

Indeterminacy and costly unemployment fluctuations in a finance constrained economy with unions*

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Abstract

In this paper, we extend the finance constrained economy proposed by Woodford (1986) to incorporate unions, imperfectly insured unemployment and transfers. We show that this simple extension of the Woodford model changes drastically its stability conditions and local dynamics around the deterministic steady state. By contrast to most related models in the literature, we find that sunspots and deterministic cycles can emerge in this economy under fairly reasonable or "realistic" values of all structural parameters. In particular, flip, Hopf and transcritical bifurcations may all occur for elasticities of input substitution that are relatively close to one, and for an arbitrarily small amount of increasing returns to scale.

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1 Introduction

In this paper, we extend the finance constrained economy proposed by Woodford (1986) to incorporate unions, imperfectly insured unemployment and transfers. Our aim is to investigate whether these new features affect the local dynamics of this economy, i.e., whether they affect the emergence of local indeterminacy, local bifurcations and deterministic or stochastic (sunspot) expectations driven business cycles near a steady state.

We pursue the line of research initiated by Grandmont et al. (1998), that studied the influence of capital-labor substitution and of the aggregate labour supply elasticity on the occurrence of endogenous deterministic or stochastic fluctuations in the standard infinite horizon, one sector Woodford (1986) model. This model considers two classes of agents "workers" and "capitalists" that optimize discounted lifetime utility. Workers consume and can work, while capitalists consume but do not work. There are two assets: money, and productivity capital. Workers face a cash in advance constraint that reflects the difficulty they have in borrowing against labor income, while capital is accepted as collateral to secure a loan. If capitalists discount the future less than workers, along equilibria close enough to the steady state, capitalists will save only in the form of capital while workers will save only in the form of money balances. All markets are competitive and output is produced from labor and capital with a constant returns to scale technology.

Grandmont et al. (1998) found that "*indeterminacy and deterministic and stochastic endogenous fluctuations did occur, but for rather low elasticities of substitution between capital and labour.*" Therefore they stated that "*It remains to be seen how additional realistic features such as increasing returns to scale, imperfect competition, and/or sluggish adjustments of wages and prices, alter the dynamics of the model, and may or may not improve the range of parameters that give rise to endogenous fluctuations*". Accordingly, Cazzavillan et al. (1998) considered increasing returns to scale in production in the benchmark Woodford (1986) model and found that this helped the emergence of indeterminacy and bifurcations. However, they need a degree of increasing returns to scale too high to be considered empirically plausible using recent estimates. See Basu and Fernald (1995) and (1997) and Burnside et al (1995). More recently Lloyd-Braga and Modesto (2003) introduced unions and wage bargaining in a Woodford model with increasing returns to scale. They found that union power influences the local dynamics of the model, and that for mild externalities (compatible with a downward sloping

marginal productivity of labour curve) indeterminacy and Hopf bifurcations emerge for reasonable values for the elasticity of substitution in production. However, with an arbitrarily small value of externalities, indeterminacy requires an infinite elasticity of substitution between capital and labour. This finding shows that the consideration of unions and wage bargaining in a Woodford type model is not enough to render the emergence of local indeterminacy and deterministic and stochastic endogenous fluctuations possible in the absence of increasing returns to scale in production.

At the same time, other works, that studied the effects of fiscal policy on the stability properties of the equilibrium in one sector real business cycle models, have shown that countercyclical taxes can reduce the required degree of increasing returns needed for indeterminacy. See for example, Schmitt-Grohé and Uribe (1997) and Guo and Lansing (1998). Therefore, these findings suggest that the introduction of taxation in a Woodford model may provide another channel through which indeterminacy and endogenous fluctuations may emerge.

Accordingly in this work, we consider an economy of the Woodford type where, as in Lloyd-Braga and Modesto (2003), wages and employment are bargained between unions and firms and where unemployment emerges as an equilibrium result. Moreover, contrary to what is generally done in the literature, we do not assume the existence of a perfect insurance mechanism against the loss of revenue associated with the risk of unemployment. Instead, we assume that there is a government that provides an imperfect insurance mechanism that guarantees a minimum real income to unemployed workers. The government runs a balanced budget and finances this programme taxing employed workers. In our view these features provide a description of the functioning of the labour market more in line with what we observe in the real world. At the same they imply, quite naturally, countercyclical taxation since taxes will be lower when unemployment is low. Technology at the firms level is of the constant returns to scale type and may exhibit constant or increasing returns to scale at the social level, via labour and capital externalities. All the other features of the model are similar to the standard Woodford (1986), except that we assume that wages/minimum income are paid by firms/government to employed /unemployed workers of period t only in the beginning of period $t + 1$. Consequently workers do not hold any wealth and money is held by firms, i.e., firms are responsible for carrying money balances in the economy from period t to $t + 1$. However, as in the Woodford model, capitalists end up holding the all amount of capital

in the economy.

We find that this simple extension of the Woodford model changes drastically its stability conditions and local dynamics around the deterministic steady state. By contrast to most related models in the literature, we find that deterministic and stochastic endogenous fluctuations, driven by self-fulfilling volatile expectations, can emerge in this economy under fairly reasonable or "realistic" values of all structural parameters. In particular, the occurrence of local indeterminacy under plausible values of the elasticity of substitution between factors does not require increasing returns to scale, being a pervasive phenomena when the technology is of a Cobb-Douglas type. Also flip, Hopf and transcritical bifurcations all occur for elasticities of input substitution that are relatively close to one, and for an arbitrarily small amount of increasing returns to scale, provided union power is sufficiently high. Moreover, these results are obtained with an average tax rate lower than the steady-state unemployment rate and therefore much lower than the one required in Schmitt-Grohé and Uribe (1997).¹ Note also that unemployment fluctuations are both involuntary and costly, since we do not assume a perfect insurance mechanism against unemployment.

The rest of the paper is organized as follows. In the next section we describe the model and obtain the (deterministic perfect foresight) dynamic equilibrium equations. In section 3 we study the existence, uniqueness and multiplicity of the steady state. In section 4 we analyze the local dynamics properties of the steady state, and also the occurrence of local bifurcations. The results obtained on local indeterminacy and bifurcations are discussed in section 5. Finally in section 6 we make a few concluding remarks.

2 The Model

We consider an infinite horizon economy composed of 5 types of agents that act under perfect foresight in the deterministic sense. Workers who consume, work and save through money holdings, facing cash in advance constraints. Capitalists who accumulate capital, but do not work, and consume out of capital earnings.² Firms that produce the single tradable good, renting capi-

¹In the benchmark case studied in Schmitt-Grohé and Uribe (1997) the labour tax rate must be above the capital share of output for indeterminacy to occur.

²Implicitly we thus follow Woodford (1986) in assuming that the discount rate of workers is less than that of capitalists. Woodford (1986) shows that in this case, at equilibrium,

tal and hiring workers. Unions that represent workers and bargain with firms over wages and employment. Finally, there is a government which provides an imperfect insurance mechanism against the loss of revenue associated with the risk of unemployment. Hence, all markets are assumed to be perfectly competitive, with the exception of the labour market where union power will prevent the wage from falling to its walrasian level.

2.1 Workers

We consider a continuum of identical infinitely lived workers of mass big enough. Preferences are given by the following utility function: $\sum_{t=1}^{\infty} \gamma^{t-1} u(c_t^w)$, where $0 < \gamma < 1$ is the constant discount factor and c_t^w is total consumption in period t .³ These workers, who are assumed to have no labour desutility, wish to supply at each period one unit of labour. However, at each period t , a worker may be either employed (state e) - receiving at the beginning of next period a nominal wage w_t - or unemployed (state u). Contrary to what is usually done in the Real Business Cycle literature (see for example Hansen, 1985), we assume that agents cannot fully insure their earnings against the risk of being unemployed. Instead, we assume that the government provides a minimum guaranteed income program, ensuring to all period $t - 1$ unemployed workers a constant real guaranteed income $b > 0$.⁴ Similarly to employed workers, we assume that these resources are only available for consumption at the beginning of period t . Since the government runs a balanced budget, these transfers are thus financed taxing in t previously employed workers (i.e., when their labour income becomes available), that must pay a real tax τ_t .⁵ Workers being only able to save in the form of money, the

capitalists will end up holding the whole stock of capital, while workers are constrained to save only in the form of money.

³We assume that u satisfies the usual properties, namely: $u(c_t)$ is a continuous real valued function in $c_t \geq 0$, with $u'(c) > 0$ and $u''(c) < 0$ for $c_t > 0$.

⁴Note that in most European countries, where such minimum guaranteed income programmes exist, they are indeed indexed to inflation, in order to ensure real purchasing power of the poor.

⁵Note that these assumptions are equivalent to assume that the government provides an *imperfect* insurance mechanism. To participate in this programme each worker pays a fixed premium τ , receiving in the event of unemployment a guaranteed minimum real income b . As usual, the premium must cover the expected value of payments, i.e., $\tau_t = B_t(1 - h_{t-1})$, that we can rewrite as $b(1 - h_{t-1}) = h_{t-1}\tau_t$ where h_t is the probability of employment in t and $b = B_t - \tau_t$.

typical budget constraint of a worker who was in state $i \in \{e, u\}$ in period $t - 1$ can be written as

$$m_{t+1}^w = m_t^w + y_t^i - p_t c_t^w \quad (1)$$

where m_t^w denotes money held at the beginning of period t , p_t is the price of output, and where $y_t^i \in \{w_{t-1} - p_t \tau_t, p_t b\}$ is a state-dependant revenue, conditioned on being in state $i \in \{e, u\}$ in period $t - 1$.

Additionally, workers are subject to a cash in advance constraint implying that, whether employed or unemployed, they can only buy the consumption good out of net earnings already received plus previously accumulated money. This implies that the following constraint

$$m_t^w + y_t^i - p_t c_t^{wi} \geq 0 \quad i = \{e, u\} \quad (2)$$

must hold for each period t . Therefore, the problem of the representative worker is to choose the positive levels of $(c_t^{wi}, m_{t+1}^w)_{t=1,2,\dots,\infty}$, $i = \{e, u\}$, in order to maximize utility subject to the constraints (1)-(2). We also assume that, when deciding how much to consume in t and how much to save in the form of money, the worker does not know yet whether he will be employed or unemployed in this period. However, he can put a probability distribution over the two states which, given that all workers are treated anonymously, consists in the period t employment (h_t) and unemployment ($1-h_t$) rates, respectively. In Appendix A.1 we give the conditions under which, at an intertemporal equilibrium, the cash in advance constraint is always binding for either employed or unemployed workers.⁶ In this case, it is easy to see from (1) and (2) that, independently of the utility function $u(c^w)$ considered, all workers will always choose the following amounts of consumption and money holdings:

$$c_t^{wi} = \frac{y_t^i}{p_t} \quad i = \{e, u\} \quad (3)$$

$$m_{t+1}^w = 0 \quad (4)$$

⁶This means that we focus on equilibria that are sufficiently close to a steady state equilibrium in which, additionally to the assumption of zero inflation ($\pi = 1$), the condition $u'(w - \tau) > \gamma h u'(w - \tau) + \gamma(1 - h)u'(b)$ is verified. See Appendix A.1

2.2 Capitalists

As in Woodford (1986) capitalists are identical and maximize $\sum_{t=1}^{\infty} \beta^{t-1} \text{Log} c_t^c$, subject to $p_t c_t^c + p_t k_{t+1}^c = p_t R_t k_t^c$, where c_t^c is consumption in period t , k_t^c is the capital stock and $R_t = (\rho_t + 1 - \delta)$ is the real gross rate of return on capital, ρ_t being the real rental rate of capital. The discount factor β is such that $\gamma < \beta < 1$, and $0 \leq \delta \leq 1$ is the capital depreciation rate. The solution to this problem may be written as:

$$c_t^c = (1 - \beta) R_t k_t^c \quad (5)$$

$$k_{t+1}^c = \beta R_t k_t^c. \quad (6)$$

2.3 Firms

Firms are assumed to operate under a private constant returns to scale technology $AF(k, l)$. However we will allow for the possibility of external effects in production that may generate increasing returns at the social level.⁷ As in Cazzavillan et. al. (1998) we represent these external effects by the function $\Psi(\bar{k}_t, \bar{l}_t)$, where \bar{k} and \bar{l} are, respectively, the average (per firm) capital stock and employment in the economy. This means that the representative firms' production function may be written as:

$$\Psi(\bar{k}_t, \bar{l}_t) AF(k_t, l_t) = \Psi(\bar{k}_t, \bar{l}_t) A l_t f(x_t) \quad (7)$$

where $x \equiv k/l$ is the capital labor ratio, $A > 0$ is a scaling factor, and where the right-hand side of (7) follows from the constant returns to scale assumption. We also make the followings assumptions on technology:

Assumption 1 $f(x)$ is a real, continuous function for $x \geq 0$, positively valued and differentiable as many times as needed for $x > 0$, with $f'(x) > 0$, $f''(x) < 0$, $f(x) - f'(x)x > 0$, $\lim_{x \rightarrow 0} \frac{f(x) - f'(x)x}{f'(x)} = 0$ and $\lim_{x \rightarrow \infty} \frac{f(x) - f'(x)x}{f'(x)} = \infty$.

⁷There exists a vast literature justifying the existence of capital and labour externalities. See for example Arrow (1962), Diamond (1982), Murphy *et al.* (1989) and Howitt and McAfee (1992). Note that such externalities allow the consideration of aggregate increasing returns to scale while maintaining the assumption of a perfectly competitive output market. This device has been widely used in macro dynamic models. See, for instance, Farmer and Guo (1994), Benhabib and Farmer (1994) and Cazzavillan et al. (1998).

We shall denote the elasticity of substitution between capital and labor by $\sigma(x) > 0$, $\frac{1}{\sigma(x)} = -\frac{f(x)f''(x)x}{f'(x)[f(x)-f'(x)x]}$.

Assumption 2 Ψ is a real, positively valued function, continuous and non-decreasing in $(\bar{k}_t, \bar{l}_t) \in \mathfrak{R}_+^2$, C^r on \mathfrak{R}_{++}^2 for r large enough and homogeneous of degree $\nu \geq 0$: $\nu = \varepsilon_{\Psi, \bar{k}}(\bar{k}_t, \bar{l}_t) + \varepsilon_{\Psi, \bar{l}}(\bar{k}_t, \bar{l}_t)$, where $\varepsilon_{\Psi, \bar{k}} \equiv \Psi'_{\bar{k}} \frac{\bar{k}}{\Psi} \geq 0$ and $\varepsilon_{\Psi, \bar{l}} \equiv \Psi'_{\bar{l}} \frac{\bar{l}}{\Psi} \geq 0$.

Hence $\varepsilon_{\Psi, \bar{k}}$ and $\varepsilon_{\Psi, \bar{l}}$ measure the contribution of capital and labor to externalities and $1 + \nu$ is the degree of social returns to scale.

Firms wish to maximize profits $\Pi_t = p_t \Psi(\bar{k}_t, \bar{l}_t) A l_t f(x_t) - w_t l_t - p_t \rho_t k_t$ but must negotiate wages and employment with unions. We assume that at each period t the sequence of decisions is the following. First, firms rent capital, k_t , on the economy-wide capital market, at a given nominal rental rate $p_t \rho_t$. Next wages, w_t , and employment, l_t , are determined through the bargaining process, and finally production takes place. As usually done in the literature we assume that workers cannot sign binding wage contracts, so that the wage and employment are determined after the capital stock decision has been made. This means that to ensure time consistency of the equilibrium the problem of the firm must be solved backwards, starting with the wage-employment bargain. Note also that, as wages are only paid in the beginning of next period, and must be paid in cash, firms will have to hold, at beginning of period $t + 1$, $m_{t+1}^f = w_t l_t$.

2.4 Unions

We assume that all workers are unionized and that unions are firm-specific, i.e., we have one union per firm. Workers are matched exogenously and uniformly with unions and cannot move between firms or unions, so that each union represents the same mass of workers. Assuming that unions wish to maximize the sum of discounted consumptions of their members we obtain (see (3)) the following objective function for the representative union:

$$\Omega_t = \sum_{i=0}^{\infty} \gamma^{i+1} \left\{ \frac{w_{t+i} l_{t+i}}{p_{t+i+1}} + b(1 - l_{t+i}) - \tau_{t+i+1} l_{t+i} \right\} \quad (8)$$

where we have normalized the mass of workers per firm to 1.

2.5 Government

As explained earlier the government guarantees a real minimal amount of income to each unemployed worker, taxing employed individuals in order to run a balanced budget. Hence we have that

$$p_t b(1 - \bar{l}_{t-1}) = p_t \tau_t \bar{l}_{t-1}. \quad (9)$$

Obviously workers, unions and firms being small take b and τ as given.

2.6 Wage, employment and capital decisions

Wage and employment are determined through an efficient bargaining procedure. This implies that l_t and w_t must solve the generalized Nash bargaining problem:

$$\underset{(w_t, l_t) \in \mathbb{R}_{++}^2}{Max} \quad (\Pi_t - \bar{\Pi}_t)^\alpha (\Omega_t - \bar{\Omega}_t)^{(1-\alpha)} \quad s.t. \quad l_t \leq n \quad (10)$$

where $0 < \alpha \leq 1$ represents the firm's bargaining power in the bargain and $(\bar{\Pi}_t, \bar{\Omega}_t)$ are the fallback payoffs of each party if no agreement in period t is reached. If negotiations fail production does not take place and all workers are unemployed. Given the sequence of decisions of firms and the assumption that no binding wage contracts can be signed, the fallback payoff of firms is: $\bar{\Pi}_t = -p_t \rho_t k_t$. Similarly we can compute the fallback payoff of a union which, using (8) is given by $\bar{\Omega}_t = \gamma b + \gamma \Omega_{t+1}$. This implies that:

$$\Pi_t - \bar{\Pi}_t = p_t \Psi(\bar{k}_t, \bar{l}_t) A l_t f(x_t) - w_t l_t \quad (11)$$

$$\Omega_t - \bar{\Omega}_t = \gamma l_t \left(\frac{w_t}{p_{t+1}} - b - \tau_{t+1} \right). \quad (12)$$

We assume that the solution l_t of problem (10) always satisfies $l_t < n$ so that the first order conditions are:

$$b + \tau_{t+1} = \frac{p_t}{p_{t+1}} \Psi(\bar{k}_t, \bar{l}_t) A \left[f(x_t) - f'(x_t) x_t \right] \quad (13)$$

$$\frac{w_t}{p_t} = \Psi(\bar{k}_t, \bar{l}_t) A \left[f(x_t) - \alpha f'(x_t) x_t \right]. \quad (14)$$

From equation (13) we can see that, whatever the union's bargaining power, the level of employment is determined by the intersection of the marginal productivity of labour curve $\Psi(\bar{k}_t, \bar{l}_t)A[f(x_t) - f'(x_t)x_t]$, with the real reservation wage schedule, $(b + \tau_{t+1})p_{t+1}/p_t$. See figure 1. By contrast, equation (14) shows that the real wage is set above the MPL (and so above the reservation wage) when $\alpha < 1$, with a markup which is increasing with the unions bargaining power. Of course, given the absence of perfect redistributive schemes, unemployed individuals are thus clearly worse off. Note also that the level of employment is influenced by expectations of future prices (shifting the reservation wage locus) which constitutes a potential channel for the emergence of expectations driven fluctuations.

The firm, anticipating the result of the bargaining process, chooses consequently $k_t > 0$ to maximize current profits that using (14) can be rewritten as

$$\Pi_t \equiv p_t \alpha \Psi(\bar{k}_t, \bar{l}_t) A f'(x_t) k_t - p_t \rho_t k_t \quad (15)$$

with $x_t \equiv k_t/l_t$ where l_t satisfies (13). This yields the following first order condition:

$$\alpha \Psi(\bar{k}_t, \bar{l}_t) A f'(x_t) = \rho_t. \quad (16)$$

Because firms operate under private constant returns to scale at equilibrium firms profits are zero so that eventually no dividends can be paid.

2.7 Equilibrium

In Definition 1 below we characterize the set of perfect foresight intertemporal equilibria of the economy considered.

Definition 1 *An intertemporal equilibrium with perfect foresight is a sequence $(k_t, l_t) \in \mathfrak{R}_{++}^2$, $t = 1, 2, \dots, \infty$ that solves the two-dimensional dynamic system*

$$k_{t+1} = \beta \left[\alpha \Psi(k_t, l_t) A f'(x_t) + (1 - \delta) \right] k_t. \quad (17)$$

$$l_{t+1} \Psi(k_{t+1}, l_{t+1}) A \left[f(x_{t+1}) - \alpha f'(x_{t+1}) x_{t+1} \right] = b \frac{[f(x_t) - \alpha f'(x_t) x_t]}{[f(x_t) - f'(x_t) x_t]}. \quad (18)$$

where $x = k/l$.

Proof. See Appendix A.2. ■

3 Steady State Analysis

In this section we discuss existence and uniqueness versus multiplicity of steady states. A steady state equilibria is a stationary solution (k, l) of (17)-(18), and, thereby, satisfies the following equations:

$$\beta\alpha\Psi(k, l)Af'(x) = \theta \quad (19)$$

$$l\Psi(k, l)A[f(x) - f'(x)x] = b, \quad (20)$$

where $x \equiv k/l$ and $0 < \theta \equiv 1 - \beta(1 - \delta) \leq 1$.

Since $\Psi(k, l)$ is a homogeneous function of degree v , we can rewrite it as:

$$\Psi(k, l) \equiv l^v \Phi(x), \varepsilon_{\Psi, L}(x) \equiv v - \varepsilon_{\Phi, x}(x), \varepsilon_{\Psi, K}(x) \equiv \varepsilon_{\Phi, x}(x) \quad (21)$$

Then, using (21), equations (19) and (20) can be, respectively, rewritten as:

$$l = \left[\frac{\theta}{A\beta\alpha} \frac{1}{\Phi(x) f'(x)} \right]^{1/v} \quad (22)$$

$$l = \left[\frac{b}{A} \frac{1}{\Phi(x) f(x) - f'(x)x} \right]^{1/(1+v)} \quad (23)$$

In what follows, we shall ensure the existence of at least one steady state $(k, l) = (k^*, l^*)$, with a given value for the employment rate, $l^* < 1$, and for the capital stock, $k^* \equiv l^*x^*$, by choosing appropriately the values of the parameters A and b . Indeed, letting $l = l^*$ and $x = x^* = k^*/l^*$ it is immediate to see that x^*, l^* satisfy (22) if and only if $A = \theta [\beta\alpha\Phi(x^*) f'(x^*) l^{*v}]^{-1} > 0$. Also, for this value of A , l^* and x^* satisfy equation (23) if and only if $b = l^*\theta [f(x^*) - f'(x^*)x^*] [f'(x^*)\beta\alpha]^{-1} > 0$. Accordingly, the following proposition holds:

Proposition 1 *Steady state existence*

Let Assumption 1 be verified and $\beta \in (0, 1)$, $\theta \in (0, 1]$, $\alpha \in (0, 1]$. Then $(k, l) = (k^, l^*)$, with $0 < l^* < 1$ and $k^* \equiv l^*x^* > 0$, is a (interior) steady state solution of the dynamic system (17)-(18) if and only if: $A = \theta [\beta\alpha\Phi(x^*) f'(x^*) l^{*v}]^{-1} > 0$ and $b = l^*\theta [f(x^*) - f'(x^*)x^*] [f'(x^*)\beta\alpha]^{-1} > 0$.*

We proceed now by studying whether more than one steady state can exist. Equating (23) and (22) we see that a steady state value of x , is a solution of the following equation:

$$Z(x) \equiv \frac{[f(x) - f'(x)x]^{v/(1+v)}}{[\Phi(x)]^{1/(1+v)} f'(x)} = \frac{A\beta\alpha}{\theta} \left(\frac{b}{A}\right)^{v/(1+v)} \equiv H. \quad (24)$$

Once we find a value x satisfying equation (24), we can easily obtain the corresponding steady state values (k, l) . Indeed, using (23) or (22) we obtain the corresponding value of l , and k is obtained by recalling the definition of x , i.e., $k = xl$. Hence, studying the existence of one or multiple steady states amounts to study the number of solutions to the equation $Z(x) = H$.

Obviously, under Proposition 1, $(k, l) = (k^*, l^*)$ is the unique steady state if $Z(x)$ is a monotonous function in $x \in (0, +\infty)$, which will be verified, since $Z(x) > 0$, if $\varepsilon_{Z,x} = Z'(x)x/Z(x)$ does not change sign. Using (21) and Assumption 1 we have that:

$$\varepsilon_{Z,x}(x) = \frac{\varepsilon_{\Psi,L}(x) - v}{1+v} + \frac{v\alpha + \alpha - 1 + s_L(x)}{\sigma(x)\alpha(1+v)} \quad (25)$$

where

$$s_L = \frac{f(x) - \alpha f'(x)x}{f(x)} \quad (26)$$

is the labour share of output satisfying, from Assumption 1, $(1-\alpha) < s_{L(x)} < 1$.

The next proposition states conditions under which $\varepsilon_{Z,x}(x)$ is always negative or always positive, so that a unique steady state exists.

Proposition 2 *Uniqueness of the steady state*

Under Proposition 1, there is a unique (interior) steady state of the dynamic system (17)-(18) if one of the following conditions holds:

- (i) $\sigma(x) > \frac{\nu\alpha + \alpha - 1 + s_L(x)}{\alpha(\nu - \varepsilon_{\psi,L}(x))}$, $\forall x > 0$;
- (ii) $\sigma(x) < \frac{\nu\alpha + \alpha - 1 + s_L(x)}{\alpha(\nu - \varepsilon_{\psi,L}(x))}$, $\forall x > 0$.

Notice that $\frac{\nu\alpha + \alpha - 1 + s_L(x)}{\alpha(\nu - \varepsilon_{\psi,L}(x))} > 1$, since under Assumptions 1 and 2 $v \geq \varepsilon_{\psi,L}(x) \geq 0$ and $\alpha - 1 + s_L(x) > 0$. Hence if technology is of a CES type with $\sigma \leq 1$ there is a unique steady state, since in this case we obtain configuration (ii) of the above Proposition.

Let us now discuss conditions under which there are at most two steady states. Clearly if the function $Z(x)$ is single caved, $\varepsilon_{Z,x}(x)$ changing sign only once from negative to positive values, then $Z(x)$ cannot cross the value H more than twice. In this case equation (24) cannot have more than two solutions and at most two steady states can exist. The same happens if $Z(x)$ is single peaked, $\varepsilon_{Z,x}(x)$ changing sign only once from positive to negative values. Also, when $\varepsilon_{Z,x}(x)$ is a monotonous function of x it can change sign at most once, from negative to positive values if it is an increasing function, and from positive to negative values if it is a decreasing function. Hence, in both cases there can be no more than two steady states. Noticing that, by (25), $\varepsilon_{Z,x}(x)$ is an increasing (decreasing) function if and only if $\varepsilon_{\Psi,L}(x) + \frac{s_L(x)}{\sigma(x)\alpha}$ is increasing (decreasing) in x , we obtain the following proposition:

Proposition 3 *Under Assumption 1, there are at most two steady states of the dynamic system (17)-(18) if one of the following conditions is satisfied:*

- (i) $\varepsilon_{\Psi,L}(x) + \frac{s_L(x)}{\sigma(x)\alpha}$ is increasing in $x > 0$;
- (ii) $\varepsilon_{\Psi,L}(x) + \frac{s_L(x)}{\sigma(x)\alpha}$ is decreasing in $x > 0$.

Consider that, for instance, (ii) is verified. Then, if $\varepsilon_{Z,x}(0) > 0$ and $\varepsilon_{Z,x}(\infty) < 0$, the function $Z(x)$ is single peaked. Therefore, under Proposition 1, two steady states will exist if and only if both $\lim_{x \rightarrow 0} Z(x)$ and $\lim_{x \rightarrow \infty} Z(x)$ take values below H , provided that $\varepsilon_{Z,x}(x^*) \neq 0$. The two steady states coincide, i.e., there is a unique steady state if $\varepsilon_{Z,x}(x^*) = 0$, i.e., if $\sigma(x^*) = \frac{\nu\alpha + \alpha - 1 + s_L(x^*)}{\alpha(\nu - \varepsilon_{\Psi,L}(x^*))}$.⁸

We consider now the simple case of an economy with a constant elasticity of substitution between factors and a constant degree of labor externalities to study whether two steady states can exist. Accordingly we have the following expressions for $f(x)$ and $\Phi(x)$:

$$f(x) = \left[(1-s)x^{\frac{\sigma-1}{\sigma}} + s \right], \text{ for } \sigma \in (0, +\infty) \text{ if } \sigma \neq 1; \quad f(x) = x^{1-s} \text{ if } \sigma = 1$$

$$\Phi(x) = x^{\nu - \varepsilon_{\Psi,L}}.$$

Using (26) and (24), it is easy to see that, in this case, $s_L(x)$ and $Z(x)$, can be written as:

⁸Later on it will be shown that in this case the steady state (k^*, l^*) undergoes a transcritical bifurcation.

$$1 - \alpha < s_L(x) = 1 - \frac{\alpha(1-s)}{(1-s) + sx^{\frac{1-\sigma}{\sigma}}} < 1 \quad (27)$$

$$Z(x) = \frac{s^{v/(1+v)}}{1-s} \left[(1-s)x^{\sigma(v-\varepsilon_{\Psi,L})-v} + sx^{\frac{\sigma-1}{\sigma}[\sigma(v-\varepsilon_{\Psi,L})-(1+v)]} \right]. \quad (28)$$

As above explained, in this economy where technology is of a CES type, two steady states may only exist if $\sigma > 1$. In this case $s_L(x)$ is a decreasing function of x and thereby we obtain configuration (ii) of Proposition 3. Also, since by (27) $s_L(0) = 1$ and $s_L(\infty) = 1 - \alpha$, we obtain that $\varepsilon_{Z,x}(0) = \frac{\varepsilon_{\Psi,L}-v}{1+v} + \frac{1}{\sigma} > 0 \iff \sigma < \frac{1+v}{v-\varepsilon_{\Psi,L}}$ and $\varepsilon_{Z,x}(\infty) = \frac{\varepsilon_{\Psi,L}(x)-v}{1+v} + \frac{v}{\sigma(1+v)} < 0 \iff \sigma > \frac{v}{v-\varepsilon_{\Psi,L}} > 1$. Hence, if $1 < \frac{v}{v-\varepsilon_{\Psi,L}} < \sigma < \frac{1+v}{v-\varepsilon_{\Psi,L}}$, the function $Z(x)$ is single peaked. Note also that, using (28), the boundary conditions are satisfied: $\lim_{x \rightarrow 0} Z(x) = \lim_{x \rightarrow \infty} Z(x) = 0$. Therefore, under Proposition 1, two steady states exist provided $\varepsilon_{Z,x}(x^*) \neq 0$, i.e. provided that $\sigma \neq \frac{\nu\alpha + \alpha - 1 + s_L(x^*)}{\alpha(\nu - \varepsilon_{\Psi,L})}$. The following proposition summarizes this result.

Proposition 4 *Multiple steady states in the CES economy*

Let $1 < \frac{v}{v-\varepsilon_{\Psi,L}} < \sigma \equiv \sigma(x) < \frac{1+v}{v-\varepsilon_{\Psi,L}}$ and $\varepsilon_{\Psi,L} \equiv \varepsilon_{\Psi,L}(x)$, for all $x > 0$. Then, under Proposition 1, there are generically (i.e., when $\sigma \neq \frac{\nu\alpha + \alpha - 1 + s_L(x^*)}{\alpha(\nu - \varepsilon_{\Psi,L})}$) two different steady states. The two steady states coincide with (k^*, l^*) when $\sigma = \frac{\nu\alpha + \alpha - 1 + s_L(x^*)}{\alpha(\nu - \varepsilon_{\Psi,L})}$.

4 Local dynamics and bifurcation analysis

In this section, we study the local stability properties of our dynamic system (17) and (18), around an interior steady state (k, l) . Equations (17) and (18), define locally a two dimensional dynamic system of the form $(k_{t+1}, l_{t+1}) = G(k_t, l_t)$, provided the elasticity of $l_{t+1} \Psi(k_{t+1}, l_{t+1}) A [f(x_{t+1}) - \alpha f'(x_{t+1})x_{t+1}]$ with respect to l_{t+1} does not vanish at the steady state under analysis.

We follow the usual procedure of linearizing the system G around the steady state and considering its associated 2x2 Jacobian matrix, J , evaluated at the steady state under study. The trace, T , and the determinant, D , of matrix J correspond respectively to the sum and product of the two roots (eigenvalues) of the associated characteristic polynomial $Q(\lambda) \equiv \lambda^2 - \lambda T + D$.

In Appendix A.3, using (17), (18), (26) and Assumptions 1 and 2, we obtain the elements of J written in terms of the parameters $1 > \theta \equiv 1 - \beta(1 - \delta) > 0$, $v > 0$, and $1 \geq \alpha > 0$ and in terms of the degree of labor externalities, $v \geq \varepsilon_{\Psi,L} \geq 0$, the elasticity of substitution between inputs, $\sigma > 0$ and the labor share, $1 > s_L > 1 - \alpha$, all evaluated at the steady state under analysis. The trace and the determinant of the Jacobian matrix J are, accordingly, given as follows:

$$T = 1 + \frac{[\theta\nu(\alpha - 1 + s_L) - (1 - s_L)(1 - \alpha) - \alpha s_L \theta \varepsilon_{\Psi,L}] \sigma}{[(\alpha - 1 + s_L) + \alpha s_L \varepsilon_{\Psi,L}] \sigma - (1 - s_L)(\alpha - 1 + s_L)} \quad (29)$$

$$D = -\frac{[\theta\nu(\alpha - 1 + s_L) - (1 - s_L)(1 - \alpha) + s_L \theta(\alpha - 1 + s_L)]}{[(\alpha - 1 + s_L) + \alpha s_L \varepsilon_{\Psi,L}] \sigma - (1 - s_L)(\alpha - 1 + s_L)} \quad (30)$$

We take $\varepsilon_{\Psi,L}$, σ , s_L , θ , α and v as parameters characterizing our economies and our aim is to study the local dynamic properties in terms of these meaningful economic parameters. We shall assume that a steady state exists in the whole range of parameter values, and, under Proposition 1, we consider that the steady state remains fixed at (k^*, l^*) for all the possible parameters' configurations.⁹

We proceed our analysis by using a geometrical method as developed in Grandmont et al. (1998), which allows us to characterize the occurrence of local indeterminacy and bifurcations, and ultimately of stochastic and deterministic endogenous fluctuations around the steady state, in terms of the relevant parameters. The idea is to analyze how the trace and determinant of our system, or equivalently the local eigenvalues, evolves in the space (T, D)

⁹When considering different values for the parameters θ (or δ and β), v , α , $\varepsilon_{\Psi,L}$ or $s_L \equiv s_L(x^*) = 1 - \alpha \frac{f'(x^*)x^*}{f(x^*)}$ we are assuming that the scale parameters A and b , given in Proposition 1, automatically adjust so that (k^*, l^*) remains a steady state solution. Note that the labor share evaluated at the steady state under consideration, s_L , is not a true parameter. However its value in real data is well known and remains almost constant across countries. Therefore we think that it is more interesting to study the dynamics in terms of s_L than in terms of $f(x)$ and $f'(x)$. Of course, doing so implies that when we consider a fixed value for s_L and change α , the value of f' evaluated at x^* is also adjusting so that s_L can indeed remain constant.

when some parameters are made to vary continuously in their admissible range.

In Figure 2 we have represented in the plane (T, D) three lines relevant for our purpose: the *line AC* ($D = T - 1$) where a local eigenvalue is equal to 1; the *line AB* ($D = -T - 1$), where one eigenvalue is equal to -1; and the *segment BC* ($D = 1$ and $|T| < 2$) where two eigenvalues are complex conjugates of modulus 1. When T and D fall in the interior of triangle ABC the steady state is a *sink* (both eigenvalues with modulus lower than one), i.e., asymptotically stable. In the present context, where only capital is a predetermined variable, the steady state is locally indeterminate¹⁰ in this case, and, as known, there are infinitely many stochastic endogenous fluctuations (sunspots) arbitrarily close to the steady state. In all other cases the steady state is locally determinate: a *saddle* (one eigenvalue with modulus higher than one and one eigenvalue with modulus lower than one) when $|T| > |D + 1|$ and a *source* (both eigenvalues with modulus higher than one) in the remaining regions.

We can also use the diagram of Figure 2 to study the occurrence of local bifurcations. When, by slightly changing a (bifurcation) parameter, the values of T and D cross the interior of the segment *BC* (a pair of complex conjugate eigenvalues crossing the unit circle) a *Hopf* bifurcation generically occurs, and deterministic cycles (periodic or quasi periodic orbits) surrounding the steady state in the state space emerge. A *flip* bifurcation occurs when the values of T and D cross the *AB* line, and deterministic cycles of period two appear. Finally, when the values of T and D cross the *AC* line, and if Propositions 1 and 3 are satisfied, a *transcritical* bifurcation generically occurs, by which two close steady states exchange stability properties.¹¹ Bifurcations may also lead to the emergence of stochastic endogenous fluctuations, even when the steady state is locally determinate. Indeed, if Hopf (flip) bifurcations are supercritical, cycles appearing when the steady state is locally determinate, there are, as shown in Grandmont et. al (1998), stochastic equilibria, remaining in a compact set that contains in its interior the stable cycle. Also, if the transcritical bifurcation involves the existence of a sink and a saddle steady state, then although no stochastic equilibria

¹⁰Indeterminacy occurs when the number of eigenvalues strictly lower than one in absolute value is larger than the number of predetermined variables.

¹¹When T and D cross the line *AC* other types of bifurcations may occur (pitchfork or saddle-node). However, since we have assumed the existence of at least one steady state (Proposition 1) and at most of two (Proposition 3) these other bifurcations are ruled out.

arbitrarily near the saddle exist, there are stochastic equilibria arbitrarily near the other close sink steady state.

Studying local dynamics by analyzing all possible cases of local indeterminacy and bifurcations as depending on the values taken by the parameters, for the whole range of their admissible values in the model, turns out to be important as long as the empirical values for these parameters are either not well known or not roughly constant across different countries, as it happens for the elasticity of substitution between inputs, σ , for the degree of unions bargaining power, $1 - \alpha$, and for the degree of externalities, $\varepsilon_{\Psi,L}$ and ν . Therefore we shall organize our discussion in terms of these parameters. We first fix α , $\varepsilon_{\Psi,L}$ and ν , and consider σ as the bifurcation parameter, by making its value to vary continuously from 0 to $+\infty$. As we shall see the occurrence of indeterminacy and bifurcations for certain values of σ depend on the values of α and also of $\varepsilon_{\Psi,L}$ and ν . However it is straightforward to see that when $\sigma = 1$ the steady state is indeterminate, for all different possible values of α , $\varepsilon_{\Psi,L}$ and ν (and of θ and s_L). Accordingly we have the following result:

Proposition 5 *The Cobb-Douglas case*

For $\sigma = 1$ the steady state is locally indeterminate.

Proof. When $\sigma = 1$ using (29) and (30) we have $T = (1 - \theta) \in (0, 1)$ and $D = 0$. Therefore the point (T, D) falls within the triangle ABC , one eigenvalue being zero and the other being $0 < 1 - \theta < 1$, both having modulus less than one. Therefore the steady state is a sink, indeterminate in the forward dynamics. ■

We now study how the trace and determinant change as σ varies from 0 to $+\infty$. From (29) and (30) the locus of points $((T_\sigma(\theta, \alpha, s_L, \varepsilon_{\Psi,l}, \nu), D_\sigma(\theta, \alpha, s_L, \varepsilon_{\Psi,l}, \nu)))$, describing the values of trace and determinant as σ takes different values, is defined through the following linear expression, the Δ line:

$$D = -\frac{(1 + \theta\nu)(1 - s_L)(1 - \alpha)\theta}{[(\alpha - 1 + s_L)\theta(1 + \nu) - (1 - s_L)(1 - \alpha)]} + \Delta'(T - 1) \quad (31)$$

$$\text{where } \Delta' = -\frac{(1 + \theta\nu)(1 - s_L)(1 - \alpha)}{[(\alpha - 1 + s_L)\theta(1 + \nu) - (1 - s_L)(1 - \alpha)]}$$

The Δ line is positively sloped when $1 - s_L < \alpha < \alpha_1 \equiv (1 - s_L) \frac{1 + \theta(1 + \nu)}{1 - s_L + \theta(1 + \nu)}$, being negatively sloped otherwise. Also, using (30) it can be seen that the determinant is always decreasing in σ . Noticing that $\Delta' = \frac{\partial D/d\sigma}{\partial T/d\sigma}$, the trace is increasing if $\alpha < \alpha_1$ and decreasing otherwise. However, given that $\sigma \geq 0$, only the part of the Δ line corresponding to non negative values of σ is relevant to our analysis. Using again (30) we see that D decreases from $D_0 = \frac{(1 - \alpha)(1 + \theta\nu)}{\alpha - 1 + s_L}$ to $-\infty$ as σ increases from zero to σ_c , and as σ increase further from σ_c to $+\infty$ the determinant decreases from $+\infty$ to $D_\infty = \frac{(1 - \alpha)(1 + \theta\nu)(1 - s_L)}{(\alpha - 1 + s_L) + \alpha s_L \varepsilon_{\Psi,1}}$. Since there is a discontinuity in the values of D (and T) as σ crosses the value σ_c , we will analyze separately the two cases: the case of $\sigma < \sigma_c$ and the case of $\sigma > \sigma_c$.

Case 1: $\sigma < \sigma_c$ (Figure 3a)

The relevant part of Δ line in this case, describing the locus (T_σ, D_σ) as σ varies from zero to σ_c , is just an half-line Δ with origin in (T_0, D_0) , the values of T and D for $\sigma = 0$, pointing downwards since D is decreasing in σ . Of course, the results on local dynamics and bifurcations will depend on the location of this half-line Δ with respect to the lines AC , BC , and AB , which is characterized by its slope, Δ' , and its initial point (T_0, D_0) . We shall therefore study how the half-line Δ moves in the space (T, D) as α varies from $1 - s_L$ to 1.

The derivative of Δ' with respect to α is always positive, and, as seen before, the denominator of Δ' becomes zero when $\alpha = \alpha_1$. Hence, the slope of the Δ line is positive and increases from $1 + \theta\nu$ to ∞ when α goes from $(1 - s_L)$ to α_1 , and it is negative and increases from $-\infty$ to zero when α goes from α_1 to 1, the line Δ becoming vertical for $\alpha = \alpha_1$. Also, as α changes, the origin of the half line Δ describes a locus of points $\{T_0(\alpha), D_0(\alpha)\}$, lying over a line Δ_0 , whose slope is given by $slope_{\Delta_0} \equiv \frac{D'_0(\alpha)}{T'_0(\alpha)} = 1 + \theta\nu$ that does not depend on α . This locus describes therefore a half-line line Δ_0 , that takes the values $(1 + \frac{\theta(\nu + s_L)}{1 - s_L}, 0)$ for $\alpha = 1$ and $(-\infty, -\infty)$ when α tends to $(1 - s_L)$. See figure 3a. As α decreases from 1 to $(1 - s_L)$, the origin (T_0, D_0) of half-line Δ shifts downwards to the left, along Δ_0 . At the same time the half-line Δ , lying over line $D = 0$, to the right of line AC , for $\alpha = 1$, rotates in the clockwise direction, becomes vertical at α_1 , and ends up lying on line Δ_0 when $\alpha = (1 - s_L)$. It is immediate to see that for high values of α , close to 1, the steady state is always a saddle since the half-line Δ do not cross line AB . As soon as the slope of Δ decreases below -1 or becomes positive, flip bifurcations occur provided the origin of half-line Δ is on the right of AB

line. See Figure 3a. There are, accordingly, two critical values to consider: α_2 the value for which the slope of the half line Δ becomes -1, and α_3 the value of α for which the half line Δ starts on the line AB . It is easy to show that $\alpha_3 < \alpha_2$. Hence for $\alpha_2 < \alpha < 1$ the steady state is a saddle, for $\alpha < \alpha_3$ the steady state is a source, while for $\alpha_3 < \alpha < \alpha_2$ the steady state is a saddle for low values of σ , and then becomes a source through a flip bifurcation for $\sigma = \sigma_F$.

Case 2: $\sigma > \sigma_c$ (Figure 3b)

The relevant part of Δ line in this case, describing the locus (T_σ, D_σ) as σ varies from $+\infty$ to σ_c , is just the half-line Δ with origin in (T_∞, D_∞) , the values of T and D for $\sigma = \infty$, pointing upwards (since D increases as σ decreases), to the right or to the left, depending on whether the slope of Δ is positive or negative (i.e., depending on whether α is lower or higher than α_1). Of course, for a given value of α , both the half-line Δ obtained when σ goes from $+\infty$ to σ_c and the half-line Δ obtained when σ goes from 0 to σ_c lie on the same line Δ , although they refer to different parts of it.

As α changes, the origin of the half line Δ describes a locus $\{T_\infty(\alpha), D_\infty(\alpha)\}$, lying on line Δ_∞ , whose slope is given by $slope_{\Delta_\infty} \equiv \frac{D'_\infty(\alpha)}{T'_\infty(\alpha)} = (1+\theta\nu)\frac{1+\varepsilon_{\psi,L}}{1+\varepsilon_{\psi,L}+\theta(1+\nu)}$, with $1 < slope_{\Delta_\infty} < slope_{\Delta_0}$, that does not depend on α . This locus describes therefore a line, that takes the values $(1 + \frac{\theta(\nu - \varepsilon_{\psi,L})}{1 + \varepsilon_{\psi,L}}, 0)$ for $\alpha = 1$ and $(1 - \frac{1 + \theta\varepsilon_{\psi,L}}{\varepsilon_{\psi,L}}, -\frac{1 + \theta\nu}{\varepsilon_{\psi,L}})$ when α tends to $(1 - s_L)$. See also figure 3b. Note that for $\nu = \varepsilon_{\psi,L}$ (no capital externalities) we have that $slope_{\Delta_\infty} = 1$ and the locus Δ_∞ coincides with the AC line.

Note that the half line Δ crosses the point $P = (1 - \theta, 0)$ for any value of α when $\sigma = 1$. (See Proposition 5). This means that the half line Δ rotates around point P , starting with a slope of zero for $\alpha = 1$. As α decreases from 1 to α_1 , the slope continuously decreases from 0 to $-\infty$, the half line Δ becoming vertical for $\alpha = \alpha_1$. Then, when α decreases from α_1 to $1 - s_L$ the slope decreases from $+\infty$ to $1 + \theta\nu$.

In this case therefore the critical values for α are the following: α_4 , the value of α such that the half line Δ crosses point B , α_2 the value of α for which the slope of the half line Δ becomes -1, α_5 , the value of α such that the half line Δ crosses the intersection of Δ_∞ with the line AB , and α_6 the value of α such that the half line Δ crosses point A . It can be shown that $\alpha_6 < \alpha_5 < \alpha_3 < \alpha_2 < \alpha_4 < 1$. Therefore, using figure 3b one can easily see that for $\alpha_4 < \alpha < 1$, as σ continuously increases from σ_c to $+\infty$, the steady-state is first a saddle, changes to a sink through a flip bifurcation when

$\sigma = \sigma_F$, eventually becoming a saddle through a transcritical bifurcation for $\sigma = \sigma_T$. When $\alpha_2 < \alpha < \alpha_4$ a change in stability conditions through a Hopf bifurcation may appear. Indeed as can be seen in figure 3b the steady state is first a saddle, undergoes a flip bifurcation for $\sigma = \sigma_F$ becoming a source, and then becomes a sink through a Hopf bifurcation for $\sigma = \sigma_H$. Finally for values of σ higher than one it undergoes a transcritical bifurcation for $\sigma = \sigma_T$ becoming a saddle. For $\alpha_5 < \alpha < \alpha_2$ the flip bifurcation disappears, the steady state is first a source, undergoes a hopf bifurcation when $\sigma = \sigma_H$ becoming a sink, and then, we obtain a saddle through a transcritical bifurcation for $\sigma = \sigma_T$. When $\alpha_6 < \alpha < \alpha_5$ the half line Δ crosses the line AB for high values of σ . Therefore the only difference between this case and the previous one is that for high values of σ the steady state changes from a saddle to a source through a flip bifurcation. Finally for $1 - s_L < \alpha < \alpha_6$ the half line Δ crosses the line AB before crossing AC so that the steady state is first a source, undergoes a Hopf bifurcation for $\sigma = \sigma_F$, becoming a sink; the flip bifurcation now occurs first, so that the steady state becomes a saddle; finally for high values of σ a transcritical bifurcation emerges for $\sigma = \sigma_T$ and the steady state eventually becomes a source

Definition 2 *Given the previous results we compute:*

$$\begin{aligned}\alpha_2 &\equiv (1 - s_L) \frac{[2(1+\theta\nu)+\theta]}{[2(1+\theta\nu)+\theta-s_L(2+\theta\nu)]} \\ \alpha_3 &= (1 - s_L) \frac{[2(2+\theta\nu)-s_L(2-\theta)]}{[2(2+\theta\nu)-s_L(2-\theta)]-s_L(2+\theta\nu)} \\ \alpha_4 &= \frac{(1-s_L)[(3-\theta)(1+\theta\nu)+1]+(1-s_L)\theta(1+v)}{(1-s_L)[(3-\theta)(1+\theta\nu)+1]+\theta(1+v)} \\ \alpha_5 &= (1 - s_L) \frac{2(2+\theta\nu)}{2(2+\theta\nu)-s_L[2+\theta\nu-(2-\theta)\varepsilon_{\psi,L}]} \\ \alpha_6 &= (1 - s_L) \frac{2+\theta\nu}{2+\theta\nu-s_L(1-\nu(1-\theta))} \\ \sigma_H &= \frac{(1-s_L)[s_L+\theta\nu(1-\alpha)]}{\theta\nu(1-s_L)(1-\alpha)+s_L\alpha(1+\varepsilon_{\psi,L})} \\ \sigma_F &= \frac{(2+\theta\nu)[(\alpha-1+s_L)-(1-s_L)(1-\alpha)]-(\alpha-1+s_L)s_L(2-\theta)}{(2+\theta\nu)[(\alpha-1+s_L)-(1-s_L)(1-\alpha)]+\alpha s_L\varepsilon_{\psi,L}(2-\theta)} \\ \sigma_T &= \frac{\nu\alpha+(\alpha-1+s_L)}{\alpha(\nu-\varepsilon_{\psi,L})}.\end{aligned}$$

All the results obtained previously on local stability and bifurcation analysis are summarized in the Propositions below.¹² Proposition 6 refers to the case where both externalities exist, while Proposition 7 is relevant when capital externalities are absent, i.e., when $\varepsilon_{\psi,L} = \nu$. Note that, despite the apparent complexity of the dynamical system, the results obtained hold under very general conditions. Indeed, besides the very general assumptions we made regarding technology and externalities (see Assumptions 1 and 2), the only supplementary condition we have to impose is that $\nu < 1/(1 - \theta)$.¹³

Proposition 6 *Both externalities case:* $\nu - \varepsilon_{\psi,L} > 0$; $\varepsilon_{\psi,L} \geq 0$

Considering that Assumptions 1 and 2 hold, and using the critical values in Definition 2, for $\nu < 1/(1 - \theta)$, the following generically holds:

- (i) *if $1 - s_L < \alpha < \alpha_6$, the steady state is a source for $\sigma < \sigma_H$, undergoes a hopf bifurcation for $\sigma = \sigma_H$, becomes a sink for $\sigma_H < \sigma < \sigma_F$, undergoes a flip bifurcation for $\sigma = \sigma_F$, becomes a saddle for $\sigma_F < \sigma < \sigma_T$. A transcritical bifurcation occur at $\sigma = \sigma_T$, and the steady state is a source for $\sigma > \sigma_T$.*
- (ii) *if $\alpha_6 < \alpha < \alpha_5$, the steady state is a source for $\sigma < \sigma_H$, undergoes a hopf bifurcation for $\sigma = \sigma_H$, becomes a sink for $\sigma_H < \sigma < \sigma_T$, undergoes a transcritical bifurcation for $\sigma = \sigma_T$, becomes a saddle for $\sigma_T < \sigma < \sigma_F$. A flip bifurcation occur at $\sigma = \sigma_F$, and the steady state is a source for $\sigma > \sigma_F$.*
- (iii) *if $\alpha_5 < \alpha < \alpha_3$, the steady state is a source for $\sigma < \sigma_H$, undergoes a hopf bifurcation for $\sigma = \sigma_H$, becomes a sink for $\sigma_H < \sigma < \sigma_T$. A transcritical bifurcation occurs at $\sigma = \sigma_T$, and the steady state is a saddle for $\sigma > \sigma_T$.*
- (iv) *if $\alpha_3 < \alpha < \alpha_4$, the steady state is a saddle if $\sigma < \sigma_F$, undergoes a flip bifurcation for $\sigma = \sigma_F$, becomes a source for $\sigma_F < \sigma < \sigma_H$, undergoes a hopf bifurcation for $\sigma = \sigma_H$, becomes a sink for $\sigma_H < \sigma < \sigma_T$. A transcritical bifurcation occurs at $\sigma = \sigma_T$, and the steady state is a saddle for $\sigma > \sigma_T$.*

¹²Note that when we consider the two cases together α_2 becomes irrelevant for the dynamics.

¹³We need this condition to ensure that $\alpha_6 > 1 - s_L$.

- (v) if $\alpha_4 < \alpha < 1$, the steady state is a saddle if $\sigma < \sigma_F$, undergoes a flip bifurcation for $\sigma = \sigma_F$, becomes a sink for $\sigma_F < \sigma < \sigma_T$. A transcritical bifurcation occur at $\sigma = \sigma_T$, and the steady state is a saddle for $\sigma > \sigma_T$.

Note that in the case where we only have labour externalities, i.e., when $\varepsilon_{\psi,L} = \nu$, Proposition 6 becomes simpler, since in this case the Δ_∞ line coincides with the line AC , so that transcritical bifurcations are no longer possible and $\alpha_5 = \alpha_6$. This means that, in this case, we obtain the following results on local stability and bifurcation analysis.

Proposition 7 *Only labour externalities case:* $v - \varepsilon_{\psi,L} = 0$; $\varepsilon_{\psi,L} \geq 0$

Considering that Assumptions 1 and 2 hold, and using the critical values in Definition 2, , for $\nu < 1/(1 - \theta)$, the following generically holds:

- (i) if $1 - s_L < \alpha < \alpha_6$, the steady state is a source for $\sigma < \sigma_H$, undergoes a hopf bifurcation for $\sigma = \sigma_H$, becomes a sink for $\sigma_H < \sigma < \sigma_F$, undergoes a flip bifurcation for $\sigma = \sigma_F$, and becomes a saddle for $\sigma > \sigma_F$.
- (ii) if $\alpha_6 < \alpha < \alpha_3$, the steady state is a source for $\sigma < \sigma_H$, undergoes a hopf bifurcation for $\sigma = \sigma_H$, and becomes a sink for $\sigma > \sigma_H$.
- (iii) $\alpha_3 < \alpha < \alpha_4$, the steady state is a saddle if $\sigma < \sigma_F$, undergoes a flip bifurcation for $\sigma = \sigma_F$, becomes a source for $\sigma_F < \sigma < \sigma_H$, undergoes a hopf bifurcation for $\sigma = \sigma_H$, and becomes a sink for $\sigma > \sigma_H$.
- (iv) if $\alpha_4 < \alpha < 1$, the steady state is a saddle if $\sigma < \sigma_F$, undergoes a flip bifurcation for $\sigma = \sigma_F$, and becomes a sink for $\sigma > \sigma_F$.

We should at this point emphasize that Proposition 7 also holds when $\varepsilon_{\psi,L} = 0$, i.e., even with strictly constant returns to scale. Hence, in our model, the emergence of interesting dynamics (bifurcations and sunspots) does not require the introduction of increasing returns.¹⁴

¹⁴However, as it will be shown below, the consideration of a small amount of increasing returns to scale may render the emergence of certain bifurcations more likely for plausible values of the other parameters.

5 Discussion of the results

In this section we discuss the results obtained with our model on local dynamics and bifurcations comparing them with those obtained in the literature. More precisely we will consider successively and distinctively two issues. First, the likelihood of being in an indeterminate equilibrium in which stochastic fluctuations are likely to emerge (meaning that the equilibrium must be a sink for a wide range of plausible parameter values). Second, the plausibility of obtaining endogenous deterministic and stochastic cycles linked to bifurcations (meaning that the bifurcation values themselves must fall within the range of admissible values in the empirical literature).

To ease the discussion we have summarized the results obtained in the previous section (Propositions 5-7) in figures 4a and 4b where we have represented in the (α, σ) plane the bifurcation values (σ_H , σ_F and σ_T) that divide the plane into different regions in which the steady state is either a sink, a source or a saddle. The situation considered in Proposition 6 is depicted in in Figure 4a, where we have set $\nu = 0.1$ and $\varepsilon_{\psi,L} = 0$, since capital externalities are required for the occurrence of transcritical bifurcations. In figure 4b we plot one of the cases considered in Proposition 7, the case of *strictly constant returns to scale* ($\nu = 0$).¹⁵

5.1 Indeterminacy

We have seen that, when the steady state is a sink, i.e., locally stable, it is locally indeterminate. This means that for any initial value of the capital stock, close enough to the steady state, there are infinitely many bounded deterministic intertemporal equilibria, that eventually converge to the steady state. In this case there are also infinitely many nondegenerate stochastic equilibria driven by self fulfilling expectations (sunspots equilibria), that stay arbitrarily close to the steady state.

Indeterminacy with constant returns to scale. One of the most striking features of figure 4 is that the equilibrium is very likely to be indeterminate even under constant returns to scale. In particular, as shown in figure 4b, when $\nu = 0$ indeterminacy occurs for a wide range of values for α

¹⁵For the other parameters we have used the values considered (quarterly calibration) in Cooley and Prescott (1995), in the first chapter of the reference book "Frontiers of Business Cycle Research" edited by T. Cooley: $\beta = 0.987$, $\delta = 0.012$ and $s_L = 0.6$.

and σ . For example when $\alpha = 1$ the equilibrium is indeterminate as soon as $\sigma > \sigma_F = 0.4$, and for $\alpha = 0.6$ indeterminacy emerges for $\sigma > \sigma_H = 0.67$.

The fact that we obtain easily indeterminacy even with constant returns to scale stands in sharp contrast with closely related models in the literature. In the original version of Grandmont et al. (1998), it is shown that indeterminacy can occur under constant returns to scale only for very low elasticities of factor substitution. In a similar framework of financed constrained households but without unions and without redistribution schemes, Cazzavillan et al. (1998) show that the likelihood of indeterminacy increases with the introduction of increasing returns to scale. However, in a calibrated version of this model, Barinci and Cheron (2000) show that with a standard Cobb-Douglas production function ($\sigma = 1$), 30% at least of IRS are required when the other parameters are calibrated according to US data at a quarterly frequency. Such strong levels of increasing returns to scale are now considered as being implausibly high given the available evidence in the empirical literature (see among others Burnside et al., 1995, and Basu and Fernald, 1995, 1997).

Hence, it might be thought in the light of these results that unions are the most obvious cause of indeterminacy. Note however in figures 4a and 4b that the region for which the equilibrium is a sink shrinks when union power increases. At very low values of α , figure 4b actually shows that for σ slightly above 1, the equilibrium becomes a saddle (so that the possibility of having sunspots driven fluctuations in the neighborhood of the steady state disappear). In Lloyd-Braga and Modesto (2003), different levels of union bargaining power are also considered, but in a framework where the reservation wage stems from the fact that unemployed workers engage in home production (so that the real amount of resources b doesn't have to be financed by taxes). It is shown that indeterminacy may occur for low values of increasing returns but only for implausibly high levels of factor substitution. It should be mentioned at this point that although the presence of unions is not a crucial factor for indeterminacy, it will turn out to be of considerable importance for the issue of having or not deterministic/stochastic irregular cycles, as we shall see below.

Indeterminacy with low tax rates. It is clear from the discussion above that taxes (combined with liquidity constraints) are rather the key ingredient giving rise to the appearance of an indeterminate equilibrium. In that respect, our model fits in the line of research that explores the role

of different balanced-budget policy rules on the stability properties of the equilibrium, see for instance Uribe and Schmitt-Grohé (1997)¹⁶. From this literature, it is known that countercyclical taxes/government expenditures are likely to favor instability. In our model this feature appears in a most natural way as government transfers, which ensure to all unemployed workers a guaranteed minimal amount of resources, obviously increase when the unemployment rate is higher. It is thus the combination of countercyclical taxes hitting workers already subject to liquidity constraints which explains why in our model the equilibrium can be indeterminate even for values of the average tax rate that can be much lower than those usually required in this literature. Indeed, using equations (9), (14) and (20), we can compute the following expression for the average tax rate at the steady state, $\frac{\tau}{w} = \frac{[f(x_t) - f'(x_t)x_t]}{[f(x_t) - \alpha f'(x_t)x_t]}(1 - l)$, that shows that it is always below the steady state unemployment rate. This contrasts for example with the reference case of Uribe and Schmitt-Grohé (1997) - Cobb-Douglas production function and wage income taxes only - where the tax rate must be higher than the capital share of output.¹⁷

5.2 Bifurcations and irregular cycles

We have seen that, from a theoretical point of view, whenever a bifurcation occurs, both deterministic and stochastic irregular cycles close to the steady state may appear. For such situations to be considered seriously as a possible explanation of actual business cycles, however, the relevant issue is not only whether bifurcations are possible but mostly if they occur for empirically plausible values of the parameters.

In that respect another noticeable feature of our model is that all three kinds of bifurcations (Hopf, flip and transcritical) appear for values of the elasticity of substitution between factors of production, σ , that are relatively close to one. In fact figure 4a (where $\varepsilon_{\psi,L} = 0$ and $\nu = 0.1$) shows that in the limit case where α tends to $1 - s_L$ all these bifurcations emerge for σ arbitrarily close to one. This is true even if we consider an arbitrarily small amount of externalities $\nu > 0$, provided $\varepsilon_{\psi,L} > 0$.¹⁸ The same happens when

¹⁶See also Guo and Lansing (1998) and Aloi, Jacobsen and Lloyd-Braga (2003).

¹⁷In our model with $\sigma = 1$, a steady state unemployment rate of 10%, and a labor share of $s_L = 0.6$, the average tax rate is 10% for $\alpha = 1$ and 5% for $\alpha = 0.5$

¹⁸It is easy to show from Definition 2 that for any value of $\nu = \varepsilon_{\psi,K} > 0$ (capital

$v = 0$, i.e., in the constant returns to scale case, but only for σ_H and σ_F , since, as we have seen, when $\varepsilon_{\psi,K} = 0$ we have that $\sigma_T = \infty$. See figure 4b. If $\varepsilon_{\psi,L} > 0$ the limit values of σ_H , σ_F and σ_T no longer tend to one when α tends to $1 - s_L$ but remain close to one if v is small. See Definition 2.

This implies that in our model, although the presence of unions is not a crucial factor for indeterminacy, union power plays a role in rendering the emergence of bifurcations more likely. In fact when union power is sufficiently high, bifurcations, and therefore endogenous deterministic and stochastic irregular cycles near the steady state, occur for empirically plausible values of the elasticity of substitution in production and for (arbitrarily) small increasing returns to scale. For example with $v = 0.1$ and $s_L = 0.6$ (the case depicted in figure 4a) for $\alpha = 0.5$ (the value usually considered in the literature) we have $\sigma_H = 0.8$, $\sigma_F = 1.59$ and $\sigma_T = 3$. On the contrary, when $\alpha = 1$ (the competitive labour market case) there are no Hopf bifurcations, and $\sigma_F = 0.4$ and $\sigma_T = 7$ so that no bifurcations, and therefore no irregular cycles, can emerge for values of σ that are (empirically) plausible.

Remark that, in previous related works in the literature, bifurcations although possible were, in a certain way, a mere theoretical phenomenon, since they only emerged for empirically implausible configurations of parameters. Indeed in Grandmont et al. (1998) in a standard Woodford (1986) model, for σ above 0.2 the steady state is always a saddle point and no local endogenous, deterministic or stochastic, fluctuations can arise with self-fulfilling expectations. Also in Cazzavillan et al. (1998), with a Woodford (1986) model with increasing returns to scale, bifurcations and endogenous fluctuations are possible for higher values of σ , but they need high values for v . In Lloyd-Braga and Modesto (2003), although the introduction of unions increases the likelihood of obtaining a Hopf bifurcation, endogenous irregular cycles are not possible with constant to scale. In our model, on the contrary, Hopf and flip bifurcations emerge for values of σ close to one provided unions are strong enough, even in the constant returns to scale case, and the same is true for transcritical bifurcations if we consider an arbitrarily small amount of externalities $v > 0$. Therefore we conclude that unions and countercyclical taxation are the features responsible for the emergence of endogenous deterministic and stochastic irregular cycles for empirically relevant values of the parameters.

externalities only), when α tends to $(1 - s_L)$, σ_H , σ_F and σ_T tend to one with always the same ordering $\sigma_T > \sigma_F > 1 > \sigma_H$.

6 Concluding Remarks

In this paper we have introduced union power, imperfectly uninsured unemployment and countercyclical taxation in the finance constrained economy proposed by Woodford (1986). Our results show that the introduction of these features changes dramatically the local dynamics of the model. We find that indeterminacy and bifurcations, and therefore endogenous deterministic or stochastic fluctuations driven by self-fulfilling expectations, emerge for empirically plausible values of the parameters. In fact indeterminacy occurs, even with constant returns to scale, for a wide range of values for the elasticity of substitution and union power containing most empirically plausible values. In particular, in the Cobb-Douglas case the steady state is always indeterminate. Also, when unions are strong, flip, Hopf and transcritical bifurcations occur for values of the elasticity of substitution close to one, and for an arbitrarily small amount of increasing returns to scale. This implies that, in contrast to most related models in the literature, sunspot driven fluctuations and periodic or irregular cycles around invariant closed orbits can occur, even for values of the elasticity of substitution in the neighborhood of one and an arbitrarily small amount of externalities. Moreover, since we do not assume a perfect insurance mechanism against unemployment, in our model, contrary to what happens in the related literature unemployment fluctuations are both involuntary and costly.

We also conclude that, while union power is crucial for bifurcations to emerge under empirically relevant configurations of the parameters, countercyclical taxation is the feature responsible for the appearance of indeterminacy in our framework. These results are in line with the findings of Schmitt-Grohé and Uribe (1997) and Guo and Lansing (1998) for real business cycle models. However, since we have only considered a specific form of taxation, linked to unemployment insurance transfers, it remains to be seen how the consideration of alternative budget-balanced policy rules changes the results here obtained. Nevertheless this is beyond the scope of the present paper and constitutes a topic for further research.

A Appendix

A.1 Binding cash-in-advance constraints in the Woodford (1986) model

The proof follows as a direct application of the general setting proposed in Dufourt et al. (2004). Consider the situation of a worker holding at date 0 a given (predetermined) amount of money m_0 . The problem of this worker may be written as

$$\text{Max}_{\{c_t, m_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \gamma^t u(c_t)$$

subject to

$$m_{t+1} = m_t + y_t^i - p_t c_t^i \quad (32)$$

$$m_t + y_t^i - p_t c_t^i \geq 0 \quad (33)$$

with m_0 given, and $y_t^j \in \{w_{t-1} - p_t \tau_t, p_t b\}$ for $i \in \{e, u\}$. This program can be set up in the following recursive form:

$$V(m_t, y_t^i) = \text{Max}_{0 \leq c_t \leq (m_t + y_t^i)/p_t} u(c_t^i) + \gamma [h_t V(m_{t+1}, y_{t+1}^e) + (1 - h_t) V(m_{t+1}, y_{t+1}^u)]$$

subject to $m_{t+1} = m_t + y_t^i - p_t c_t^i$, $i \in \{e, u\}$, or writing constraints explicitly:

$$\begin{aligned} V(m_t, y_t^i) = & \text{Max}_{\{c_t, m_{t+1}\}} u(c_t^i) + \lambda_t^i (m_t + y_t^i - p_t c_t^i - m_{t+1}) \\ & + \mu_t^i (m_t + y_t^i - p_t c_t^i) + \gamma [h_t V(m_{t+1}, y_{t+1}^e) + (1 - h_t) V(m_{t+1}, y_{t+1}^u)] \end{aligned}$$

The first order conditions are given by:

$$u'(c_t^i) = p_t (\lambda_t^i + \mu_t^i) \quad (34)$$

$$\lambda_t^i = \gamma [h_t (\lambda_{t+1}^e + \mu_{t+1}^e) + (1 - h_t) (\lambda_{t+1}^u + \mu_{t+1}^u)] \quad (35)$$

We are looking for the conditions under which a consumer will choose to consume all its available (after tax) monetary resources $m_t + y_t^i$ under

all possible states (employed or unemployed) so that constraint (33) is always binding. This means that we are looking for the sequences of revenues and probability distributions over employment and unemployment that are consistent with $\mu_t^i > 0$ and $c_t^i = (m_t + y_t^j)/p_t$ for all $t = 0..\infty$ and all $i \in \{e, u\}$

Substituting $\mu_t^i > 0$ for all $t = 0..\infty$ and all $i \in \{e, u\}$ in (34)-(35) yields

$$\frac{u'(c_t^i)}{p_t} - \mu_t^i = \gamma \left[h_t \frac{u'(c_{t+1}^e)}{p_{t+1}} + (1 - h_t) \frac{u'(c_{t+1}^u)}{p_{t+1}} \right]$$

or equivalently

$$\frac{u'(c_t^i)}{p_t} > \gamma \left[h_t \frac{u'(c_{t+1}^e)}{p_{t+1}} + (1 - h_t) \frac{u'(c_{t+1}^u)}{p_{t+1}} \right] \quad (36)$$

for all $i \in \{e, u\}$, and where, for all $t = 0..\infty$, we have $c_t^e = (w_{t-1} - p_t \tau_t)/p_t$ and $c_t^u = b$.

Condition (36) is in particular verified at the steady state with $\pi = 1$ (and therefore in the vicinity of this steady state) if

$$u'(w - \tau) > \gamma [h u'(w - \tau) + (1 - h) u'(b)]$$

and

$$u'(b) > \frac{\gamma}{\pi} [h u'(w - \tau) + (1 - h) u'(b)]$$

As long as $w - \tau > b$ (a condition that is implied by the wage bargaining process), only the first condition is actually required.

A.2 Obtaining the dynamic system

At the symmetric equilibrium we have $k_t = \bar{k}_t$ and $l_t = \bar{l}_t$. Also, equilibrium in the capital services market requires that $k_{t+1} = k_{t+1}^c$. Therefore, recalling the definition of R_t and using (6) and (16), we obtain the first equilibrium condition in the main text: eq. (17).

Using (9) in equilibrium (13) becomes:

$$\frac{b}{\bar{l}_t} = \frac{p_t}{p_{t+1}} \Psi(k_t, l_t) A \left[f(x_t) - f'(x_t) x_t \right] \quad (13')$$

Money market clearing requires that money demand for all t must equal $M > 0$, the constant quantity of outside money in the economy. We have

seen that money demand of workers is zero. See (4). However, firms need to hold money at the beginning of period $t + 1$ in order to pay wages to employed workers in t . Therefore equilibrium in the money market requires that $m_{t+1} = l_t w_t = M$. Hence, by (14) the following relation must be satisfied at equilibrium in every period t :

$$\frac{M}{p_t} = l_t \Psi(k_t, l_t) A \left[f(x_t) - \alpha f'(x_t) x_t \right]. \quad (37)$$

Dividing (37) for period t by (37) for period $t + 1$, using (13') to substitute p_{t+1}/p_t and rearranging terms, we finally obtain the second equilibrium condition: eq (18). Note that the output market also clears by Walras law.

A.3 The elements of matrix J

Using (17), (18), (26) and, remembering from Assumptions 1 and 2 the definition of σ and that $\varepsilon_{\Psi, k} = v - \varepsilon_{\Psi, l}$, we obtain the following expressions for the elements of J :

$$\begin{aligned} J_{kk} &= 1 + \theta \left[v - \varepsilon_{\Psi, l} - \frac{(\alpha - 1 + s_L)}{\alpha \sigma} \right] \\ J_{kl} &= \theta \left[\varepsilon_{\Psi, l} + \frac{(\alpha - 1 + s_L)}{\alpha \sigma} \right] \frac{k}{l} \\ J_{lk} &= \frac{l}{k} \left\{ \frac{(1 - s_L)(1 - \alpha)(\sigma - 1)}{(\alpha - 1 + s_L)(\sigma - 1 + s_L) + \alpha \sigma s_L \varepsilon_{\Psi, l}} \right. \\ &\quad \left. - \frac{(1 - s_L)[(1 - \alpha)(\sigma - 1) + s_L] + \alpha \sigma s_L (v - \varepsilon_{\Psi, l})}{(\alpha - 1 + s_L)(\sigma - 1 + s_L) + \alpha \sigma s_L \varepsilon_{\Psi, l}} J_{kk} \right\} \\ J_{ll} &= - \frac{(1 - s_L)(1 - \alpha)(\sigma - 1)}{(\alpha - 1 + s_L)(\sigma - 1 + s_L) + \alpha \sigma s_L \varepsilon_{\Psi, l}} \\ &\quad - \frac{l}{k} \frac{(1 - s_L)[(1 - \alpha)(\sigma - 1) + s_L] + \alpha \sigma s_L (v - \varepsilon_{\Psi, l})}{(\alpha - 1 + s_L)(\sigma - 1 + s_L) + \alpha \sigma s_L \varepsilon_{\Psi, l}} J_{kl} \end{aligned}$$

where $0 < \theta \equiv 1 - \beta(1 - \delta) < 1$.

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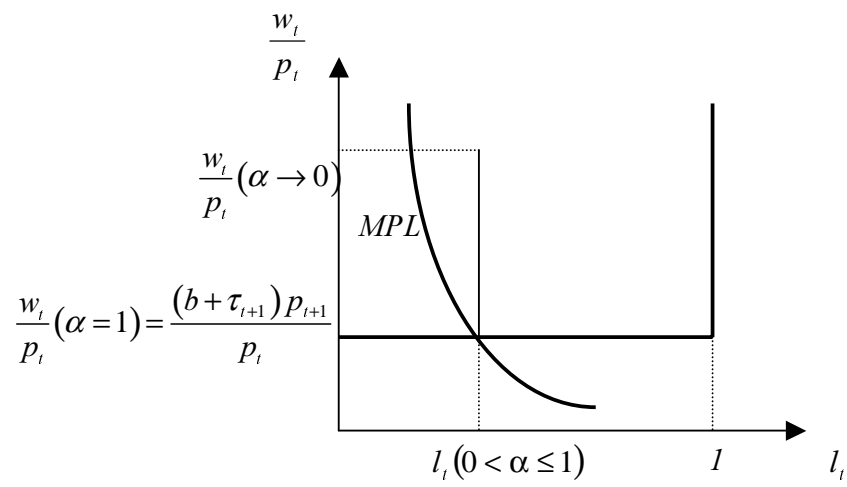


Figure 1: Labor market at temporary equilibrium

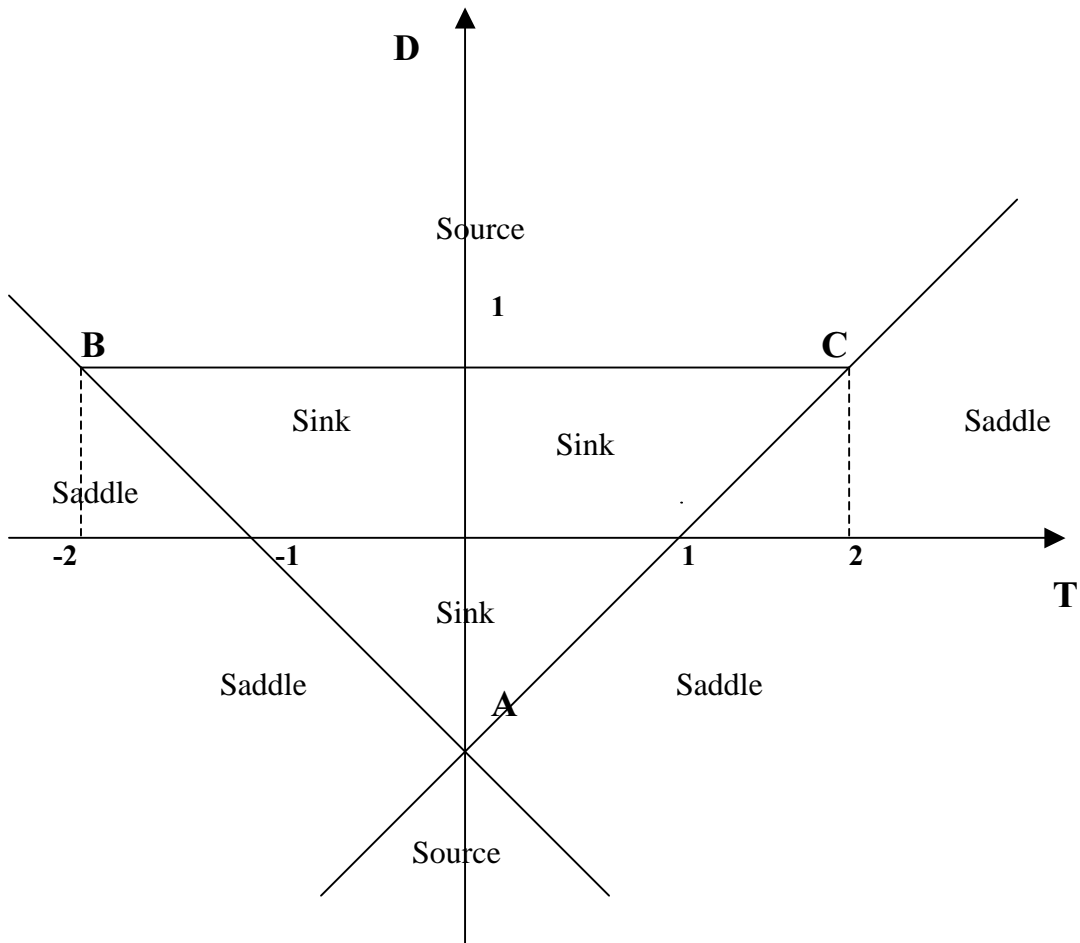


Figure 2

