Macroeconomic risks and the term structure of interest rates

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Abstract

We study a SDGE model of the macroeconomy and examine the implication of shocks for the term structure of interest rates, both in terms of dynamics and risk premium. We solve the model using collocation methods, rathern than relying on linearization, to allow risk to have full impact on the equilibrium.

1 Introduction

The finance literature maintains that a time varying risk premium on bonds is required to match key features of actual bond data. In other words, the so-called expectations hypothesis (which holds that the yield on a long bond should correspond to the expected sum of future short rates) fails to hold in the data by an extent which varies over time. It is therefore natural to ask whether realistic risk premia can be generated within a microfounded general equilibrium model.

This paper explores the implications for the term structure of real and nominal interest rates of a general equilibrium model with nominal rigidities. A key feature of the paper is that we do not log-linearize the model around the steady state, which would prevent any analysis of risk premia, but rather solve it using non-linear methods.

When trying to understand nominal term structure movements, it is natural for macroeconomists to focus on the conduct of monetary policy, as the latter dictates what happens with the short end of the curve, and expectations of future policy affect the long end. There have already been attempts to integrate microfounded macro-models with term structure

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models. However, such attempts have often been conducted based on a linearization of the macroeconomic model. This solution method has the unfortunate effect of limiting the role of risk, as the second moments of the shocks driving the economy (and hence the term structure) play no role on equilibrium outcomes in the case of a first order expansion (and a limited role in a second order expansion). The result is that the dynamics of longer bond yields are typically a puzzle for macro-models, as the risk component is empirically important.

This paper investigates the extent of this puzzle, based on a relatively standard model specification in the neo-Keynesian tradition. The main differencing feature of our analysis is that we use collocation methods to solve the model. Collocation is designed to provide a good global approximation to the nonlinear solution for a certain range of the state variables and it therefore allows us to carry out a full analysis of risk premia. These methods have the additional benefit that they should work relatively well for the case of large shocks, in contrast to local approximations around the non-stochastic steady state. To match interest rate movments, our notion is that the size of the shocks required are typically not "small".

The ultimate aim of this research agenda is to build a sufficiently rich macroeconomic model to replicate features of the risk premium that appear to characterize actual data. This draft focuses on the more modest objective of analyzing two simple examples of risk sources, a technology shock (which has an effect on the real term structure) and a shock to the target rate of inflation (which is a nominal shock). We limit our objective for two reasons. First, as will be clear below, the computational cost of having many state variables is much higher than in the linearized case. Second, for intuition, it makes sense to start of in a simple framework where things are reasonably transparent.

Albeit interesting, we limit the scope and do not include the issue of the zero bound of nominal interest rates. These methods are in principle well suited also for such constraints, but the required degree of accuracy of the solution needed for bond pricing is difficult to obtain and we leave it for future work.

The model does deliver some intuitive results, but the magnitudes of premia are very small indeed. We plan to extend the analysis in several directions, introducing habit persistence and/or cost push shocks to break the positive correlation of inflation and output that currently prevails in the model. The latter feature, as it turns out, has the effect that nominal bonds actually hedge inflation risk, as the return on the bond is low (high inflation) in the good states (when output is high).

[... To be completed ...]

2 The model

The essence of the model can be split into three components. First, the choice of utility function will pin down the structure of the stochastic discount factor (henceforth SDF), that will (under the assumption that utility is additively separable w.r.t consumption) be the discounted ratio of marginal utility of consumption across time. Second, the real side of the model will pin down concepts like the natural real rate and natural output, that will depend on factors like technology shocks. Under the assumption that monetary policy tracks this real rate, the nominal side of the economy will be insulated against these shocks. The final component is represented by shocks that have nominal effects. Under the assumption of flexible prices, there is a complete separation of the nominal and real side. However, we also examine the case of some nominal rigidities, which breaks this dichotomy and introduces real effects of the nominal shocks.

For comparison, note that the typical treatment in finance is to model the first step above. Then, something is assumed about the joint distribution of asset returns and the SDF. In our general equilibrium model, in contrast, this distribution will be endogenous (up to the exogenous distribution of the structural shocks, of course) and will depend on the last two steps in the program above.

In short, the real side of the model will feature a continuum of households consuming a basked of goods, and providing labour services to firms. The firms are monopolistic competitors and only use labour for production. The nominal side includes some scope for suboptimal policy shocks, which can have real effects through the presence of sticky prices. We also use the indexation version of the Calvo pricing introduced in Christiano et. al. (2003) for two reasons. First, inflation will be driven to some extend by lagged inflation, which is an empirically plausible hypothesis. Second, the firms not allowed to update their prices optimally will nevertheless change based on aggregate past inflation, which means that the welfare consequences of the suboptimal policy shocks we will consider (shocks to the target rate of inflation) will be limited, as the distribution prices will be more compressed.

Finally, the complete markets assumption allows us to solve the model without considering explicitly the nominal bonds. The solution (i.e. how aggregate output and inflation respond to the state variables and the shocks in the economy) will pin down the stochastic behavior of the pricing kernel (SDF). This will then be used in a second step to price the nominal bonds, hence allowing us to recover the term structure.

2.1 Consumers

The exposition of the variant of the Christiano et. al. model borrows heavily from Woodford (2003).

Consumers maximize the discounted sum of the period utility

$$U(C,L) = u(C) - \int_0^1 v(L(i))\,di$$

where C is a consumption index satisfying

$$C = \left(\int_0^1 C(i)^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}},$$

and by assumption workers provide $L(i)$ hours of labor to firm i explaining the integral. We could introduce money by adding real balances to utility, which would pin down a money demand function, but we abstract from this issue. The budget constraint is hence given by

$$P_t C_t + B_t = \int_0^1 w_t(i)\, L_t(i)\,di + \int_0^1 \Pi_t(i)\,di + W_t$$

with the price level P_t defined as the minimal cost of buying one unit of C_t, hence equal to

$$P_t = \left(\int_0^1 p(i)^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

In the expression above, B_t denotes end of period holdings of a complete portfolio of state contingent assets. W_t denotes the beginning of period value of the assets, w_t is the nominal wage rate, $L_t(i)$ is the supply of labor to firm i. Complete markets imply the existence of a unique pricing kernel, or stochastic discount factor, or state price deflator, denoted $Q_{t,t+1}$. This implies that

$$B_t = E_t\left(Q_{t,t+1} W_{t+1}\right).$$

The first order conditions w.r.t intertemporal consumption allocation and labour supply are

$$\begin{aligned} Q_{t,t+1} &= \beta \frac{u_c\left(C_{t+1}\right) P_t}{u_c\left(C_t\right) P_{t+1}} \\ \frac{w_t\left(i\right)}{P_t} u_c\left(C_t\right) - v_L\left(L\left(i\right)\right) &= 0. \end{aligned}$$

The demand for an individual good is

$$C_t\left(i\right) = \left(\frac{P_t\left(i\right)}{P_t}\right)^{-\theta} C_t.$$

2.2 Firms

Turning to the firms, the production function is given by

$$Y_t\left(i\right) = A_t f\left(L_t\left(i\right)\right).$$

Since output is demand determined, and since there is only one factor of production, the firm does not have a choice in how much labor to employ, it simply has to employ enough to produce the amount demanded at the chosen price. To this end, note that the production function implies that to produce $Y_t\left(i\right)$, we need

$$L_t\left(i\right) = f^{-1}\left(\frac{Y_t\left(i\right)}{A_t}\right)$$

units of labor. This means that the total nominal cost function for the firm will be given by (TC is not to be confused with the product of the tax rate and consumption!!!)

$$TC_t\left(i\right) = w\left(i\right) f^{-1}\left(\frac{Y_t\left(i\right)}{A_t}\right).$$

The real marginal cost, mc_t, is given by

$$mc_t = \frac{1}{P_t}\frac{\partial TC_t}{\partial Y_t\left(i\right)} = \frac{w\left(i\right)}{P_t A_t}\frac{1}{f'\left(f^{-1}\left(\frac{Y_t(i)}{A_t}\right)\right)}.$$

Since labor market clearing implies that the wage has to equate supply and demand for labor, an alternative expression is

$$mc_t\left(Y_t^i, Y_t\right) = \frac{1}{A_t}\frac{v_L\left(f^{-1}\left(\frac{Y_t^i}{A_t}\right)\right)}{u_c\left(Y_t\right)}\frac{1}{f'\left(f^{-1}\left(\frac{Y_t^i}{A_t}\right)\right)} \quad (1)$$

where we again have used that $C_t = Y_t$.

Assuming that the firms hire workers at market given wage rates, since supply has to equal demand, the firms problem in the case of flexible prices is (taking the aggregate price level as given)

$$\max_{P_t^i} P_t^i Y_t^i \left(P_t^i\right) - TC_t \left(Y_t^i \left(P_t^i\right)\right),$$

$$Y_t^i = \left(\frac{P_t^i}{P_t}\right)^{-\theta} Y_t \tag{2}$$

The first order condition in the case of flexible prices is leads to the familiar result that prices are set as a markup on marginal cost,

$$\begin{aligned} \frac{P_t^i}{P_t} &= \mu mc_t \\ \mu &\equiv \frac{\theta}{\theta - 1} \end{aligned} \tag{3}$$

Note that we can directly see that if all prices are flexible, such that $P_t^i = P_t$, then (2) implies that $Y_t^i = Y_t$ and (3) together with (1) pins down aggregate output as the solution to the equation

$$mc_t \left(Y_t^n, Y_t^n\right) = \mu^{-1}. \tag{4}$$

Woodford labels this the natural rate of output, i.e. the level that would prevail if the economy was at the efficient equilibrium, which in the present setting corresponds to the flexible price equilibrium. Marginal costs will be an increasing function of θ or, correspondingly, production is an increasing function of θ. The limiting case when θ tends to infinity corresponds to perfect competition, hence the preceding result confirms the standard intuition that monopolistic competition leads to inefficiently low output.

2.3 Nominal side; rigidities and monetary policy

Turning to the case of sticky prices, with Calvo contracts and a probability of keeping the price fixed equal to ζ, the firms problem is instead to maximize the expected profits, taking into account that there is a probability that they will not be allowed to adjust prices again in future periods. The optimal problem is then

$$\max_{P_t^i} \mathrm{E}_t \sum_{s=t}^{\infty} \zeta^{s-t} Q_{t,s} \left(P_s^i Y_s^i \left(P_s^i\right) - TC_s \left(Y_s^i \left(P_s^i\right)\right)\right),$$

where, as in Smets and Wouters (2003)

$$P_s^i = P_t^i \left(\frac{P_{s-1}}{P_{t-1}}\right)^{\iota},$$

since firms not changing prices optimally are assumed to modify them using a rule of thumb that indexes them to lagged inflation. The degree of indexation is measured by ι, where $0 \leq \iota \leq 1$ ($\iota = 0$ takes us back to the standard Calvo world; $\iota = 1$ implies perfect indexation as in Christiano, Eichenbaum and Evans, 2001). Note that this feature will reduce the welfare loss arising from temporary deviations from zero in the inflation target, at least when $\iota = 1$. The pricing decision of the firm has the first order condition

$$\begin{aligned} \frac{P_t^*}{P_t} &= \mu \frac{\mathrm{E}_t \sum_{s=t}^{\infty} (1-\zeta)^{s-t} Q_{t,s} \left(\frac{P_s}{P_t}\right)^{1+\theta} \left(\frac{P_{s-1}}{P_{t-1}}\right)^{-\iota\theta} mc_s Y_s}{\mathrm{E}_t \sum_{s=t}^{\infty} (1-\zeta)^{s-t} Q_{t,s} \left(\frac{P_s}{P_t}\right)^{\theta} \left(\frac{P_{s-1}}{P_{t-1}}\right)^{\iota(1-\theta)} Y_s} \\ &\equiv \mu \frac{K_{2,t}}{K_{1,t}}. \end{aligned} \tag{5}$$

where we used the assumption that firms are perfectly symmetric. Hence, all firms that do get to change price will set it equal and we denote this optimal price p_t^*. Furthermore, the average level of prices in the group that does not change prices is equal to the average price level from the last period (since, by a law of large number type of argument, those firms are drawn from the pool of all firms). Going back to the definition of the price level, this implies

$$\begin{aligned} P_t &= \left[\int_0^1 p\left(i\right)^{1-\theta}\right]^{\frac{1}{1-\theta}} \\ &= \left[\left(1-\zeta\right) P_t^{*1-\theta} + \zeta \left(P_{t-1} \left(\frac{P_{t-1}}{P_{t-2}}\right)^{\iota}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}. \end{aligned}$$

Note that the last equation can usefully be rewritten as

$$\frac{P_t^*}{P_t} = \left(\frac{1-\zeta\left(\frac{\Pi_{t-1}^{\iota}}{\Pi_t}\right)^{1-\theta}}{1-\zeta}\right)^{\frac{1}{1-\theta}} \tag{6}$$

using the definition $\Pi_t \equiv \frac{P_t}{P_{t-1}}$. Note that lagged inflation now becomes a state variable. Since the ratio of price levels can be re-expressed in terms of the product of inflation rates, we can now combine the above equations to a system of four equations and four unknown processes, Y_t, Π_t, $K_{1,t}$ and $K_{2,t}$. We note that the solution is facilitated by the possibility of recursifying the evolution for $K_{1,t}$ and $K_{2,t}$, which is done by leading the expressions, multiply them with an appropriate scaling factor and taking expectations at t to obtain

$$K_{1,t} = Y_t + \Pi_t^{\iota(1-\theta)} \zeta \mathrm{E}_t \left[Q_{t,t+1} \Pi_{t+1}^{\theta} K_{1,t+1}\right] \tag{7}$$

$$K_{2,t} = mc_t Y_t + \zeta \Pi_t^{-\theta\iota} \mathrm{E}_t \left[Q_{t,t+1} \Pi_{t+1}^{1+\theta} K_{2,t+1}\right] \tag{8}$$

$$Y_t^i = Y_t^i\left(\Pi_t, Y_t\right).$$

We close the model with a policy rule in terms of inflation, a la Taylor. However, before doing so, we need to discuss the concept of the natural real rate of interest.

2.4 Natural rate of interest and monetary policy

Woodford defines the real natural rate of interest as the one that would emerge in the efficient equilibrium characterized by flexible prices (and a steady state government subsidy offsetting the markup distortion).

The real rate is the one corresponding to the real pricing kernel evaluated at the natural rate of output, i.e.

$$\frac{1}{R_t} \equiv \beta \mathrm{E}_t\left(\frac{u_c\left(Y_{t+1}^n\right)}{u_c\left(Y_t^n\right)}\right)$$

Given that (4) defines $Y_t^n\left(A_t\right)$, the above equation together with the law of motion for the technology shock defines $R_t\left(A_t\right)$.

The central bank will set the gross riskless (in nominal terms) one period rate, I_t. Letting

$$I_t = R_t v\left(\Pi_t, \varepsilon_t\right) \tag{9}$$

we have a complete characterization of the model. If the model is log-linearized around the steady state, there is no need to assume particular functions for utility and production, since the only thing that will matter is elasticities of those functions evaluated at the steady state. In our case, however, the explicit functional form are necessary since we do not approximate the model around the steady state. We therefore proceed to make some explicit functional choices. The finance literature has focused on the fact that the real discount factor is only (under some assumptions) a function of the ratio of marginal utilities. Looking at the variability of aggregate consumption data, which is low, this means that the scope for matching volatile variables, such as asset prices, is limited. The HJ bounds provide a graphic illustration of the variance implied.[... to be extended ...]. The main result of their exercise was that when trying to explain the equity premium puzzle, an extreme degree of risk aversion had to be imposed on the "standard" utility function. One of the ways proposed to improve the model's performance has been to assume habit persistence, which allows a smaller variability in consumption to have a magnified effect on the ratio of marginal utilities (e.g. Constantinides, 1990, Campbell and Cochrane, 2003). This approach has also been used in the monetary policy literature, see Amato and Laubach (2003), Christiano et. al. (2001), Smets and Wouters (2003). We nevertheless start with the CRRA case to provide a benchmark.

2.5 Bonds

Given the stochastic discount factor derived above, bonds can be priced following standard methods. For the price of a real asset with maturity n at time t, N_t, we know that

$$N_t = E_t\left(q_{t,t+n}N_{t+n}\right)$$

where N_t and N_{t+n} are the prices of the asset today and in n periods, respectively, $q_{t,t+n} \equiv q_{t,t+1}q_{t+1,t+2}...q_{t+n-1,t+n}$ and $q_{t,t+1}$ is the real stochastic discount factor $q_{t,t+1} = \beta\frac{u_c\left(Y_{t+1}^n\right)}{u_c\left(Y_t^n\right)}$. Using the recursive definition of $q_{t,t+n}$, we can write this as $q_{t,t+n} \equiv \beta^n\frac{u_c\left(Y_{t+n}^n\right)}{u_c\left(Y_t^n\right)}$. The asset pricing equation can then be rewritten as

$$N_t = \beta^n E_t\left[\frac{u_c\left(Y_{t+n}^n\right)}{u_c\left(Y_t^n\right)}N_{t+n}\right]$$

For an n-period bond, written $N_{n,t}$ and such that $N_{t+n} = 1$, we obtain

$$N_{n,t} = E_t\left(\beta^n\frac{u_c\left(Y_{t+n}^n\right)}{u_c\left(Y_t^n\right)}\right)$$

This equation can be solved recursively given the sole solution for the natural rate of output.

Similarly, nominal bonds can be priced using the nominal discount factor $Q_{t,t+1}$. For an n-period bond, written $B_{n,t}$ and such that $B_{t+n} = 1$, we obtain

$$B_{n,t} = E_t\left(\beta^n\frac{P_t}{P_{t+n}}\frac{u_c\left(Y_{t+1}^n\right)}{u_c\left(Y_t^n\right)}\right)$$

In general, we can rewrite the bond pricing equation as

$$B_{n+1,t} = E_t\left(Q_{t,t+1}B_{n,t+1}\right)$$

3 Solving the model

We adopt the following functional forms

$$\begin{aligned} u(C) &= \frac{C^{1-\gamma}}{1-\gamma} \\ v(L) &= \chi L^{\phi} \\ f(L) &= L^{\alpha}. \\ f^{-1}(y) &= y^{\frac{1}{\alpha}} \end{aligned}$$

Repeating the first order conditions with these functional choices, we have

$$Q_{t,t+1} = \beta \frac{P_t}{P_{t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

and hence the condition for the riskless rate is, using $C_t = Y_t$,

$$\frac{1}{I_t} = E_t \left(\beta \Pi_{t+1}^{-1} \left(\frac{Y_{t+1}}{Y_t} \right)^{-\gamma} \right) \tag{10}$$

Using the functional forms in (1) results in the real marginal cost expression

$$mc_t = \frac{\chi\phi}{\alpha} A_t^{-\frac{\phi}{\alpha}} Y_t^{\gamma + \frac{\phi-\alpha}{\alpha}} \left(\frac{1 - \zeta \Pi_t^{\theta-1} \Pi_{t-1}^{\iota(1-\theta)}}{(1-\zeta)} \right)^{\frac{\theta(\alpha-\phi)}{\alpha(1-\theta)}} \tag{11}$$

where we have used equation (6).

Finally, let us assume that the rule is

$$\begin{aligned} I_t &= R_t^n \left(\frac{\Pi_t}{\Pi_t^*} \right)^{\psi} \\ \Pi_t^* &= (\Pi_t^*)^{\rho_{\Pi}} e^{\eta_t} \end{aligned} \tag{12}$$

where Π_t^* is a time varying inflation target. The presence of a time varying inflation target aims to capture the fact that central banks do not strive to keep inflation always and exactly on target, but are willing to tolerate some deviation from the target within a narrow range.

Equations (10), (11), (12), (7), (8), (5) and (6) define an equation system in the four unknown processes $\Pi_t, Y_t, K_{1,t}$ and $K_{2,t}$ and (once some substitutions have been carried out) four equations. Once this system is solved, the rest of the variables can be calculated recursively (if needed). We will only need the evolution for output and inflation in order to price bonds, since these processes pin down the stochastic discount factor.

3.1 Special case of flexible prices

Using (11) and (4), with $P_t^i = P_t$, we find that

$$Y_t^n = \left(\frac{\alpha}{\phi\chi\mu}\right)^{\frac{\alpha}{\phi+\alpha(\gamma-1)}} A^{\frac{\phi}{\phi+\alpha(\gamma-1)}}. \tag{13}$$

where we let the superscript n indicate "natural". Next, we find the real interest rate consistent with zero inflation by combining the Euler equation

$$\frac{1}{R_t} = \beta E_t \left(\left(\frac{Y_{t+1}^n}{Y_t^n} \right)^{-\gamma} \right). \tag{14}$$

with the solution for Y_t^n above, and use the fact that the shocks are log-normal to find

$$R_t = \beta^{-1} A_t^{\frac{\gamma(\rho-1)\phi}{\phi+\alpha(\gamma-1)}} e^{-\frac{\sigma^2}{2}\left(\frac{\gamma\phi}{\phi+\alpha(\gamma-1)}\right)^2}.$$

Incidentally, we can note that the real interest rate is the inverse of the discount factor in the non-stochastic steady state. This is not true, however, in the stochastic steady state, where it instead will be scaled by a function of the variance of the technology shocks.

When technology is at its steady state value of 1, the natural rate boils down to $R = \beta^{-1} e^{-\frac{\sigma^2}{2}\left(\frac{\gamma\phi}{\phi+\alpha(\gamma-1)}\right)^2}$. Since the second term in this expression is always negative, the natural rate in the stochastic equilibrium will be lower than under certainty. To understand this result, recall that households are risk averse and their utility is therefore concave in consumption (output). Concavity implies that households will systematically – that is even in steady state – expect the utility of $t+1$ consumption to be lower than the utility of current consumption. Expected marginal utility will therefore be higher in the future, thus making households willing to shift consumption forward in time. Since this is not feasible in equilibrium, the natural rate must fall to discourage people from saving.

3.2 The solution algorithm

In order to solve the model described by Equations (10), (11), (12), (7), (8), (5) and (6) for the four unknown processes $\Pi_t, Y_t, K_{1,t}$ and $K_{2,t}$ we resort to collocation methods, as described in Judd (1998) and Miranda and Fackler (2002). A rational expectations solution to this model is one that makes all jump variables functions of the states (sometimes the states have to include the shocks). In the LQ case, we can guess that the jumps are *linear* functions of all the states and then use an undetermined coefficients method to find the solution. In the current case, they will be non-linear functions of the states.

We have (at least) two choices to proceed. We can either parameterize the jump variable, Π_t, or the functions inside the expectations operators of the various equations. The latter approach would be similar in spirit to the approach in Marcet and Sargent (), with the difference that the expectations operator would not be part of the parameterized function. As there are no kinks in the decision rules, we proceed to parameterize those directly.

Define

$$S_t = \left[\begin{array}{ccc} A_t & \Pi_t^* & \Pi_{t-1} \end{array}\right]'.$$

Next, use Chebychev polynomials as basis functions and define the three unknown functions

$$\begin{aligned}
\Pi_t &= \sum_{j=1}^{n^3} c_j^{\Pi} \phi_j \left(S_t\right) \\
Y_t &= \sum_{j=1}^{n^3} c_j^{Y} \phi_j \left(S_t\right) \\
K_{1,t} &= \sum_{j=1}^{n^3} c_j^{K} \phi_j \left(S_t\right) \\
K_{2,t} &= \frac{\theta - 1}{\theta} \left(\frac{1 - \zeta \Pi_t^{\theta-1} \Pi_{t-1}^{\iota(1-\theta)}}{(1-\zeta)} \right)^{\frac{1}{1-\theta}} K_{1,t}.
\end{aligned}$$

Given a guess of the projection coefficients in the above polynomials, we have a complete description of the evolution of the states. The first two are trivial and the third (lagged inflation) is obtained by using the parameterized inflation evaluated at the states. I.e.

$$S_{t+1} = g\left(S_t, \xi_{t+1}\right)$$

where ξ_{t+1} contains the two innovations to technology and the target.

Next, we replace all occurrences of the jump variables in the system with the parameterized functions. This will reduce our problem to a root finding problem where we choose c_j^{Π}, c_j^{Y} and c_j^{K} to make all system equations hold with equality. To identify the $3n^3$ coefficients, we evaluate the system at n^3 nodes (since the three equations makes this $3n^3$ conditions). If we pick n values of A_t and evaluate the above equation, we have a system of n equations and n unknown c_j coefficients. The last step is to discretize the distribution of the shocks, such that there are k values with associated ω_k probabilities, instead of the continuos distribution. We chose a three point discretization of the Gaussian shocks, and then use (**??**) to find the range of values of A that makes sure that we stay within the range (i.e. the upper bound is the solution of $\log A_{\max} = \rho \log A_{t\max} + \max(\varepsilon)$). Finally, we chose the Chebychev nodes spaced according to the Chebychev prescription.

3.3 Bond pricing and macroeconomics

[... Discussion of how the formulation of the risk premium in finance is related to the pricing kernel that comes out of the utility based approach, with reference to Bliss and ?, Singelton, Dai (2003), Wachter (2003?) to be added. Touch on incomplete markets(?) ...]

3.3.1 The term structure of natural rates

Our model is sufficiently simple to allow us to derive the whole term structure of natural rates analytically. Given the functional forms of utility and production functions, we know that the price of a real bond with maturity n at time t, $N_{n,t}$, must be such that

$$N_{n,t} = \beta^n \left(Y_t^n\right)^\gamma E_t\left[\left(Y_{t+n}^n\right)^{-\gamma}\right]$$

Substituting out natural output and evaluating expectations, it follows that

$$N_{n,t} = \beta^n A_t^{\frac{\gamma\phi(1-\rho^n)}{\gamma\alpha-\alpha+\phi}} e^{\frac{1}{2}\frac{1-\rho^{2n}}{1-\rho^2}\left(\frac{\gamma\phi}{\phi-\alpha(1-\gamma)}\right)^2\sigma_\varepsilon^2}$$

which characterize the whole term-structure of natural rate prices. Note that, if we focus on continuously compounded returns defined as $\overline{R}_{n,t} = -\frac{1}{n}\ln N_{n,t}$, we obtain

$$\overline{R}_{n,t} = a_n + b_n \ln A_t$$

where

$$\begin{aligned} a_n &\equiv \ln\beta^{-1} - \frac{1}{2n}\frac{1-\rho^{2n}}{1-\rho^2}\left(\frac{\gamma\phi}{\phi+\alpha(\gamma-1)}\right)^2\sigma_\varepsilon^2 \\ b_n &\equiv -\frac{1}{n}\frac{\gamma\phi(1-\rho^n)}{\phi+\alpha(\gamma-1)} \end{aligned}$$

The continuously compounded yield is therefore affine in $\ln A_t$. The term structure of natural rates behaves therefore as an affine 1-factor model, so that, for example, it can never become hump-shaped.

Note that the yield to maturity on a 1-period bond, $N_{1,t}^{-1}$, is the natural rate of interest. Provided technology shocks are stationary with $\rho < 1$, the steady state yield on a bond with infinite maturity is $\lim_{n\to\infty} \overline{R}_{n,t} = \beta^{-1}$. Since the second term in a_n must be negative by construction, it follows that the whole term structure of natural rates is lower than the value it would assume under certainty, namely β^{-1}.

Note also that $\frac{\partial a_n}{\partial n} > 0$, hence term premia are increasing in maturity. Intuitively, the possible occurrence of a technology shock in the future implies that the real return on any real bond with maturity longer than 1 period is uncertain. More precisely, since the production function is convex, expected $t+1$ output is always larger than current output, thus the marginal utility of consumption is expected to decrease over the future. Agents would therefore be inclined to frontload consumption to the present, but this is infeasible in equilibrium. The return on real bonds must therefore increase, the more so for bonds of longer maturity, in order to increase agents' willingness to defer consumption to the future.

To relate this result to the affine term structure literature, note that the real pricing kernel $q_{t,t+1}$ can be rewritten as $\ln q_{t,t+1} = -\ln(R_t) - \ln E_t\left[\left(Y_{t+1}^n\right)^{-\gamma}\right] - \gamma \ln Y_{t+1}^n$, or, after rewriting natural output in terms of its determinants,

$$\ln q_{t,t+1} = -\ln(R_t) - \frac{1}{2}\left(\frac{\gamma\phi}{\phi+\alpha(\gamma-1)}\right)^2 \sigma_\varepsilon^2 - \frac{\gamma\phi}{\phi+\alpha(\gamma-1)}\sigma_\varepsilon \upsilon_{t+1}$$

where υ_{t+1} is a white noise shock with unit variance.

This formulation is entirely analogous to the one used in the affine literature, i.e. $\ln q_{t,t+1} = -\ln(R_t) - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{1,t+1}$, where λ_t are exogenously postulated vectors representing the market prices of risk. Our model allows us to derive endogenously the determinants of the prices of risk as

$$\lambda_t = \frac{\gamma\phi}{\phi+\alpha(\gamma-1)}\sigma_\varepsilon$$

Intuitively, the price of risk is higher, the higher values of the relative risk aversion parameter γ or of the curvature of the disutility of labour ϕ, and the lower the weight of labour in the production function α. The positive link between degree of risk aversion and prices of risk is immediately intuitive. The role of the disutility of labour derives from the fact that shocks may have an undesirable impact in forcing households to work more or less than desired. The importance of labour in the production function is also indirectly linked to households' aversion to work. The larger the share of labour in production, the smaller the marginal impact on labour of a certain productivity shock. Risk neutrality

($\gamma = 0$) or indifference with respect to the labour-work choice ($\phi = 0$) would be sufficient to drive the price of risk to zero.

While derived endogenously rather that postulated exogenously, the above market price of risk are constant over time. This has been shown to be a drawback for affine models, in the sense of hampering their ability to explain the features of the data which represent a puzzle for the expectations hypothesis (Dai and Singleton, 2003).

3.3.2 Nominal bonds

We can rewrite the recursion for nominal bonds as

$$B_{n+1,t} = E_t \left(Q_{t,t+1} B_{n,t+1} \right)$$

where

$$Q_{t,t+1} = \beta \Pi_{t+1}^{-1} \left(\frac{Y_{t+1}}{Y_t} \right)^{-\gamma}$$

The solution of the model gives us the equilibrium dynamics of Π_t and Y_t and hence also for $Q_{t,t+1}$. Thus, we can solve for the bond prices recursively starting from the short term nominal interest rate, i.e. $B_{1,t}$. Given $B_{1,t}$, the recursion allows us to derive $B_{2,t}$ as $B_{2,t} = E_t \left(Q_{t,t+1} B_{1,t+1} \right)$. Given $B_{2,t}$ we can then derive $B_{3,t}$ and so on and so forth. Once we know bond prices, we can derive holding period returns, yields to maturity and forward rates using standard definitions.

4 Equilibrium dynamics

Benchmark parameters are set to

α	ζ	γ	θ	ι	β	χ	ϕ
0.7	0.7	2	11	0.1	0.99	$\frac{1}{\phi}$	2

This gives a markup on marginal cost of about 10%. The degree of indexation should be higher, based on evidence in Smets and Wouters, Woodford, but we start off with this benchmark calibration. The degree of non-linearity is of course increasing in many of these parameters, such that we could "boost" the effect a bit. For example, it is typically found that γ needs to be very high indeed in order to match the equity premium puzzle. We will rather address this issue with a habit formulation...

4.1 Macroeconomic dynamics

Given our model solution, we can perform stochastic simulations to derive the moments of the distributions of all variables in the model. We set the parameter values so that the nonlinear effects on the unconditional mean of inflation are negligeble This ensures that the policy rule does deliver, on average, the level of inflation that is optimal in the model.

The natural (risk-free) rate is, nevertheless, significantly lower than its certainty equivalent value, given by the reciprocal of the time discount parameter, for the reasons explained above. For our benchmark parameter values, the discrepancy is equal to about 15 basis points, and it can be larger for larger values of γ and ϕ, or for lower values of α. Since the nominal interest rate tracks the natural rate following technology shocks, it will also tend to deviate from its certainty equivalent value in the stochastic steady state. In this model, it tends to incorporate a negative risk premium, and therefore to be even lower than the natural rate. Intuitively, inflation target shocks are the only shocks that cause a discrepancy between natural and nominal interest rates. As we show below, these shocks tend to cause an increase in output and consumption at the same time as the real return on short-term bonds falls (and viceversa for negative inflation target shocks). Hence, short-term nominal bonds provide a hedge against consumption fluctuations induced by inflation target shocks, which explains the negative premium required on their return.

As in linear models, we can also describe macroeconomic dynamics in terms of impulse response functions. The responses of the macroeconomic variables to an innovation in the state of technology or in the inflation target are shown in Figures 1 and 2.

The shock would tend to put downward pressure on prices but, following a corresponding change in the natural rate, monetary policy reacts with a 25 basis points easing that ensures the constancy of inflation following the shock.

A 20 basis points increase in the inflation target has no effect on the natural rate, but tends to produce an immediate increase in inflation. However, firms do not overreact to the news, since they know that the cost of not adjusting prices immediately is mitigated by the indexation scheme. As a result, inflation increases less than the target, the nominal interest rate falls as prescribed by the policy rule and output rises because of the monetary policy easing. Over time, inflation picks up due to the indexation scheme and the nominal interest rate raises above the steady state and returns to the baseline from above.

[... To be completed ...]

4.2 Yield dynamics

The average nominal term structure of interest rates and volatilities is shown in Figure 2 (based on a 50000 period stochastic simulation). As already mentioned above, smaller values of α increase the term premium over the whole maturity spectrum. Intuitively, a smaller α increases the sensitivity of labour demand to technology shocks, and hence the risk for households of having to work more (or less) than desired. Regarding the average difference, it is clear that it is empirically way too low. The reason is that in the model, there is (as of yet!) no shock that will create the type of interaction of inflation and output that will warrant a nominal premium. A natural way of introducing such a component would be to examine a "cost push" shock. This will be introduced as a time-varying elasticity of substitution between goods, which in turn will result in a time-varying markup. Policy responses to such shocks will tend to drive output in the opposite direction of inflation, and hence create a premium effect.

The term structure of volatilities is reasonably in line with empirical facts, as it is clearly downward sloping.

5 Conclusions

The model does deliver some intuitive results, but the magnitudes of premia are very small indeed. We plan to extend the analysis in several directions, introducing habit persistence and/or cost push shocks to break the positive correlation of inflation and output that currently prevails in the model. The latter feature, as it turns out, has the effect that nominal bonds actually hedge inflation risk, as the return on the bond is low (high inflation) in the good states (when output is high). [to be completed]

A The marginal cost is obtained by

$$\begin{aligned}
mc_t &= \frac{1}{P_t}\frac{\partial TC_t}{\partial Y_t^i} = \frac{w_t(i)}{P_t A_t \alpha}\left(\frac{Y_t^i}{A_t}\right)^{\frac{1-\alpha}{\alpha}} \\
&= \frac{\chi\phi\left(\left(\frac{Y_t^i}{A_t}\right)^{\frac{1}{\alpha}}\right)^{\phi-1}}{Y_t^{-\gamma}A_t\alpha}\left(\frac{Y_t^i}{A_t}\right)^{\frac{1-\alpha}{\alpha}} \\
&= \frac{\chi\phi Y_t^{\gamma}}{A_t\alpha}\left(\frac{Y_t^i}{A_t}\right)^{\frac{\phi-\alpha}{\alpha}} \\
&= \alpha^{-1}\chi\phi A_t^{-\frac{\phi}{\alpha}}Y_t^{\gamma}Y_t^{i\frac{\phi-\alpha}{\alpha}} \\
&= \alpha^{-1}\chi\phi A_t^{-\frac{\phi}{\alpha}}Y_t^{\gamma}\left(\left(\frac{P_t^i}{P_t}\right)^{-\theta}Y_t\right)^{\frac{\phi-\alpha}{\alpha}} \\
&= \alpha^{-1}\chi\phi A_t^{-\frac{\phi}{\alpha}}Y_t^{\gamma+\frac{\phi-\alpha}{\alpha}}\left(\frac{1-\zeta\Pi_t^{\theta-1}}{(1-\zeta)}\right)^{\frac{\theta(\alpha-\phi)}{(1-\theta)\alpha}}
\end{aligned}$$

B Bond pricing

The price of a one period bond is defined as

$$B_{t,t+1} = \mathrm{E}_t\left(Q_{t,t+1}1\right),$$

and hence coincides with the inverse of I_t. Define S_t to be the vector of state variables. Furthermore, the solution of the model will imply a relation

$$S_{t+1} = g\left(S_t, \varepsilon_{t+1}\right)$$

Given the complete markets assumption, we can first solve the model to obtain the stochastic discount factor $Q_{t,t+1}$. The expected value of this will only be a function of the current states, so that

$$B_{t,t+1} = B_{t,t+1}\left(S_t\right).$$

We will now construct a recursive formula for bonds of longer maturity.

$$\begin{aligned}
B_{t,t+2} &= \mathrm{E}_t\left(Q_{t,t+2}1\right) \\
&= \mathrm{E}_t\left(Q_{t,t+1}B_{t+1,t+2}\left(S_{t+1}\right)\right) \\
&= \mathrm{E}_t\left(Q_{t,t+1}\left(g\left(S_t,\varepsilon_{t+1}\right)\right)B_{t+1,t+2}\left(g\left(S_t,\varepsilon_{t+1}\right)\right)\right)
\end{aligned}$$

The last equation is only a function of current states once the shocks are integrated out, such that

$$B_{t,t+2} = B_{t,t+2}\left(S_t\right).$$

$$\begin{aligned}
B_{t,t+3} &= \mathrm{E}_t\left(Q_{t,t+3}1\right) \\
&= \mathrm{E}_t\left(Q_{t,t+1}B_{t+1,t+3}\left(S_{t+1}\right)\right) \\
&= \mathrm{E}_t\left(Q_{t,t+1}\left(g\left(S_t,\varepsilon_{t+1}\right)\right)B_{t+1,t+3}\left(g\left(S_t,\varepsilon_{t+1}\right)\right)\right).
\end{aligned}$$

Hence, by induction,

$$B_{t,t+n} = \mathrm{E}_t\left(Q_{t,t+1}\left(g\left(S_t,\varepsilon_{t+1}\right)\right)B_{t+1,t+n}\left(g\left(S_t,\varepsilon_{t+1}\right)\right)\right),$$

and we can hence build the term structure of interest rates starting from the low end.

For the simple model with only the technology shock, assuming the interest rate is set such that inflation is zero at all times, the only state variable is the technology shock.

$$A_t = A_{t-1}^{\rho}e^{\varepsilon_{t+1}}.$$

In this case, the term structure is completely real, and we

$$\begin{aligned}
B_{t,t+1}\left(A_t\right) &= \left(R_t^n\right)^{-1} \\
&= \beta e^{\frac{\sigma^2}{2}\left(\frac{\gamma\phi}{\phi+\alpha(\gamma-1)}\right)^2} A_t^{\frac{\gamma(1-\rho)\phi}{\phi+\alpha(\gamma-1)}}
\end{aligned}$$

which, by (13), is

$$\begin{aligned}
Q_{t,t+1} &= \beta\left(\frac{A_{t+1}}{A_t}\right)^{\frac{-\gamma\phi}{\phi+\alpha(\gamma-1)}} \\
&= \beta\left(A_t^{\rho-1}e^{\varepsilon_{t+1}}\right)^{\frac{-\gamma\phi}{\phi+\alpha(\gamma-1)}} \\
&= \beta A_t^{\frac{-\gamma\phi(\rho-1)}{\phi+\alpha(\gamma-1)}} e^{\frac{-\gamma\phi}{\phi+\alpha(\gamma-1)}\varepsilon_{t+1}}
\end{aligned}$$

Hence,

$$\begin{aligned}
B_{t,t+2} &= \mathrm{E}_t\left(Q_{t,t+1}B_{t+1,t+2}\left(S_{t+1}\right)\right) \\
&= \mathrm{E}_t\left(Q_{t,t+1}B_{t+1,t+2}\left(S_{t+1}\right)\right) \\
&= \beta^2 e^{\frac{\sigma^2}{2}\left(\frac{\gamma\phi}{\phi+\alpha(\gamma-1)}\right)^2}\mathrm{E}_t\left(A_t^{\frac{\gamma\phi(1-\rho)}{\phi+\alpha(\gamma-1)}} e^{\frac{-\gamma\phi}{\phi+\alpha(\gamma-1)}\varepsilon_{t+1}} A_{t+1}^{\frac{\gamma(1-\rho)\phi}{\phi+\alpha(\gamma-1)}}\right) \\
&= \beta^2 e^{\frac{\sigma^2}{2}\left(\frac{\gamma\phi}{\phi+\alpha(\gamma-1)}\right)^2}\mathrm{E}_t\left(A_t^{\frac{\gamma\phi(1-\rho)}{\phi+\alpha(\gamma-1)}} e^{\frac{-\gamma\phi}{\phi+\alpha(\gamma-1)}\varepsilon_{t+1}} \left(A_t^{\rho}e^{\varepsilon_{t+1}}\right)^{\frac{\gamma(1-\rho)\phi}{\phi+\alpha(\gamma-1)}}\right) \\
&= \beta^2 e^{\frac{\sigma^2}{2}\left(\frac{\gamma\phi}{\phi+\alpha(\gamma-1)}\right)^2}\mathrm{E}_t\left(A_t^{\frac{\gamma\phi(1-\rho)}{\phi+\alpha(\gamma-1)}} e^{\frac{-\gamma\phi}{\phi+\alpha(\gamma-1)}\varepsilon_{t+1}} \left(A_t^{\rho}e^{\varepsilon_{t+1}}\right)^{\frac{\gamma(1-\rho)\phi}{\phi+\alpha(\gamma-1)}}\right).
\end{aligned}$$

When there are shocks that do have an effect on output and inflation, the general program will be as follows.

1. Solve the model and obtain output and inflation as functions of the states.
2. Fix the values of interest for the states and create a grid using gridmake(statevalues).
3.

$$\begin{aligned} B_{t,t+1}\left(S_t\right) &= \mathrm{E}_t\left(Q_{t,t+1}\right) \\ &= \beta Y_t^{\gamma}\mathrm{E}_t\left(Y_{t+1}\left(S_{t+1}\right)^{-\gamma}\left(\Pi_{t+1}\left(S_{t+1}\right)\right)^{-1}\right) \end{aligned}$$

where the expectation is evaluated using the discretized shocks in the same manner as in the model solving. The result is a vector of one period nominal bond prices. Next, the recursion can be evaluated using

$$\begin{aligned} B_{t,t+2} &= \mathrm{E}_t\left(Q_{t,t+1}B_{t+1,t+2}\left(S_{t+1}\right)\right) \\ &= \beta Y_t^{-\gamma}\mathrm{E}_t\left(Y_{t+1}\left(S_{t+1}\right)^{-\gamma}\left(\Pi_{t+1}\left(S_{t+1}\right)\right)^{-1}B_{t+1,t+2}\left(S_{t+1}\right)\right). \end{aligned}$$

A good way to do this is to build a matrix where each column contains one maturity for all the values of the shocks. The benefit is that then the term structure for that configuration of shocks is simply the row of the matrix.

References [To be completed]

References

[1]

[2] Ang, A. and M. Piazzesi (2003), "A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables," *Journal of Monetary Economics*, forthcoming.

[3] Christiano, L., M. Eichenbaum and C. L. Evans, (2001), "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *NBER Working Paper* #8403.

[4] Judd, K. (1998), *Numerical Methods in Economics.* MIT Press.

[5] Piazzesi, M. (2001), "Macroeconomic jump effects and the yield curve", mimeo, UCLA.

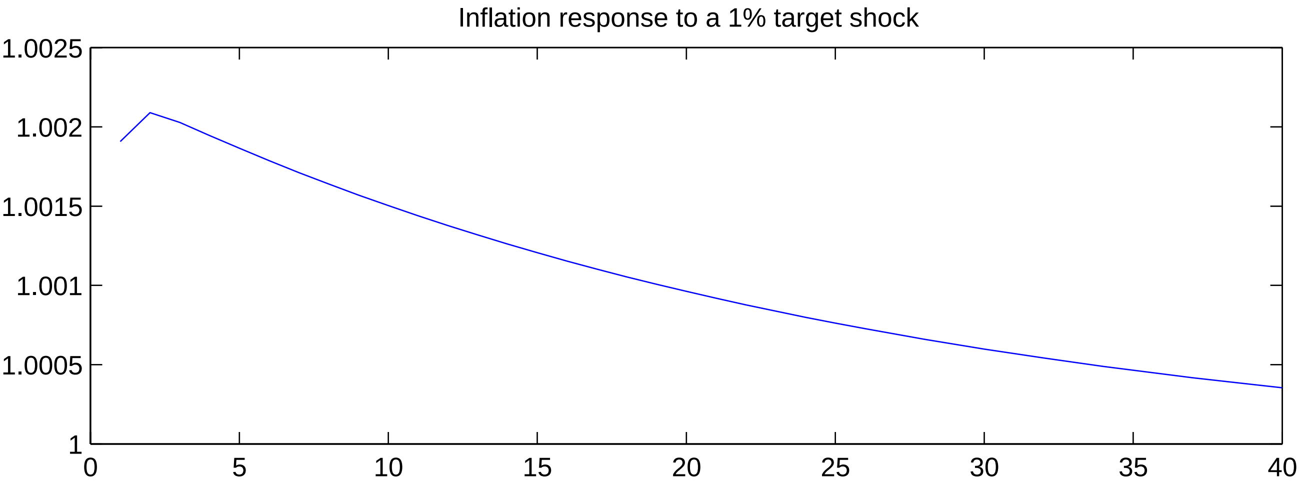
[6] Miranda, J. and P. Fackler (2002), *Applied Computational Economics and Finance*, MIT Press.

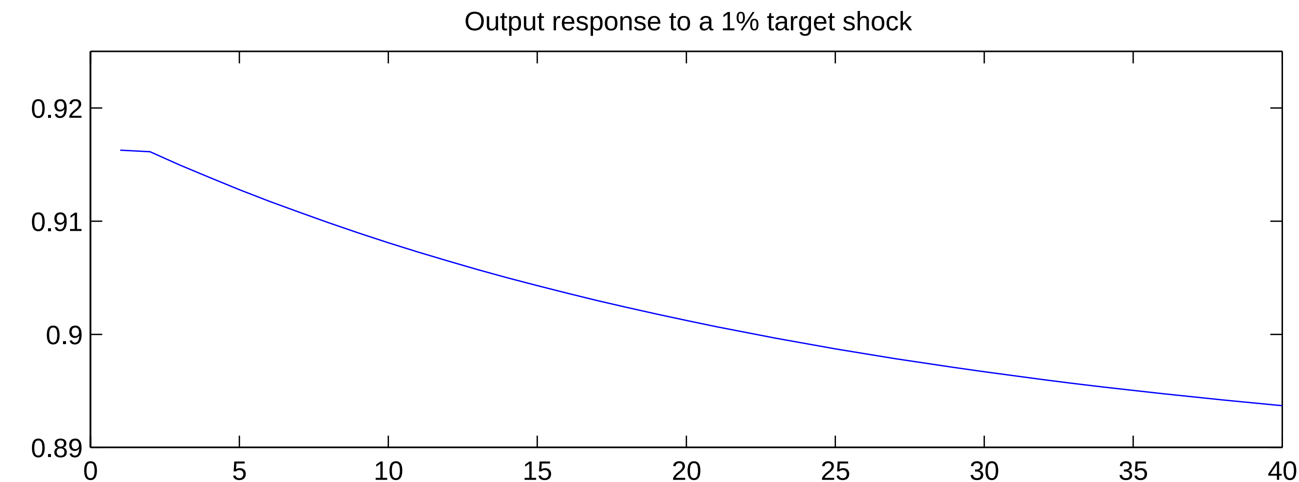
[7] Rudebusch, G. and T. Wu (2003), "A no-arbitrage model of the term structure and the macroeconomy," mimeo, Federal Reserve Bank of San Francisco, August.

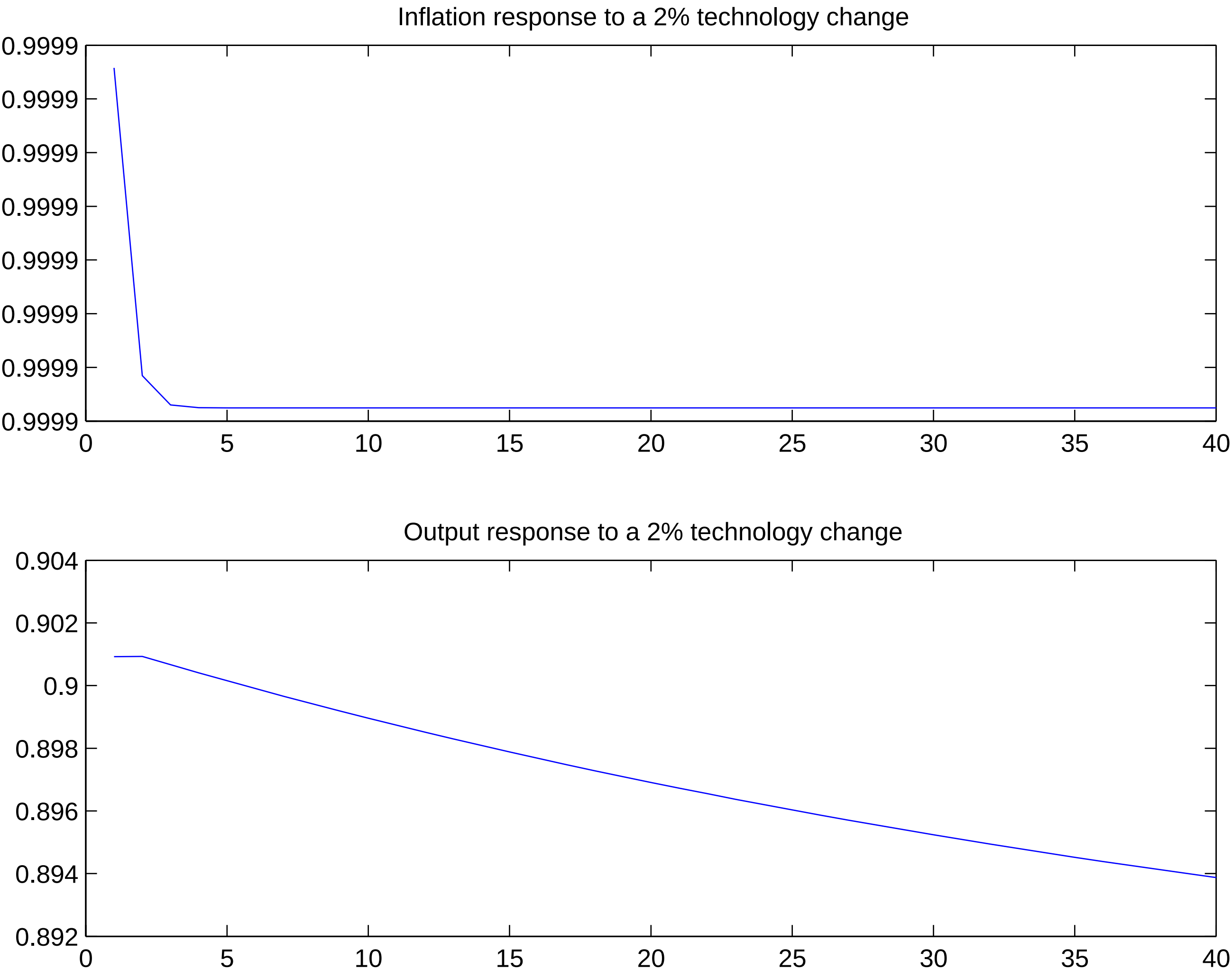
[8] Woodford, M. (2003), *Interest and Prices*, Princeton University Press.

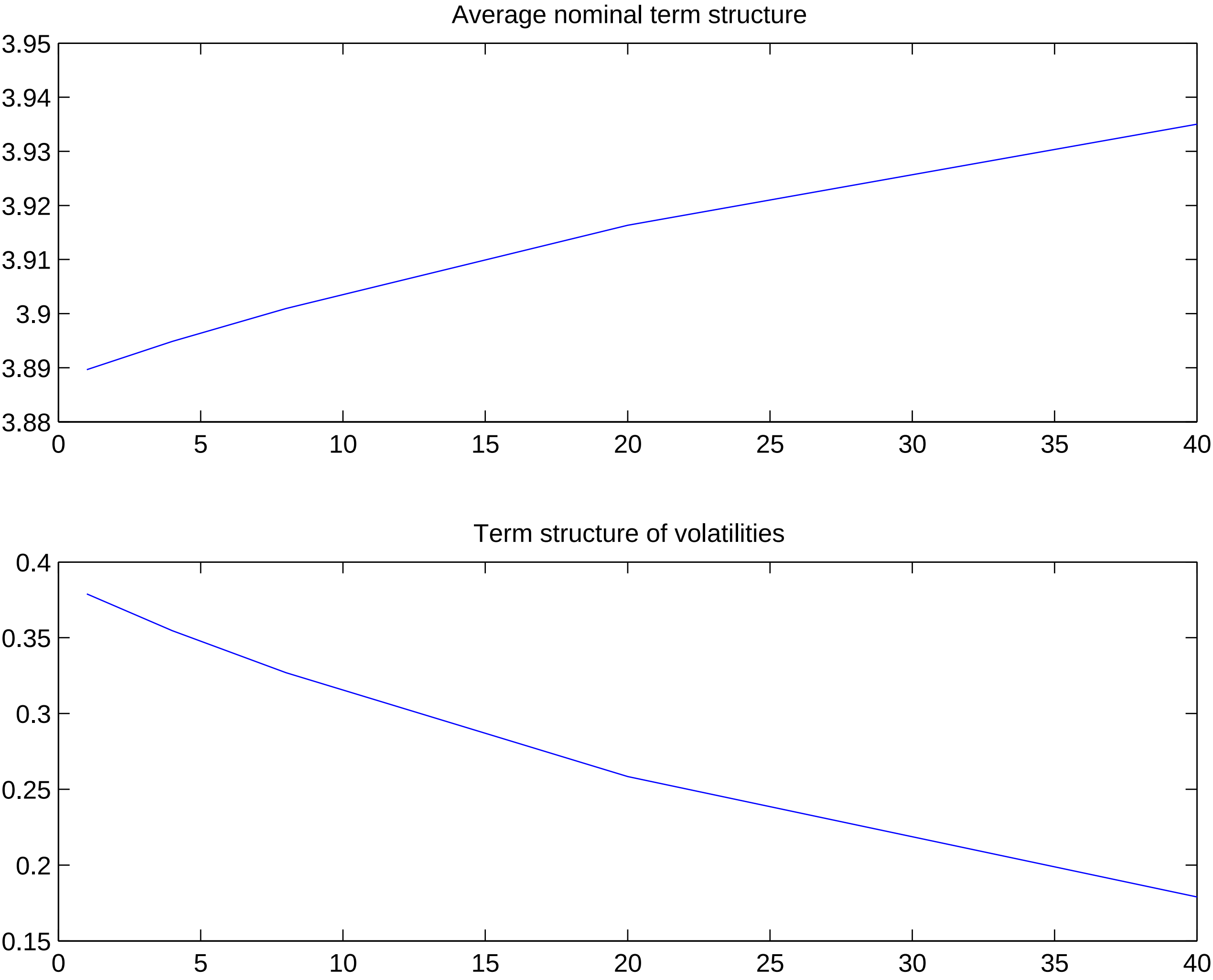
[9] Wu, T. (2002), "Macro Factors and the Affine Term Structure of Interest Rates", mimeo, Federal Reserve Bank of San Francisco.

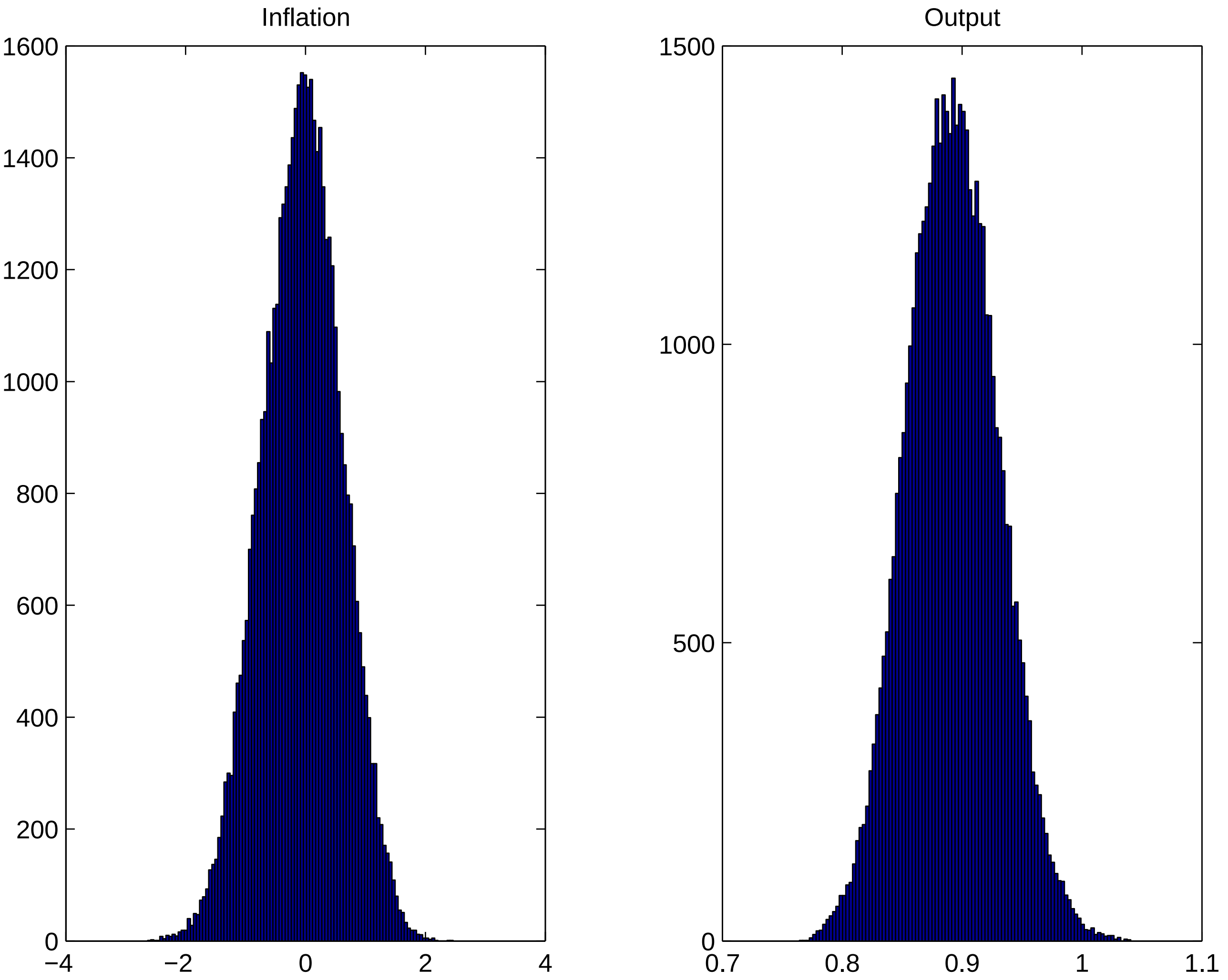
[10]


Inflation response to a 1% target shock
1.0025
1.002
1.0015
1.001
1.0005
1
0
5
10
15
20
25
30
35
40


Output response to a 1% target shock
0.92
0.91
0.9
0.89
0
5
10
15
20
25
30
35
40


Inflation response to a 2% technology change
0.9999
0.9999
0.9999
0.9999
0.9999
0.9999
0.9999
0.9999
0
5
10
15
20
25
30
35
40
Output response to a 2% technology change
0.904
0.902
0.9
0.898
0.896
0.894
0.892
0
5
10
15
20
25
30
35
40


Average nominal term structure
3.95
3.94
3.93
3.92
3.91
3.9
3.89
3.88
0
5
10
15
20
25
30
35
40
Term structure of volatilities
0.4
0.35
0.3
0.25
0.2
0.15
0
5
10
15
20
25
30
35
40


Inflation
Output
1600
1400
1200
1000
800
600
400
200
0
−4
−2
0
2
4
1500
1000
500
0
0.7
0.8
0.9
1
1.1