vector for A_t) and each function to be interpolated.

The second benefit is that the memory access in line 22 for the y-arrays (the next period value and policy functions, V_{t+1}, $C^{\star}_{t+1}$, and $B^{\star}_{t+1}$) are well aligned because of the assumption that the x-vector is ordered.

Specially, I propose to parallelize over the grid of B^k, and then for each ψ^q apply Algorithm 1 to the full $\mathcal{G}_A^k$-vector to find $V_+^{1:\#_A,kq}$, $C_+^{1:\#_A,kq}$, and $B_+^{1:\#_A,kq}$. Calculating $W^{1:\#_A,kq}$ and $Q^{1:\#_A,kq}$ is then just

$$W^{1:\#_A,k} = \sum_{q=1}^{\#_q} \omega^q V_+^{ikq} \tag{3.19}$$

$$Q^{1:\#_A,k} = \beta \sum_{q=1}^{\#_q} \omega^q \mathcal{T}'_{M,A}(\mathcal{G}_A^k, B^k, \psi^q) u'_C(C_+^{1:\#_A,kq}, B_+^{1:\#_A,kq}). \tag{3.20}$$

3.6 7777777777777777777777777Extensions

Consider the following four potentially interesting extensions:

1. Kinky $\mathcal{T}_M$. If $\mathcal{T}_M$ is not differentiable everywhere in A_t there will be multiple Euler equations. If there are k kinks where $\mathcal{T}_M$ is not differentiable in A_t, there will be $k+1$ intervals where $\mathcal{T}_M$ is differentiable in A_t. Consequently there will be $k+1$ mutually exclusive Euler equations. Interior optimal consumption choices not at kinks must still satisfy one of these, and the proposed NEGM can therefore still be applied. In the example in Section 4, $\mathcal{T}_M$ is not differentiable everywhere in A_t.

2. Utility depend on A_t i.e. $u(C_t, A_t, B_t)$.

The Euler equation will change, but, as long as u'_C remain invertible in C_t, the proposed NEGM can still be used.

3. Utility is non-separable in N_t and/or D_t i.e. $u(N_t, C_t, D_t, B_t)$.

This in general implies that the model in (3.7) would also need to have N_t and D_t as states. Instead of $C^{\star}_t(L_t, B_t)$ it would be necessary to compute $C^{\star}_t(L_t, N_t, D_t, B_t)$. This is potentially much more costly. The proposed NEGM will therefore only be beneficial if special model structure can be used to e.g. remove the dependence on B_t.

Algorithm 1: Vectorized Linear Interpolation of Multiple Functions

input : d (number of dimensions)
$\#_y$ (number of functions to be interpolated)
$\#_x$ (number of evaluation points)
x (vector of evaluation points in the first dimension)
z (vector of fixed evaluation points in all other dimensions)
$y_1, y_2, \ldots, y_k$ (d-dimensional arrays of known values)
$\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_d$ (grid vectors, ordered asc.)
add (table with 2^d with all permutations of vectors of d 0s and 1s)

output: r (matrix of interpolated values initialized to zero)

for $j = 2$ **to** d **do**
    find pos_left$[j]$ such that $\mathcal{G}_j^{\text{pos\_left}[j]} \leq z[j-1] < \mathcal{G}_i^{\text{pos\_left}[j]+1}$
    rel_diff$[j] = (z[j-1] - \mathcal{G}_j^{\text{pos_left}[j]})/(\mathcal{G}_j^{\text{pos_left}[d]+1} - \mathcal{G}_j^{\text{pos_left}[j]})$

for $i = 1$ **to** 2^d **do**
    initialize $\omega[i] = 1.0$
    for $d = 2$ **to** $dim(y)$ **do**
        $\text{pos}[j, i] = \text{pos\_left}[d] + \text{add}[j, i]$
        if $add[j, i] = 1$ **then**
            $\omega[i] \mathrel{*}= \text{rel\_diff}[j]$
        else
            $\omega[i] \mathrel{*}= (1.0 - \text{rel\_diff}[j])$

for $k = 1$ **to** $\#_x$ **do**
    find pos_left_vec$[k]$ such that $\mathcal{G}_1^{\text{pos\_left\_vec}[k]} \leq x[k] < \mathcal{G}_1^{\text{pos\_left\_vec}[k]+1}$
    rel_diff_vec$[k] = (x[k] - \mathcal{G}_1^{\text{pos_left_vec}[k]})/(\mathcal{G}_1^{\text{pos_left_vec}[k]+1} - \mathcal{G}_1^{\text{pos_left_vec}[k]})$

for $h = 1$ **to** $\#_y$ **do**
    for $i = 1$ **to** 2^d **do**
        for $k = 1$ **to** $\#_x$ **do**
            $\text{pos}[1, i] = \text{pos\_left\_vec}[k] + \text{add}[1, i]$
            if $add[i, 0] = 1$ **then**
                $\overline{\omega} = \text{rel\_diff\_vec}[k] \cdot \omega[i]$
            else
                $\overline{\omega} = (1.0 - \text{rel\_diff\_vec}[k]) \cdot \omega[i]$
            $r[i, h] \mathrel{+}= \overline{\omega} \cdot y^h[\text{pos}[1, i], \text{pos}[2, i], \ldots, \text{pos}[d, i]$

4. The transition to N_{t+1} depend on A_t, i.e. $\mathcal{T}_N(A_t, B_t, \psi_{t+1})$. If $\mathcal{T}_N$ is continuous and differentiable wrt. to A_t it will still be possible to derive an equation like the standard Euler equation, where the RHS only contains post-decision states and LHS only contains consumption. The fast vectorized interpolation scheme can, however, not be straightforwardly used as changes in A_t then affect both M_{t+1} and N_{t+1}.

4 Benchmark Model

In this Section, I outline a benchmark two-asset consumption-saving to test the NEGM in practice. The model have both a liquid and an illiquid asset, and is similar to the model in Kaplan and Violante (2014). Note that the model nests the canonical buffer-stock consumption model of Deaton (1991, 1992) and Carroll (1997, 1992, 2012) as a special case when there are no transaction costs.

Demographics. The economy is populated by a continuum of households living for T periods, $t \in \{1, 2, \dots, T\}$. All households retire deterministically at the end of period T_R.

Preferences. All households have CRRA preferences given by

$$u(C_t) = \zeta_t \frac{C_t^{1-\rho}}{1-\rho}$$

$$\zeta_t = \begin{cases} 1 & \text{if } t \leq T_R \\ \zeta & \text{else} \end{cases},$$

7where C_t is consumption expenditures, ρ is the CRRA coefficient, and ζ is a taste shifter scaling the utility function in retirement.

Earnings. The labor earnings of working households are given by

$$Y_t = P_t \xi_t, \; t \leq T_R \tag{4.1}$$

$$P_t = G_t P_{t-1} \psi_t, \; t \leq T_R, \tag{4.2}$$

where

$$\log \psi_t \sim \mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2)$$
$$\log \xi_t \sim \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2),$$

and G_t is the growth factor of income. After retirement households receive a pension benefit given by

$$Y_t = P_t, \ t \geq T_R + 1 \tag{4.3}$$
$$P_t = \kappa P_{T_R}, \ t \geq T_R + 1. \tag{4.4}$$

Saving and borrowing. Households can save and borrow in a *liquid* asset, A_t, and save in an *illiquid* asset, B_t. When saving in the liquid asset the household earns a risk-free gross return of R, and while borrowing in the liquid asset they pay a gross interest rate of $R_- > R$, i.e

$$R(A_t) = \begin{cases} R & \text{if } A_t > 0 \\ R_- & \text{else.} \end{cases}$$

The household can while working borrow up to a fraction of his permanent income,

$$A_t \geq -\omega P_t, \ t < T_R. \tag{4.5}$$

He can, however, not retire in or die in debt

$$A_{T_R} \geq 0$$
$$A_T \geq 0.$$

Savings in the illiquid asset earns a risk-free gross return of $R_B > R$. To transact in the illiquid asset the household must pay a *fixed adjustment cost* of $\lambda > 0$.[9]

[9] It is also possible allow λ to be stochastic as often considered in the firm investment literature, see Caballero and Engel (1999). This also includes the case in Bayer, Lütticke, Pham-Do and Tjaden (2015) where there in each period is a fixed probability that the household is not allowed to adjust its illiquid assets (i.e. that the adjustment costs are infinite).

The beginning-of-period liquid and illiquid assets, denoted M_t and N_t, in sum evolve as7

$$
\begin{aligned}
M_{t+1} &= R(A_t)A_t + Y_{t+1} \\
N_{t+1} &= R_B B_t.
\end{aligned}
$$

Retirement. Assuming that there are no transaction costs in retirement, the problem from T_R can be solved analytically. The solution is

$$
C^{\star}_{T_R}(M,N,P) = \min\left\{ M_{T_R}, \frac{\gamma_1\left[M+N+(1+\gamma_0)\,R_b^{-1}\kappa P_{T_R}\right]}{R_b^{-1}(\beta R_b\zeta)^{\frac{1}{\sigma}}+\gamma_1} \right\} \tag{4.6}
$$

$$
V_{T_R}(M,N,P) = \frac{C^{*}_{T_R}(M,N,P)^{1-\sigma}}{1-\sigma} + \beta\frac{\zeta\gamma_2 C^{1-\sigma}_{T_R+1}}{1-\sigma}, \tag{4.7}
$$

where

$$
\begin{aligned}
C_{T_R+1} &\equiv \begin{cases} (\beta R_b\zeta)^{\frac{1}{\sigma}} C^{\star}_{T_R}(M,N,P) & \text{if } C^{\star}_{T_R}(M,N,P) < M+N \\ \gamma_1(1+\gamma_0\kappa)P_{T_R} & \text{else} \end{cases} \\
\gamma_0 &\equiv \frac{1-(R_b^{-1})^{T-T_R}}{1-R_b^{-1}} - 1 \\
\gamma_1 &\equiv \frac{1-R_b^{-1}(\beta R_b)^{1/\sigma}}{1-\left[R_b^{-1}(\beta R_b)^{1/\sigma}\right]^{T-T_R}} \\
\gamma_2 &\equiv \frac{1-\left(\beta^{\frac{1}{\sigma}} R_b^{\frac{1-\sigma}{\sigma}}\right)^{T-T_R}}{1-\left(\beta^{\frac{1}{\sigma}} R_b^{\frac{1-\sigma}{\sigma}}\right)}.
\end{aligned}
$$

Hereby we see that the taste shifter, ζ, can be interpreted as a measure of the post-retirement saving motive. If $\zeta = 0$ then there is no motive to save for retirement. If $\zeta = 1$ then consumption smoothing is the only reason to save for retirement. If $\zeta > 1$ there are additional reasons to save for retirement such as e.g. bequests or non-modeled uncertainty and constraints.

4.1 Recursive Formulation and Calibration

The recursive formulation of the model is

$$
\begin{aligned}
V_t(M_t, N_t, P_t) &= \max_{C_t, B_t} u(C_t, B_t) + \beta \mathbb{E}_t[V_{t+1}(\bullet)] \qquad (4.8) \\
\text{s.t.} & \\
A_t &= M_t - C_t + (N_t - B_t) - \mathbf{1}_{B_t \neq N_t} \lambda \\
M_{t+1} &= R(A_t) A_t + Y_{t+1} \\
N_{t+1} &= R_B B_t \\
B_t &\geq 0 \\
A_t &\geq -\omega P_t,
\end{aligned}
$$

7where the income process was specified in equations (4.1)–(4.2), and β is the discount factor.

I choose the parameters given in Table 1. Except for the preference parameters, I choose the same parameters as in Druedahl and Martinello (2017) where the NEGM is applied in practice to a two-asset model of Danish households. The income growth factor is chosen to match the income profile they find.

4.2 7Relation to General Case

Let $z_t \in \{0, 1\}$ denote the choice of whether to adjust or not. The model can then alternatively be written as a maximum over z_t*-specific value functions* conditioning of the discrete choice of whether to adjust or not,

$$
V_t(M_t, N_t, P_t) = \max_{z_t \in \{0,1\}} v_t(M_t, N_t, P_t, z_t). \qquad (4.9)
$$

Table 1: Parameters

Parameter	Description	Value
T	Life span after age 25	60
T_R	Working years	35
β	Discount factor	0.945
ρ	CRRA-coefficient	2
ζ	Taste-shifter	1.1
G_t	Growth factor of income	see text
σ_ψ	Std. of permanent shock	0.073
σ_ξ	Std. of transitory shock	0.085
κ	Retirement replacement rate	0.90
R	Return of *liquid* assets, *saving*	1.02
R_-	Return of *liquid* assets, *borrowing*	1.078
R_B	Monetary return of *illiquid* assets	1.057
ω	Borrowing constraint	0.25
λ	Fixed adjustment cost	$0.02 \cdot \mathbb{E}[P_t]$

The value function for *no-adjustment* is

$$\begin{aligned} v_t(M_t, N_t, P_t, 0) &= \max_{C_t} u(C_t, B_t) + \beta \mathbb{E}_t[V_{t+1}(\bullet)] \\ &\text{s.t.} \\ A_t &= M_t - C_t \\ B_t &= N_t \\ A_t &\geq -\omega P_t. \end{aligned} \tag{4.10}$$

Defining cash-on-hand conditional on adjustment as

$$X_t = M_t + N_t - \lambda, \tag{4.11}$$

the value function for *adjustment* is

$$\begin{aligned} v_t(M_t, N_t, P_t, 1) & & \\ = \tilde{v}_t(X_t, P_t) &= \max_{C_t, B_t} u(C_t, B_t) + \beta \mathbb{E}_t[V_{t+1}(\bullet)] \\ & \text{s.t.} \\ A_t &= X_t - C_t - B_t \\ B_t &\geq 0 \\ A_t &\geq -\omega P_t. \end{aligned} \tag{4.12}$$

Denote the optimal choice functions by $C_t^\star(\bullet, 0)$, $C_t^\star(\bullet, 1)$ and $B_t^\star(\bullet, 1)$. The optimal discrete choice is denoted $z_t^\star(\bullet)$.

Using the variational argument presented for the general model class, it can be proven that an optimal interior consumption choice, C_t, must satisfy one the following Euler equations

$$C_t^{-\rho} = Q_t(A_t, B_t, P_t) \equiv \begin{cases} \beta R \mathbb{E}_t \left[C_{t+1}^{-\rho}\right] & \text{if } A_t > 0 \\ \beta R_{-} \mathbb{E}_t \left[C_{t+1}^{-\rho}\right] & \text{if } A_t \in (-\omega P_t, 0). \end{cases} \tag{4.13}$$

7Consequently, the problem in (4.10) can be solved for the *non-adjusters* by EGM. Further defining the post-decision value function as

$$W_t(A_t, B_t, P_t) = \mathbb{E}_t[V_{t+1}(\bullet)] \tag{4.14}$$

the problem for adjusters can be written as.

$$\begin{aligned} v_t(X_t, P_t) &= \max_{B_t \in [0, X_t + \omega P_t]} u(C_t, B_t) + \beta W_t(A_t, B_t, P_t) \\ & \text{s.t.} \\ M_t &= X_t - B_t \\ N_t &= B_t \\ C_t &= C_t^\star(M_t, N_t, P_t, 0) \\ A_t &= M_t - C_t^\star \end{aligned} \tag{4.15}$$

This problem is much easier than the original problem in (4.12) because it is reduced by one dimension and does not require any numerical integration, but just interpolation of $C_t^{\star}(M_t, N_t, P_t, 0)$ and $W_t(A_t, B_t, P_t)$.

4.3 Implementation

Interpolation. It is not necessary to ever construct the over-arching value function, $V_t(M_t, N_t, P_t)$. The post-decision value function can instead be found directly from the next period z_t-specific functions as

$$W_t(A_t, B_t, P_t) = \beta \mathbb{E}_t \left[\begin{cases} v_{t+1}(\bullet, 0) & \text{if } z_{t+1}^{\star}(\bullet) = 0 \\ \tilde{v}_{t+1}(\bullet) & \text{if } z_{t+1}^{\star}(\bullet) = 1 \end{cases} \right] \tag{4.16}$$

and likewise with $Q_t(A_t, B_t, P_t)$.

Grids. I have separate grids for P_t, M_t, N_t, A_t and X_t while the grid for B_t is the same as that for N_t. All grids vary by t, and the assets grids vary by the current element in P_t, but are otherwise tensor grids.

1. The grid for A_t is chosen to explicitly include $\{-\omega P_t, -\omega P_t + \epsilon, -\epsilon, \epsilon\}$, where ϵ is a small number, such that the borrowing constraint and the kink at $A_t = 0$ is well-approximated. A dense grid for A_t is costly as numerical integration of the next period value function and consumption function is needed for each element.

2. A dense grids for N_t is costly for the same reason as A_t.

3. The grid for M_t is only used in the upper envelope algorithm, and it is therefore feasible for this grid to be very dense.

4. The grid for X_t is only used for the adjusters. Consequently it is feasible to have a rather dense grid.

5. A dense grid for P_t is costly both for the same reason as A_t and because it implies that the adjuster problem have to be solved more times.

In general, all grids are specified such that they are relatively more dense for smaller values, and this even more so for small P_t. The largest node in each grid is proportional to P_t.

7Numerical integration. For evaluating expectations Gauss-Hermit quadrature with 6 points for each shock, $\#_\psi = \#_\xi = 6$, is used.

Multi-start. For solving the problem in (4.15) $\#_k = 3$ multi-start values for B_t are used.

Language. The code is written in `C++` (`OpenMP` is used for parallelization) with an interface to `MATLAB` for setting up grids and printing figures. The optimization problems are solved by the *Method of Moving Asymptotes* from Svanberg (2002), implemented in `NLopt` by Johnson (2014). The code was run on a *Intel(R) Core(TM) i7-4770 CPU* with 8 processors (4 cores) and 32 GB of RAM.

5 Speed and Accuracy

To measure the accuracy of the proposed NEGM, I, following Judd (1992) and Santos (2000), calculate the $\log_{10}$ of the *relative absolute Euler error* across households optimally making an interior consumption choice. Specifically, I calculate

$$\overline{\mathcal{E}} \equiv \frac{\sum_{t\in\{5,10,15,20,30\}}\sum_{i=1}^{N}\mathcal{E}_{it}\mathbf{1}_{A_{it}\in[\omega P_t+\epsilon;-\epsilon]\cup[\epsilon;\infty[}}{\sum_{t\in\{5,10,15,20,30\}}\sum_{i=1}^{N}\mathbf{1}_{A_{it}\in[\omega P_t+\epsilon;-\epsilon]\cup[\epsilon;\infty[}} \tag{5.1}$$

where

$$\begin{aligned}\mathcal{E}_{it} &\equiv \log_{10}(|\Delta_{it}/C_{it}|)) \\ \Delta_{it} &\equiv \begin{cases} C_t^{-\rho} - \beta R\mathbb{E}_t[C_{t+1}^{-\rho}] & \text{if } A_{it} > \epsilon \\ C_t^{-\rho} - \beta R_{-}\mathbb{E}_t[C_{t+1}^{-\rho}] & \text{if } A_{it} \in [\omega P_t + \epsilon; -\epsilon], \end{cases}\end{aligned}$$

and $\epsilon = 0.02$. A value of $\overline{\mathcal{E}}$ of -2 and -4 are interpreted as average approximation errors of respectively 1 and 0.01 percent of consumption. I will henceforth refer to $\overline{\mathcal{E}}$ as simply the average Euler error.

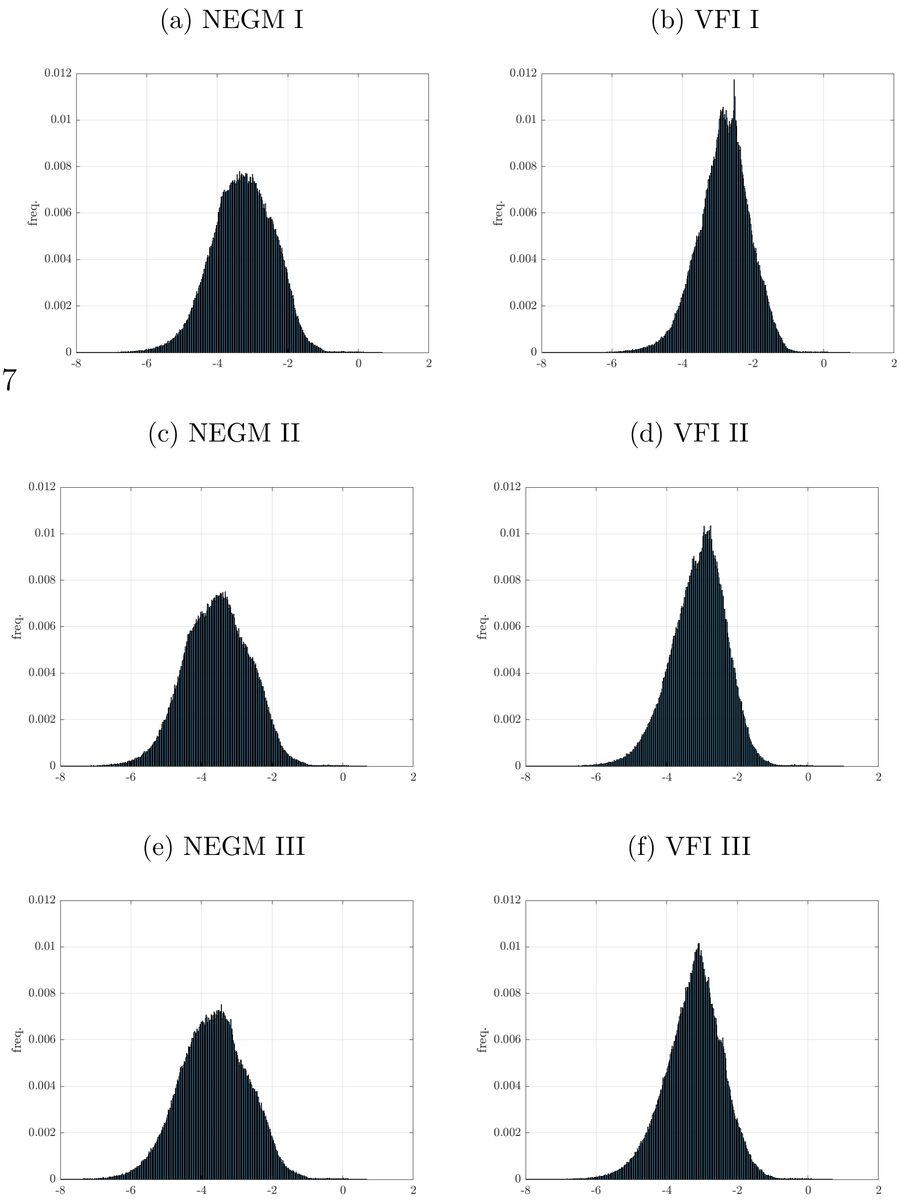
As a comparision, I use a pure VFI algorithm with grids set up in the exact same way as in the NEGM, and with three multi-starts for the one-dimensional non-adjuster problem and six multi-starts for the two-dimensional adjuster problem.

7

Table 2: Speed and Accuracy

	NEGM			VFI		
	I	II	III	I	II	III
Solution time (minutes)	0.14	0.48	1.13	12.41	28.85	51.31
without vectorization	0.39	1.41	3.30			
Avg. Euler error ($\overline{\mathcal{E}}$)	−3.28	−3.56	−3.65	−2.83	−3.10	−3.23
	grids					
$\#_M$	100	200	400	50	75	100
$\#_N$	50	100	150	50	75	100
$\#_P$	60	60	60	60	60	60
$\#_X$	100	150	250	50	100	150
$\#_A$	50	100	150			
Total grid points (in thousands):						
post-decision functions	150	600	1350	3	5	6
non-adjuster functions	300	1200	3600	150	338	600
adjuster functions	6	9	15	3	6	9
	life-cycle profiles					
A_t/P_t (median) at age 35	−0.06	−0.07	−0.07	−0.06	−0.07	−0.07
A_t/P_t (median) at age 45	−0.02	−0.03	−0.03	−0.03	−0.03	−0.03
A_t/P_t (median) at age 55	0.00	0.00	0.00	0.00	0.00	0.00
$(A_t + B_t)/P_t$ (median) at age 35	−0.03	−0.04	−0.04	−0.04	−0.04	−0.04
$(A_t + B_t)/P_t$ (median) at age 45	0.64	0.63	0.63	0.63	0.63	0.63
$(A_t + B_t)/P_t$ (median) at age 55	1.53	1.52	1.51	1.51	1.51	1.51
	MPC and group shares					
MPC						
25th	0.10	0.10	0.10	0.12	0.12	0.11
50th	0.21	0.21	0.21	0.23	0.23	0.22
75th	0.32	0.31	0.30	0.32	0.33	0.33
Share of $A_t = -\omega P_t$	0.08	0.09	0.09	0.07	0.08	0.09
Share of $A_t \in (-\omega P_t, 0)$	0.48	0.48	0.47	0.50	0.49	0.48
Share of $A_t = 0$	0.17	0.18	0.19	0.16	0.17	0.18
Share of $A_t \in \{0, -\omega P_t\}$ and $B_t > 0$	0.14	0.16	0.16	0.13	0.14	0.15
Share of adjusters ($z_t = 1$)	0.08	0.08	0.08	0.08	0.08	0.08

Figure 1: Histograms: Relative absolute Euler error ($\mathcal{E}_{it}$)



7

Speed and accuracy measures for both the NEGM and the VFI are shown in Table 2. The table shows the speed in minutes, the average Euler error, the chosen grid sizes, and various simulation outcomes for a population of 100.000 households. For both the NEGM and VFI, results are reported for three different grid sizes, referred to as NEGM I-III and VFI I-III. The average Euler errors are calculated for a constant sample coming from the simulation of NEGM II.

We see that even NEGM I, with the most sparse grids, imply a better average Euler error than VFI III. This is the case even though the solution time for NEGM I is 0.16 minutes, while the solution time for VFI III is more than 50 minutes. Even NEGM II, with somewhat denser grids, is about 100 times faster than VFI III, and clearly superior in terms of accuracy. Looking at the simulated life-cycle profiles and implied marginal propensities to consume (MPCs), we see that all the specifications deliver very similar results.

Figure 1 shows the distribution of the Euler errors for each specification. We see that the left tail of very small errors is thicker for the NEGM, but that the dispersion is also generally larger for the NEGM. For the considered grid sizes, both VFI and NEGM can not get rid of a thin right tail with a few rather large errors. The reason is that the level of next period consumption can vary substantially across the choice of whether to adjust or not in the next period; small errors in this dimension can therefore result in a large Euler error.

Finally, the second row of Table 2 report the solution time, when a standard multi-linear interpolation algorithm is used instead of the vectorized algorithm proposed in this paper. In NEGM III, where the grid of A_t is large, the solution time triples. In NEGM I, where the grid of A_t has fewer grid points, the solution time still more than doubles.

All these results are naturally specific to the chosen model. If a model with more stochastic elements, and thus heavier numerical integration, had been chosen, the relative speed-up of NEGM, would have been even larger. Likewise, the speed-up would have been larger if more quadrature points had been used in the chosen model. On the other hand, the relative speed-up of NEGM decreases if the VFI algorithm can suffice with fewer multi-starts. For non-convex optimization problems it is, however, very hard to determine when enough multi-starts are used. A low number of multi-starts may be fine for a specific set of parameters, but not for another set of parameters. For calibration and estimation problems it is therefore typically necessary to choose a high number of multi-starts to be on the safe side.

6 Conclusions

In this paper, I have presented a new solution method for solving multi-dimensional consumption-saving models with non-convexities. I have shown that nesting am endogenous grid method step inside an otherwise standard value function iteration can improve the solution time considerably. The speed-up is achieved both by simplifying the involved optimization problems and by reducing the need for numerical integration. Using a novel vectorized interpolation algorithm, the time cost of the numerical integration required in the endogenous grid method step is reduced substantially. For the benchmark two-asset consumption-saving model, a speed-up factor in excess of 100 relative to VFI was observed for given level of accuracy.

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