Near-Rational Exuberance

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7 June 2004
Preliminary and Incomplete

Abstract

We study how the use of judgement or "add-factors" in macroeconomic forecasting may disturb the set of equilibrium outcomes when agents learn using recursive methods. We isolate conditions under which new phenomena, which we call exuberance equilibria, may exist in standard macroeconomic environments. These equilibria may display sunspot-like behavior, but without a requirement that the underlying rational expectations equilibrium is indeterminate. We suggest ways in which policymakers might avoid unintended outcomes by adjusting policy to minimize the risk of exuberance equilibria.

[*]Any views expressed are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of St. Louis or the Federal Reserve System.

1 Introduction

1.1 Judgement variables in forecasting

Judgement is a fact of life in macroeconomic forecasting. It is widely understood that even the most sophisticated econometric forecasts are adjusted before presentation. This adjustment is so pervasive that it is widely known as the use of "add-factors"—subjective changes to the forecast which depend on the forecaster's assessment of special factors that are not well summarized by the variables that are included in the econometric model. This is so much the case that recently, some authors have argued that economists need to take explicit account of the effects of judgement on the behavior of macroeconomic systems. One example where judgement explicitly enters the analysis in a monetary policy context is Svensson (2003). A forthright discussion of how prominently judgement enters into actual macroeconomic forecasting is contained in Reifschneider, Stockdon, and Wilcox (1997). As they state, "... [econometric] models are rarely, if ever, used at the Federal Reserve without at least the *potential* for intervention based on judgement. Instead, [the approach at the Federal Reserve] involves a mix of strictly algorithmic methods ("science") and judgement guided by information not available to the model ("art") (p. 2, italics in original).

We wish to think of the news or add-factor that modifies the forecast as a qualitative, unique, commonly understood economy-wide variable: In sum, a "judgement variable." A good example of a judgemental adjustment is suggested by Reifschneider, Stockdon, and Wilcox (1997), when they discuss the "financial headwinds" that were thought to be inhibiting U.S. economic growth in the early to mid-1990s. As they discuss, the "headwinds" add-factor was used to adjust forecasts over a period of many quarters. It was communicated to the public prominently in speeches by Fed Chairman Alan Greenspan. It was thus widely understood throughout the economy and was highly serially correlated. This is the type of variable we have in mind, although by no means would we wish to restrict attention to this particular example. We think add-factoring is occurring continuously.

We take it for granted that it is a matter of conventional wisdom among economists that judgement is all to the good in macroeconomic forecasting. Models are, of course, crude approximations of reality and must be supplemented with other information not contained in the model.

1.2 Feedback from judgement

Our focus in this paper is on how the add-factor or judgemental adjustment of forecasts may create more problems than it solves. In particular, we show how such a practice can lead to the possibility of self-fulfilling fluctuations. To be clear, we examine the extreme case where the judgement variable is not intrinsically related to economic fundamentals at all. Thus our results come from a situation where the forecasting judgement being added is, fundamentally speaking, not useful in forecasting the variables of interest.

We study systems with well-defined rational expectations equilibria. We replace rational expectations with adaptive learning using the methodology of Evans and Honkapohja (2001). We then investigate the equilibrium dynamics of the system if the econometric models of the agents are supplemented with judgement. To define an exuberance equilibrium, we first require that the perceived evolution of the economy corresponds to the actual evolution by imposing a consistent expectations equilibrium concept on the model as developed by Hommes and Sorger (1998). Under this requirement, the autocovariance generating functions of the perceived and actual laws of motion correspond exactly. Secondly, we require individual rationality in individual agents' choice to include the judgement variable in their forecasting model, given that all other agents are using the judgement variable and hence causing it to influence the actual dynamics of the macroeconomy. Finally, we require learnability or expectational stability. When all three of these requirements are met, we say that an exuberance equilibrium exists. In our exuberance equilibria, all agents would be better off if the judgement variable were not being used, but as it is being used, no agent wishes to discontinue its use. We view this as a Nash equilibrium in beliefs.

1.3 Near Rationality

Though the use of judgement has been justified as a Nash equilibrium, our equilibrium does not correspond exactly to a rational expectations equilibrium. This is because the judgement variable is assumed to be unavailable in the statistical part of the forecasting. We think of this as reflecting the separation of the econometric forecasting unit from the actual decision makers. The latter have the econometric forecast as an input to which they are free to add the judgement variable.

In other words, we are assuming that the judgement variable is not one that can be converted by the decision makers into a statistical time series, which can formally be utilized in an econometric forecasting model. In a similar vein the decision makers face a dichotomy in their use of judgement: they either incorporate the variable as an add-factor or they ignore it and directly use the econometric forecast. This inability of the decision makers to transmit to the econometric forecasters in a quantitative way the judgement aspects behind their final economic decisions is the source of the deviation from full "rational expectations" and the reason for our use of the term "near rationality."

1.4 Main findings

We isolate conditions under which exuberance equilibria exist in widely studied dynamic frameworks which depend on expectations. Among the frameworks we study is the canonical New Keynesian model of Woodford (2003) and Clarida, Gali, and Gertler (1999).

We also study the relationship between indeterminacy and exuberance. We show that exuberance is a clear possibility even in the case where the underlying rational expectations equilibrium is determinate. Thus an interesting and novel finding is the possibility of "sunspot-like" equilibria, but without requiring that the underlying rational expectations equilibrium of the model is indeterminate. In a sense, we find "sunspot-like" equilibria without indeterminacy. This may lead one to view the possibility of exuberance equilibria as particularly worrisome, as exuberance equilibria may exist even in otherwise benign circumstances.

In the policy-oriented New Keynesian application, our findings suggest a new danger for policy makers: Choosing policy to induce determinacy and learnability may not be enough, because the policy maker must also avoid the prospect of exuberance equilibria. We show how policy may be designed to avoid this danger.

2 Judgement and consistent expectations equilibria

2.1 A multivariate linear model

Our results depend on the idea that agents participating in macroeconomic systems are learning using recursive algorithms, and that the systems under learning eventually converge. In many cases, as discussed in Evans and Honkapohja (2001), this convergence would be to a rational expectations equilibrium. The crucial aspect for the present paper is that once agents have their macroeconometric forecast from their regression model, the forecast is then judgementally adjusted.

To fix ideas, consider an economy which may be described by

$$y_t = \beta y^e_{t+1} + u_t \tag{1}$$

where y_t is a vector of the economy's state variables, β is a conformable matrix of parameters, and u_t is a vector of stochastic noise terms. For convenience we have dropped any constants in this equation. The term y^e_{t+1} represents the possibly non-rational expectations of the private sector agents; since agents are initially learning, these expectations may initially be non-rational. The judgement vector in the economy follows

$$(I - \rho L)\, \xi_t = \eta_t \tag{2}$$

where I is a conformable identity matrix, ρ is a conformable matrix with roots inside the unit circle, L is a lag operator, ξ_t is a vector of judgement variables, and η_t is a vector of stochastic noise terms. We assume that u_t and η_t evolve independently, so that the judgement variables have no fundamental effect on the economy described by equation (1). This is obviously an extreme assumption but it is also the one that we think is the most interesting for the purpose of illustrating our main points, as it is the starkest case. It would be interesting to allow for some correlation between u_t and η_t in future research.

The hallmark of the recursive learning literature is the assignment of a *perceived law of motion* to the agents, so that we can view them as using recursive algorithms to update their forecasts of the future based on actual data produced by the system in which they operate.[1] A key aspect of this assignment is to keep the perceived law of motion consistent with the actual law of motion of the system, which will be generated by the interaction of equation (1) with the agents expectations formation process. With judgement in the model, it will be apparent below that the ARMA perceived law of motion

$$y_t = by_{t-1} + v_t - av_{t-1}, \tag{3}$$

can be consistent with the actual law of motion. Here b and a are conformable matrices and v_t is a vector of stochastic noise terms. We can write this as

$$y_t = \theta(L) v_t, \tag{4}$$

where

$$\theta(L) = (I - bL)^{-1} (I - aL). \tag{5}$$

Agents must form expectations using their perceived law of motion, which implies

$$\begin{aligned} E_t^{\star} y_{t+1} &= by_t - av_t & (6) \\ &= [b\theta(L) - a] v_t. & (7) \end{aligned}$$

In what follows, we will sometimes call (6) the *macroeconometric forecast*. It is based on the econometric model, the perceived law of motion, alone,

[1] We can think of this as corresponding to the existence of a forecasting community using econometric-based models to guide the expectations of private sector and governmental agents. Forecasting communities like this exist in all industrialized nations.

and is the traditional description of the expectations formation process in the learning literature.

The novel feature of this paper is that we allow judgement to be added to the macroeconometric forecast:

$$y_{t+1}^{e} = E_t^{\star} y_{t+1} + \xi_t. \tag{8}$$

Our goal is to understand the implications of this add-factor judgement on the nature of equilibrium in the economy, and on the convergence of the learning algorithm to equilibrium. We stress that if the judgement vector is null, the model corresponds to a version of systems analyzed extensively in Evans and Honkapohja (2001), and that the conditions for convergence to rational expectations equilibrium in that case are well-established.

2.2 The actual law of motion

Since expectations in the economy are being formed via equation (8), and since these expectations affect the evolution of the economy's state vector through equation (1), we deduce an actual law of motion for this system as

$$\begin{aligned} y_t &= \beta \left[b\theta (L) - a \right] v_t + \beta (I - \rho L)^{-1} \eta_t + u_t \\ &= \beta \left[b\theta (L) - a \right] \theta (L)^{-1} y_t + \beta (I - \rho L)^{-1} \eta_t + u_t \\ &= \beta \left[b - a\theta (L)^{-1} \right] y_t + \beta (I - \rho L)^{-1} \eta_t + u_t. \end{aligned} \tag{9}$$

We can then write

$$\left\{ I - \beta \left[b - a\theta (L)^{-1} \right] \right\} y_t = \beta (I - \rho L)^{-1} \eta_t + u_t. \tag{10}$$

If we define $M(L) \equiv I - \beta \left[b - a\theta (L)^{-1} \right]$, it follows that

$$y_t = M(L)^{-1} \beta (I - \rho L)^{-1} \eta_t + M(L)^{-1} u_t. \tag{11}$$

Judgement naturally influences the evolution of the state vector because it influences the views of economic actors concerning the future. The critical question is then whether there are conditions under which the agents would continue to use the add-factored forecast (8) when the economy is evolving according to equation (11). That is, could the agents come to perceive that the judgement vector is in fact useful in forecasting the state vector, even though by construction there is no fundamental relationship? We now turn to this question.

2.3 Consistent expectations

To analyze the effects of judgemental adjustment, we employ the concept of *consistent expectations equilibrium* as defined by Hommes and Sorger (1998). Their core idea is that the agents should see no difference between their own perceptions of how the economy evolves and the actual data from the economy. One way to develop conditions under which such an outcome may occur is to require that the autocovariance generating function of the perceived law of motion corresponds exactly to the autocovariance generating function of the actual law of motion. For the perceived law of motion, or PLM, equation (4), we can write the autocovariance generating function as[2]

$$G_{PLM}(z) = \theta(z)\,\Sigma_v \theta\left(z^{-1}\right)^T, \tag{12}$$

where Σ_v is the variance-covariance matrix associated with v,

$$\begin{aligned} \theta(z) &= (I - bz)^{-1}(I - az), & (13) \\ \theta\left(z^{-1}\right) &= \left(I - bz^{-1}\right)^{-1}\left(I - az^{-1}\right), & (14) \end{aligned}$$

z is a complex scalar, and T indicates transpose. For the actual law of motion, or ALM, the autocovariance generating function is the sum of two such functions

$$G_{ALM}(z) = G_\eta(z) + G_u(z) \tag{15}$$

by the independence of η and u. Defining $N(L) \equiv M(L)^{-1}\beta(I - \rho L)^{-1}$ as the term that modifies η_t in equation (11), it follows that

$$\begin{aligned} G_\eta(z) &= N(z)\,\Sigma_\eta N\left(z^{-1}\right)^T & (16) \\ G_u(z) &= M(z)^{-1}\,\Sigma_u\left[M\left(z^{-1}\right)^{-1}\right]^T & (17) \end{aligned}$$

[2]See Brockwell and Davis (1991, pp. 417-420), or Hamilton (1994, pp. 266-268).

where Σ_η and Σ_u are the variance-covariance matrices associated with η and u, $M(z) \equiv I - \beta\left[b - a\theta(z)^{-1}\right]$, and $N(z) \equiv M(z)^{-1}\beta(I-\rho z)^{-1}$. A consistent expectations equilibrium is then characterized by $G_{PLM}(z) = G_{ALM}(z)$, or

$$\theta(z)\Sigma_v\theta\left(z^{-1}\right)^T = N(z)\Sigma_\eta N\left(z^{-1}\right)^T + M(z)^{-1}\Sigma_u\left[M\left(z^{-1}\right)^{-1}\right]^T. \quad (18)$$

This condition can be rewritten as

$$\theta(z)\Sigma_v\theta\left(z^{-1}\right)^T = \\ M(z)^{-1}\left[\beta(I-\rho z)^{-1}\Sigma_\eta\left(\left(I-\rho z^{-1}\right)^{-1}\right)^T\beta^T + \Sigma_u\right]\left(M\left(z^{-1}\right)^{-1}\right)^T. \quad (19)$$

Thus we can write

$$M(z)\theta(z)\Sigma_v\theta\left(z^{-1}\right)^T M\left(z^{-1}\right)^T = \\ \beta(I-\rho z)^{-1}\Sigma_\eta\left(\left(I-\rho z^{-1}\right)^{-1}\right)^T\beta^T + \Sigma_u, \quad (20)$$

or

$$\left[(I-\beta b)\theta(z) - a\right]\Sigma_v\left[\theta\left(z^{-1}\right)^T(I-\beta b)^T - a^T\right] = \\ \beta(I-\rho z)^{-1}\Sigma_\eta\left(\left(I-\rho z^{-1}\right)^{-1}\right)^T\beta^T + \Sigma_u. \quad (21)$$

We will use this condition extensively in the remainder of the paper, in order to isolate conditions under which a consistent expectations equilibrium exists.[3]

The formulation just presented defines an exact notion of a consistent expectations equilibrium. In the applications we will also make use of the notion of an *approximate* consistent expectations equilibrium. In the latter concept agents use a PLM that is an approximation to but not fully consistent with the ALM in the sense that they do not exactly match all of their autocovariances. However, for an approximate consistent expectations equilibrium we do require that the equilibrium PLM is the best choice within the

[3]We have verified existence of a solution to (21) in some numerical examples.

restricted class considered. For example, if the agents use a VAR(1) forecasting model, the parameters used must provide the best linear projections of the variables being forecast.

2.4 Incentives to include judgement

When all agents in the model are making use of the judgementally adjusted forecast described in equation (8), they induce an actual law of motion for the system which is described by equation (11). An individual agent may nevertheless decide that it is possible to make more efficient forecasts by simply ignoring the judgemental adjustment. If this is possible, then it is not individually rational for all agents to use the add-factored forecast. We check this individual forecast efficiency condition by comparing the variance of the forecast error for the judgemental forecast (8) to the variance of the forecast error with judgement not included, the macroeconometric forecast (6), under the condition that all other agents are using the judgementally adjusted forecast and thus are inducing the actual law of motion (11). This can be written as follows. Given that

$$y_{t+1} = M(L)^{-1} \beta (I - \rho L)^{-1} \eta_{t+1} + M(L)^{-1} u_{t+1}, \tag{22}$$

under what conditions does

$$Var\left[y^e_{t+1} - y_{t+1}\right] < Var\left[E^\star_t y_{t+1} - y_{t+1}\right] \tag{23}$$

hold? We will show below that this condition can be met under admissible conditions on model parameters.

2.5 Learnability

Since we have made an assumption that the agents in the model are learning using recursive algorithms, we also need to impose learnability of any proposed equilibrium as a condition for plausibility.

2.6 Exuberance equilibrium

An exuberance equilibrium can now be defined. Given the actual law of motion (11), a exuberance equilibrium exists if (1) a consistent expectations equilibrium exists, (2) individual agents rationally decide to include the judgement vector in their forecasts by checking that condition (23) is met, and (3) the consistent expectations equilibrium is learnable in the sense of Evans and Honkapohja (2001).

3 Exuberance equilibria in the scalar model

3.1 The scalar model

Are their any conditions under which an exuberance equilibrium could exist? There are, and we argue that the conditions are in fact worrisomely plausible. In order to obtain some intuition, we turn first to the analysis of the scalar case.

We maintain the same notation as above, but we let y_t, ξ_t, u_t, η_t, and v_t be scalar-valued, and we proceed with the understanding that β, ρ, a, and b are scalars, and that $I = 1$ in the scalar case. The macroeconometric forecast can then be written as

$$\begin{aligned} E_t^{\star} y_{t+1} &= by_t - av_t \\ &= b\theta(L) v_t - av_t \\ &= \left(b\frac{1-aL}{1-bL} - a \right) v_t \\ &= \left(\frac{b-a}{1-bL} \right) v_t. \end{aligned} \tag{24}$$

The add-factored forecast is

$$y_{t+1}^{e} = E_t^{\star} y_{t+1} + \xi_t. \tag{25}$$

This induces an actual law of motion for the economy

$$\begin{aligned} y_t &= \beta y^e_{t+1} + u_t \\ &= \beta \left[E^\star_t y_{t+1} + \xi_t \right] + u_t \\ &= \beta \left(\frac{b-a}{1-bL} \right) v_t + \frac{\beta}{1-\rho L} \eta_t + u_t \\ &= \beta \left(\frac{b-a}{1-bL} \right) \left(\frac{1-bL}{1-aL} \right) y_t + \frac{\beta}{1-\rho L} \eta_t + u_t. \end{aligned} \tag{26}$$

Solving for y_t implies that the actual law of motion is

$$y_t = \frac{1-aL}{\beta(a-b)+1-aL} \left(\frac{\beta}{1-\rho L} \eta_t + u_t \right). \tag{27}$$

3.2 Conditions for consistent expectations

To apply the idea of consistent expectations equilibrium, we require that the autocovariance generating functions of the perceived and actual laws of motion are the same. The autocovariance generating function for the perceived law of motion in the scalar case is given by

$$G_{PLM}(z) = \sigma_v^2 \frac{(1-az)(1-az^{-1})}{(1-bz)(1-bz^{-1})}. \tag{28}$$

For a consistent expectations equilibrium we require that $G_{ALM}(z) = G_{PLM}(z)$. We note that this implies an ALM that takes an ARMA(1,1) form with identical coefficients to those of the PLM.[4]

For the actual law of motion,

$$G_{ALM}(z) = G_\eta(z) + G_u(z) \tag{29}$$

where

$$G_\eta(z) = \frac{\sigma_\eta^2 \beta^2 (1-az)(1-az^{-1})}{\left[\beta(a-b)+1-az\right]\left[\beta(a-b)+1-az^{-1}\right](1-\rho z)(1-\rho z^{-1})}, \tag{30}$$

[4]See Brockwell and Davis (1991, p. 90, Remark).

and

$$G_u(z) = \frac{\sigma_u^2 (1 - az)(1 - az^{-1})}{[\beta (a - b) + 1 - az][\beta (a - b) + 1 - az^{-1}]}. \tag{31}$$

The sum of these two functions is therefore

$$G_{ALM}(z) = \frac{(1 - az)(1 - az^{-1})}{(1 - \rho z)(1 - \rho z^{-1})} \times \left\{ \frac{\beta^2 \sigma_\eta^2 + (1 - \rho z)(1 - \rho z^{-1}) \sigma_u^2}{[\beta (a - b) + 1 - az][\beta (a - b) + 1 - az^{-1}]} \right\}. \tag{32}$$

It can be seen from the form of $G_{ALM}(z)$ that, for arbitrary a and b, the ALM is an ARMA(2,2) process. As we will now show, there are choices of a and b that yield $G_{PLM}(z) = G_{ALM}(z)$. These choices of a and b also have the property that the corresponding ALM takes an ARMA(1,1) form that matches the PLM. This is possible if a and b are chosen so that there is a common factor in the numerator and denominator of the expression on the right-hand side of $G_{ALM}(z)$.

We now set $G_{PLM}(z) = G_{ALM}(z)$, under the condition that $b = \rho$ so that the poles of the autocovariance generating functions agree. This yields

$$\sigma_v^2 [\beta (a - \rho) + 1 - az] \left[\beta (a - \rho) + 1 - az^{-1}\right] = \beta^2 \sigma_\eta^2 + (1 - \rho z) \left(1 - \rho z^{-1}\right) \sigma_u^2. \tag{33}$$

This equation can be written as

$$\sigma_v^2 \left\{[1 + \beta (a - \rho)]^2 + a^2\right\} - \sigma_v^2 a [\beta (a - \rho) - 1] \left(z + z^{-1}\right) = \beta^2 \sigma_\eta^2 + \sigma_u^2 \left(1 + \rho^2\right) - \sigma_u^2 \rho \left(z + z^{-1}\right). \tag{34}$$

For the autocovariances of the perceived and actual laws of motion to be equal, the coefficients on the powers of z in this equation must be equal. By equating coefficients on powers of z, we obtain two equations, which are given by

$$\sigma_v^2 \left\{[1 + \beta (a - \rho)]^2 + a^2\right\} = \beta^2 \sigma_\eta^2 + \sigma_u^2 \left(1 + \rho^2\right) \tag{35}$$

and

$$\sigma_v^2 a [\beta (a - \rho) + 1] = \sigma_u^2 \rho. \tag{36}$$

We wish to solve for a value of a such that $|a| < 1$. Solving equation (36) for σ_v^2 and substituting the result into equation (35), and in addition defining $s \equiv \beta^2\sigma_\eta^2 + \sigma_u^2(1+\rho^2)$, we obtain the quadratic equation

$$c_2 a^2 + c_1 a + c_0 = 0 \equiv f(a) \tag{37}$$

with

$$\begin{aligned} c_2 &\equiv s\beta - \rho\left(1+\beta^2\right)\sigma_u^2, \\ c_1 &\equiv s(1-\rho\beta) - 2\rho\beta(1-\rho\beta)\sigma_u^2, \\ c_0 &\equiv -\rho(1-\rho\beta)^2\sigma_u^2. \end{aligned}$$

We deduce that $f(0) < 0$, and that

$$f(1) = \sigma_\eta^2\beta^2\left[1 + (1-\rho)\beta\right] + \sigma_u^2(1-\rho\beta)(1+\beta)\left[(\rho-1)^2\right] > 0. \tag{38}$$

In addition,

$$f(-1) = \sigma_\eta^2\beta^2\left[(1+\rho)\beta - 1\right] + \sigma_u^2(1-\rho\beta)\left[(\beta-1)\left(\rho^2+2\rho+1\right)\right]. \tag{39}$$

For $\sigma_\eta^2 \to 0$, $f(-1) < 0$, while for $\sigma_u^2 \to 0$, $f(-1) > 0$ provided $\beta > (1+\rho)^{-1}$, which holds as $\beta \to 1$ and $\rho \to 1$. We also note that for $\sigma_\eta^2 \to 0$, $a = \rho$ solves equation (37), while for $\sigma_u^2 \to 0$, $a = 0$ is a solution. We conclude that there is always a unique positive value of $a \in (0, \rho)$ that solves the quadratic equation (37). While there can be a second, negative root, we do not focus on it in the discussion below.[5] Figure 1 illustrates the roots of the quadratic $f(a)$ for three cases where the relative variance $R = \sigma_\eta^2/\sigma_u^2$ is changing from zero to 2, with β and ρ fixed. Table 1 provides some values of the positive root of $f(a)$ for selected parameter values.

Since a value of $a \in [0, \rho]$ always exists, the conditions for a consistent expectations equilibrium can always be met in the scalar case. We now ask whether individual rationality holds with respect to inclusion of the judgement variable in making forecasts.

[5] We conjecture that the learnability condition will not be met for a negative root.

Table 1. Value of a.						
	$\beta = .75$		$\beta = .85$		$\beta = .95$	
	$\rho = .75$	$\rho = .95$	$\rho = .75$	$\rho = .95$	$\rho = .75$	$\rho = .95$
$R = 0.1$	.59	.54	.53	.51	.46	.23
$R = 1$	.26	.21	.20	.13	.14	.06
$R = 10$	.05	.04	.03	.02	.02	.01

Table 1: Values of the positive root of f(a) for selected parameters.

3.3 Incentives to include judgement in the scalar case

In this section, we use the condition from the consistent expectations calculation that $b = \rho$. We then note that $v_t = \left(\frac{1-\rho L}{1-aL}\right) y_t$. The econometric forecast is therefore given by

$$E_t^{\star} y_{t+1} = \frac{\rho - a}{1 - \rho L} v_t = \frac{\rho - a}{1 - aL} y_t \tag{40}$$

whereas the judgementally adjusted forecast is given by

$$y_{t+1}^{e} = \frac{\rho - a}{1 - aL} y_t + \frac{1}{1 - \rho L} \eta_t. \tag{41}$$

The question from an (atomistic) individual agent's point of view is then whether they should use (40) or (41) as a basis for their expectations of the future state of the economy.

We assume for the purposes of this calculation that all other agents in the economy are making use of the judgementally adjusted forecast. This induces an actual law of motion, as depicted in equation (27), which is

$$y_t = \frac{1 - aL}{\beta (a - \rho) + 1 - aL} \left(\frac{\beta}{1 - \rho L} \eta_t + u_t \right). \tag{42}$$

By substituting equation (42) into both (40) and (41), we can write the two types of forecasts in terms of the shocks u_t and η_t. These expressions become

$$E_t^{\star} y_{t+1} = \frac{\rho - a}{\beta (a - \rho) + 1 - aL} \left(\frac{\beta}{1 - \rho L} \eta_t + u_t \right) \tag{43}$$

in the case of no judgement, and

$$y^e_{t+1} = \frac{\rho - a}{\beta(a-\rho) + 1 - aL}\left(\frac{\beta}{1-\rho L}\eta_t + u_t\right) + \frac{1}{1-\rho L}\eta_t \tag{44}$$

in the case of the judgementally adjusted forecast. The actual state of the economy at time $t+1$ is, from equation (42),

$$y_{t+1} = \frac{1-aL}{\beta(a-\rho)+1-aL}\left(\frac{\beta}{1-\rho L}\eta_{t+1} + u_{t+1}\right). \tag{45}$$

We can therefore compute forecast errors in each of the two cases. When computing these forecast errors, we save on clutter by ignoring the terms involving u, as these will be the same whether or not the agent judgementally adjusts the forecast. The forecast error in the case of no judgement can be written as

$$FE_{NJ} \equiv [y_{t+1} - E^\star_t y_{t+1}]\,|_{u=0} = \frac{\beta}{1+\beta(a-\rho)}\frac{1}{\left[1-\left(\frac{a}{1+\beta(a-\rho)}\right)L\right]}\eta_{t+1} \tag{46}$$

whereas in the case of a judgementally adjusted forecast it is

$$FE_J \equiv \left[y_{t+1} - y^e_{t+1}\right]|_{u=0} = \frac{\beta}{1+\beta(a-\rho)} \times \frac{1-\left(a+\beta^{-1}\right)L + a\beta^{-1}L^2}{1-\left(\frac{a+\rho[1+\beta(a-\rho)]}{1+\beta(a-\rho)}\right)L + \left(\frac{a\rho}{1+\beta(a-\rho)}\right)L^2}\eta_{t+1}. \tag{47}$$

Is it possible for the variance of the judgementally adjusted forecast (47) to be lower than the variance of the econometric forecast (46)? It is. Consider the special case when $\sigma^2_\eta \to 0$ so that the positive root $a \to \rho$. Then

$$FE_{NJ} = \frac{\beta}{1-\rho L}\eta_{t+1} \tag{48}$$

whereas

$$FE_J = \frac{\beta\left(1-\beta^{-1}L\right)}{1-\rho L}\eta_{t+1}. \tag{49}$$

The difference between the variances of these two forecast errors is then[6]

$$Var[FE_J] - Var[FE_{NJ}] = \frac{\beta^{-1}}{1-\rho^2}\left(\beta^{-1} - 2\rho\right). \tag{50}$$

This can be less than zero if and only if

$$\rho\beta > \frac{1}{2}. \tag{51}$$

We conclude that individuals will decide to use the judgementally adjusted forecast in cases where ρ is relatively large, meaning that the serial correlation in the judgement variable is substantial, and when β is simultaneously relatively high, meaning expectations are relatively important in determining the evolution of the economy.

We remark that these conditions are exactly the ones that correspond to the most likely scenario for this type of model.[7]

Another, polar opposite, special case is one where $\sigma_u^2 \to 0$ so that the positive root $a \to 0$. Then

$$FE_{NJ} = \frac{\beta}{1-\rho\beta}\eta_{t+1} \tag{52}$$

whereas

$$FE_J = \frac{\beta}{(1-\rho\beta)}\frac{\left(1-\beta^{-1}L\right)}{(1-\rho L)}\eta_{t+1}. \tag{53}$$

The difference between the variances of these two forecast errors is then

$$Var[FE_J] - Var[FE_{NJ}] = \frac{\left(\beta^{-1}-\rho\right)^2}{1-\rho^2}. \tag{54}$$

This can never be less than zero given maintained assumptions. We conclude that it cannot be individually rational for agents to use a judgementally adjusted forecast in the scalar case when the relative variance of the judgemental variable is very large.

[6] See, for instance, Harvey (1981, p. 40). The variance of $x_t = [(1+\theta L)/(1-\phi L)]\,\epsilon_t$ is $\left[\left(1+\theta^2+2\phi\theta\right)/\left(1-\phi^2\right)\right]\sigma_\epsilon^2$.

[7] The case with $a \to \rho$ is a near-common factor representation of the time series. As we will show below, it is not necessary for a to equal ρ for our results to hold.

By continuity we deduce from these two special cases that there are values of $R = \sigma_\eta^2/\sigma_u^2 \in (0, \infty)$ such that $a \in (0, \rho)$ and agents rationally choose to use a judgementally adjusted forecast, given that all other agents are doing so.

The general case involves the variance of an ARMA$(2, 2)$ process. For a process in this class written generically as

$$x_t = \frac{1 + \theta_1 L + \theta_2 L^2}{1 - \phi_1 L - \phi_2 L^2} \epsilon_t, \tag{55}$$

the variance of x_t is given by

$$Var\left(x_t\right) = \frac{x_{num}}{x_{den}} \sigma_\epsilon^2. \tag{56}$$

where

$$x_{num} = \frac{\left(1 + \phi_2\right) \phi_1 \left(\theta_1 + \theta_2 \phi_1 + \theta_2 \theta_1\right)}{1 - \phi_2} + \left(\theta_1 + \theta_2 \phi_1\right)\left(\phi_1 + \theta_1\right) + \left(1 + 2\theta_2 \phi_2 + \theta_2^2\right) \tag{57}$$

and

$$x_{den} = 1 - \frac{\phi_1^2}{1 - \phi_2} - \frac{\phi_2 \phi_1^2}{1 - \phi_2} - \phi_2^2 \tag{58}$$

Considering the forecast error in the case without judgement included, equation (46), we set $\theta_1 = \theta_2 = \phi_2 = 0$ and $\phi_1 = a/\left[1 + \beta\left(a - \rho\right)\right]$ in equation (56). For the case with judgement, we set

$$\theta_1 = -\left(1 + a\beta\right)\beta^{-1}, \tag{59}$$

$$\theta_2 = a\beta^{-1}, \tag{60}$$

$$\phi_1 = \frac{a + \rho\left[1 + \beta\left(a - \rho\right)\right]}{1 + \beta\left(a - \rho\right)}, \tag{61}$$

$$\phi_2 = \frac{-a\rho}{1 + \beta\left(a - \rho\right)}. \tag{62}$$

As we have seen, the value of a can be influenced independently of the values of ρ and β by choice of the relative variance R. But we must also ensure that the parameters of the ARMA$(2,2)$ are consistent with stationarity of the process. This consideration places an additional restriction on exuberance equilibria, namely, that the roots of $1 - \phi_1 z - \phi_2 z^2$ both lie outside the unit circle.[8]

Figure 2 illustrates the findings for the general case. The figure is drawn for $\beta = .9$ and $\rho = .9$, which corresponds to what might be regarded as a realistic case. The variances of the forecast errors with and without judgement are plotted on the vertical axis, while the value of a is plotted on the horizontal axis. Each value of a between zero and ρ corresponds to a different relative variance $R = \sigma_\eta^2/\sigma_u^2$, and larger values of R are associated with smaller values of a. We have already seen from the examination of special cases that as $R \to \infty$, $a \to 0$ and we expect the forecast error variance of the econometric forecast to be smaller. This result is borne out in the figure. In addition, we expect the variance of the judgementally adjusted forecast to be lower when $R \to 0$, in which case $a \to \rho$. This is also borne out in the figure. But the figure also shows intermediate cases, and indicates that a does not have to be particularly close to ρ for the individual rationality condition to be met. In fact, the two forecast error variances are equal at $a \approx .21$, which is far from the value of ρ in this example, which is .9. We conclude that the conditions for exuberance equilibria to exist are quite likely to be met for a wide range of relative variances R provided both β and ρ are relatively close to one.

This intuition can be partially verified by checking cases where β and ρ are not so large. Based on condition (51) above, one might conjecture that the individual rationality constraint is binding at values $\rho\beta < 1/2$. In fact, at $\rho = .7$ and $\beta = 7$, an exercise like the one behind Figure 2 shows that there are no values of a that make the judgementally adjusted forecast preferable to the econometric forecast.

[8] See Hamilton (1994, pp. 59-60). The stationarity condition often fails to hold for the negative value of a discussed above.

3.4 Learnability in the scalar case

To study learning we initially analyze the case where the agents have AR(p) PLMs. These PLMs do not lead to an exact CEE but for p large enough the fixed points of learning will be good approximations to the corresponding exact CEE. The PLM is specified as

$$y_t = \sum_{i=1}^{p} b_i y_{t-i} + v_t. \tag{63}$$

This leads to forecasts with judgement

$$y^e_{t+1} = \sum_{i=0}^{p-1} b_{i+1} y_{t-i} + \xi_t, \tag{64}$$

where ξ_t is the judgement term, and hence to the ALM

$$y_t = (1-\beta b_1)^{-1} \left\{ \sum_{i=0}^{p-1} \beta b_{i+1} y_{t-i} + \beta(1-\rho L)^{-1}\eta_t + u_t \right\} \tag{65}$$

or equivalently

$$\begin{aligned} y_t \;=\;& (\rho + \beta(1-\beta b_1)^{-1} b_2) y_{t-1} + \\ & \beta(1-\beta b_1)^{-1} \left\{ \rho b_p y_{t-p} + \sum_{i=2}^{p-1} (b_{i+1} - \rho b_i) y_{t-i} \right\} + \\ & (1-\beta b_1)^{-1}(\beta \eta_t + u_t - \rho u_{t-1}). \end{aligned} \tag{66}$$

Note that the ALM is an ARMA(p,1) process and this is the way in which the AR(p) PLM can only give an approximate CEE.

Let $b = (b_1, \ldots, b_p)$ and let $P[y_t \,|Y_{t-1}] = T(b) Y_{t-1}$ be the linear projection of y_t on Y_{t-1} where $Y'_{t-1} = (y_{t-1}, \ldots, y_{t-p})$. Using standard results on linear projections,

$$T(b) = \left(E y_t Y'_{t-1}\right) (E Y_{t-1} Y'_{t-1})^{-1}. \tag{67}$$

An approximate CEE $\bar{b}$ satisfies the equation $\bar{b} = T(\bar{b})$. To compute $T(b)$ one can write the system in 1st order form

$$z_t = B z_{t-1} + D \begin{pmatrix} u_t \\ \eta_t \end{pmatrix} \tag{68}$$

with $z_t = (y_t, y_{t-1}, \ldots, y_{t-p}, u_t, u_{t-1}, \eta_t)'$. The relevant values for $\left(Ey_tY'_{t-1}\right)$ and $(EY_{t-1}Y'_{t-1})$ can be obtained from the equation

$$vec(var(z_t)) = [I - B \otimes B]^{-1} vec(Dvar \begin{pmatrix} u_t \\ \eta_t \end{pmatrix} D'). \tag{69}$$

Here $vec(K)$ is the vectorization of a matrix K and $\otimes$ is the Kronecker product. The equilibrium $\bar{b}$ can then by the "E-stability" algorithm

$$b_s = b_{s-1} + \gamma(T(b_{s-1}) - b_{s-1}), \tag{70}$$

where γ is chosen to be a small positive constant.

This procedure will automatically give us learnable equilibrium in the following sense. The econometricians are estimating an AR(p) PLM for y_t and are assumed to update their parameter estimates over time using recursive least squares (RLS). As previously explained, the policy makers add their judgement adjustment to the econometricians' forecast and, together with the variable u_t, the current value of y_t is determined. $T(b)$ denotes the true coefficients projection for a given forecast coefficients b. Under RLS learning it can be shown that econometrician's estimates b_t at time t on average move in the direction $T(b_t)$. Equation (70) describes this adjustment in notional time s. Using the techniques of Evans and Honkapohja (2001), it can be shown that RLS learning converges locally to $\bar{b}$ if it is a locally asymptotically stable fixed point (70) for sufficiently small $\gamma > 0$.

To illustrate this procedure we have computed AR(3) approximate CEE for $\beta = 0.9$, $\rho = 0.7$ and $R = 1$. The exact ARMA(1,1) CEE computed earlier were $b = 0.7$ and either $a = 0.180814$ or $a = -0.660017$. Directly computing the approximate AR(3) CEE we obtain the solution

$$b_1 = 0.517259,\ b_2 = 0.0929807,\ b_3 = 0.0157536. \tag{71}$$

These are reasonable approximations to the first three terms of the series expansion of $(1 - bz)/(1 - az)$ for the positive value of a. The above method of computation indicates that the ARMA(1,1) CEE with positive a is the learnable equilibrium. In contrast, we were unable to find AR(p) approximate

CEEs near the exact CEE with $a = -0.660017$ which were stable under the E-stability algorithm. These results justify our earlier decision to focus on the ARMA(1,1) CEE for which a is positive.[9]

4 A monetary policy example

4.1 New Keynesian macroeconomics

We wish to study exuberance equilibria in a simple microfounded macroeconomic model suggested by Woodford (2003) and Clarida, Gali, and Gertler (1999). We use a simple, three-equation version given by

$$x_t = x^e_{t+1} - \sigma^{-1}\left[r_t - \pi^e_{t+1}\right] + u_{x,t}, \quad (72)$$

$$\pi_t = \kappa x_t + \delta \pi^e_{t+1} + u_{\pi,t}, \quad (73)$$

$$r_t = \varphi_\pi \pi_t + \varphi_x x_t. \quad (74)$$

In these equations, x_t is the output gap, π_t is inflation, and r_t is the nominal interest rate. All variables are represented as percentage point deviations from steady state, where the steady state is normalized to zero. The terms $u_{x,t}$ and $u_{\pi,t}$ represent stochastic disturbances to the economy. The parameter σ^{-1} is related to the elasticity of intertemporal substitution in consumption of a representative household. The parameter κ is related to the degree of price stickiness in the economy, and δ is the discount factor of a representative household. The third equation describes the Taylor-type policy rule in use by the policy authority, in which the parameters φ_π and φ_x are assumed to be positive.[10] In the formulation (72)-(74), only the private sector forms expectations about the future. Other, forwarding-looking forms of the Taylor-type policy rule are possible, but we do not analyze them here.

[9] We note that ARMA(1,1) PLMs could be directly studied for stability under learning using recursive prediction error or pseudo-linear regression algorithms.

[10] Depending on how one interprets the microfoundations of the model, these equations, derived under a rational expectations benchmark, can differ under learning. For a micro interpretation where the equations are altered under learning, and additional analysis, see Preston (2003).

Substituting (74) into (72) and writing the system in matrix form gives

$$y_t = \beta y^e_{t+1} + C u_t \tag{75}$$

where $y_t = [x_t, \pi_t]'$, $y^e_{t+1} = \left[x^e_{t+1}, \pi^e_{t+1}\right]'$, $u_t = [u_{x,t}, u_{\pi,t}]'$ with covariance matrix

$$\Sigma_u = \begin{bmatrix} \sigma^2_{u,11} & \sigma^2_{u,12} \\ \sigma^2_{u,21} & \sigma^2_{u,22} \end{bmatrix}, \tag{76}$$

$$\beta = \frac{1}{\sigma + \varphi_x + \kappa\varphi_\pi} \begin{bmatrix} \sigma & 1 - \delta\varphi_\pi \\ \kappa\sigma & \kappa + \delta\left(\sigma + \varphi_x\right) \end{bmatrix} \tag{77}$$

and

$$C = \frac{1}{\sigma + \varphi_x + \kappa\varphi_\pi} \begin{bmatrix} \sigma & -\varphi_\pi \\ \kappa\sigma & \sigma + \varphi_x \end{bmatrix}. \tag{78}$$

We wish to endow the private sector with a perceived law of motion potentially consistent with the rational expectations equilibrium of the model. This is the ARMA process given by

$$y_t = b y_{t-1} + v_t - a v_{t-1}, \tag{79}$$

where $v_t = [v_{x,t}, v_{\pi,t}]'$ with covariance matrix

$$\Sigma_v = \begin{bmatrix} \sigma^2_{v,11} & \sigma^2_{v,12} \\ \sigma^2_{v,21} & \sigma^2_{v,22} \end{bmatrix}, \tag{80}$$

$$b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \tag{81}$$

and

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{82}$$

Equation (79) can be written as

$$y_t = \theta\left(L\right) v_t \tag{83}$$

where

$$\theta\left(L\right) = \left(I - bL\right)^{-1}\left(I - aL\right). \tag{84}$$

Expectations based on the perceived law of motion, or econometric model, are

$$E_t^\star y_{t+1} = [b\theta(L) - a]\, v_t. \tag{85}$$

The judgementally adjusted forecast is

$$y_{t+1}^e = E_t^\star y_{t+1} + \xi_t \tag{86}$$

where

$$\xi_t = \rho \xi_{t-1} + \eta_t, \tag{87}$$

with $\xi_t = \left[\xi_{x,t}, \xi_{\pi,t}\right]'$, $\eta_t = \left[\eta_{x,t}, \eta_{\pi,t}\right]'$ with covariance matrix

$$\Sigma_\eta = \begin{bmatrix} \sigma_{\eta,11}^2 & \sigma_{\eta,12}^2 \\ \sigma_{\eta,21}^2 & \sigma_{\eta,22}^2 \end{bmatrix}, \tag{88}$$

and

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}. \tag{89}$$

It follows from a modified version of equation (11) that the actual law of motion is

$$y_t = M(L)^{-1} \beta (I - \rho L)^{-1} \eta_t + M(L)^{-1} C u_t \tag{90}$$

where

$$M(L) = I - \beta \left[b - a\theta(L)^{-1}\right]. \tag{91}$$

We now turn to defining the consistent expectations condition for this model.

4.2 Consistent expectations

The consistent expectations condition for the model is given by a version of equation (21), namely

$$[(I - \beta b)\theta(z) - a]\, \Sigma_v \left[\theta\left(z^{-1}\right)^T (I - \beta b)^T - a^T\right] = \beta (I - \rho z)^{-1} \Sigma_\eta \left(\left(I - \rho z^{-1}\right)^{-1}\right)^T \beta^T + C \Sigma_u C^T. \tag{92}$$

In order to determine the values of $a_{i,j}$, $b_{i,j}$, and $\sigma_{v,i,j}^2$, $i, j = 1, 2$, we wish to examine this equation in detail. The condition equates two 2×2 matrices and thus dictates four equations in the complex scalar z. We now turn to examining these equations.

[NOTE: We have a few promising numerical examples of solutions. A New Keynesian numerical example still needs to be done and further analysis of the key equation should be attempted. The examples thus far suggest that the hypothesis $b = \rho$ is probably not justified.]

4.3 Learnability and approximate CEE

We now consider an approximate CEE based upon a VAR(1) PLM. This is an extension of the approach used in the univariate case to compute approximate CEE based on AR(p) PLMs. The PLM is $y_t = by_{t-1} + v_t$ and the implied ALM is

$$y_t = (I - \beta b)^{-1}\beta(\rho\xi_{t-1} + \eta_t) + (I - \beta b)^{-1}Cu_t. \tag{93}$$

This can be put in state-space form

$$z_t = Bz_{t-1} + De_t, \tag{94}$$

where $z_t' = (y_t', y_{t-1}', \xi_t')$ and $e_t' = (\eta_t', u_t')$. The second moments for computing the linear projection can be obtained from the equation

$$vec(var(z_t)) = (I - B \otimes B)^{-1}vec(Dvar(e_t)D'). \tag{95}$$

The projection is $P[y_t \,|y_{t-1}] = T(b)y_{t-1}$, where

$$T(b) = E(y_t y_{t-1}')(Ey_{t-1}y_{t-1}')^{-1}. \tag{96}$$

The approximate CEE matrix $\bar{b}$ can be computed as in the scalar case as fixed points of the E-stability algorithm (70). This procedure also guarantees stability of the equilibrium under RLS learning.

4.4 Incentives to include judgement

To compute the incentive to use judgement we write the forecasts in the form

$$y_{t+1}^e = by_t + k\xi_t, \tag{97}$$

where $k = 0$ or 1. The ALM is $y_t = (I - \beta b)^{-1}(\beta \xi_t + C u_t)$, where b is the equilibrium value $\bar{b}$, and the forecast error is

$$\begin{aligned} FE &= y_{t+1} - y_{t+1}^e \\ &= (I - \beta b)^{-1}(\beta \rho \xi_t + \beta \eta_{t+1} + C u_{t+1}) \\ &\quad - b(I - \beta b)^{-1}(\beta \xi_t + C u_t) - k \xi_t. \end{aligned} \tag{98}$$

Separating out terms involving ξ_t, the forecast error can be written as

$$FE = (N - kI)\xi_t + t.i.k, \tag{99}$$

where

$$N = (I - \beta b)^{-1}\beta \rho - b(I - \beta b)^{-1}\beta \tag{100}$$

and $t.i.k$ refers to terms that are independent of k.

The relevant parts of the mean-squared forecast error matrices without and with judgement are

$$M(0) = N \Sigma_\xi N', \tag{101}$$

$$M(1) = (N - I)\Sigma_\xi (N - I)', \tag{102}$$

where $\Sigma_\xi = var(\xi_t)$. In the multivariate set-up these quantities are matrices and their comparison can be done in several ways. The strongest concept involves the definiteness of the difference of the two matrices and a natural weaker concept involves comparing the difference between the variances of the individual components of the forecast error. With this in mind we make the following definitions:

(1) The CEE exhibits *strong exuberance* if $M(0) - M(1)$ is a positive definite matrix.

(2) The CEE exhibits *weak exuberance* if the diagonal elements of $M(0) - M(1)$ are positive.

Correspondingly, the equilibrium can be said to exhibit strong or weak non-exuberance if positive is replaced by negative in definitions (1) and (2).

It is easy to compute that

$$M(0) - M(1) = (N - \frac{1}{2}I)\Sigma_\xi + \Sigma_\xi(N' - \frac{1}{2}I). \tag{103}$$

Using a theorem by Lyapunov (see Theorem 2.2.1 in Horn and Johnson, Vol.II, p. 96) it follows that the eigenvalues of N must have real parts greater than $\frac{1}{2}$ in any strongly exuberant CEE. If the CEE exhibits strong non-exuberance, then the eigenvalues of N must have real parts less than $\frac{1}{2}$.

In general, there does not seem to be any simple criterion for weak exuberance or non-exuberance. However, one limiting case is revealing. If $\Sigma_u >> \Sigma_\xi$ and Σ_ξ and Σ_u are diagonal, then it can be shown that weak exuberance arises if and only if $\beta_{ii}\rho_i > \frac{1}{2}$ for $i = 1, 2$. This follows because $\Sigma_u(i,i) \to \infty$ implies that the CEE coefficients $\bar{b} \to 0$ and the (i,i) element of $M(0) - M(1) \to (2\rho_i\beta_{ii} - 1)\Sigma_\xi(i,i)$.

4.5 A calibration

We now return to the New Keynesian model to examine the examine the approximate CEE with exuberance. We use the Woodford (2003) calibration $\sigma = 0.157$, $\kappa = 0.024$, $\delta = 0.99$. For the exuberance variable we assume $\rho = \text{diag}(0.99, 0.95)$ and $\Sigma_\xi = (I-\rho)^{-1}\,\text{diag}(0.1, 0.1)(I-\rho)^{-1}$. The variances of the fundamental shocks are assumed to be $\Sigma_u = \text{diag}[50, 50]$.

The policy parameters φ_π and φ_x are varied and we are interested in values of φ_π and φ_x that might be consistent with exuberance equilibrium. As an initial example we set $\varphi_\pi = 1.05$ and $\varphi_x = 0.05$. These values satisfy the Taylor principle and deliver a determinate REE in the usual set-up, see Bullard and Mitra (2002). In the approximate CEE the coefficients of the vector autoregression are approximately [NOTE: do a longer run]

$$b = \begin{pmatrix} 0.0167 & 0.0757 \\ 0.0864 & 0.450 \end{pmatrix}. \tag{104}$$

The matrix $M(0) - M(1)$ is computed to be

$$\begin{pmatrix} 1.4327 & -0.2717 \\ -0.2717 & 0.7999 \end{pmatrix}, \tag{105}$$

which has eigenvalues 1.5334 and 0.6993. Hence the CEE is strongly exuberant.

When φ_π is increased to 1.5 the CEE is no longer strongly exuberant but it remains weakly exuberant. If φ_π is made sufficiently big even weak exuberance can be eliminated.

5 Conclusion

We have studied how a new phenomenon, exuberance equilibrium, may arise in standard macroeconomic environments. We assume that agents are learning in the sense that they are using econometric models to forecast the future values of variables they care about. Unhindered, this learning process would converge to a rational expectations equilibrium in the economies we study. We investigate the idea that agents may be tempted to include judgemental adjustments to their forecasts if all others in the economy are similarly judgementally adjusting their forecasts. The judgemental adjustment, or add factor, is a pervasive and widely-acknowledged feature of actual macroeconometric forecasting in industrialized economies. We show the conditions under which such add-factoring can become self-fulfilling, altering the actual dynamics of the economy significantly, but in a way that remains consistent with the econometric model of the agents.

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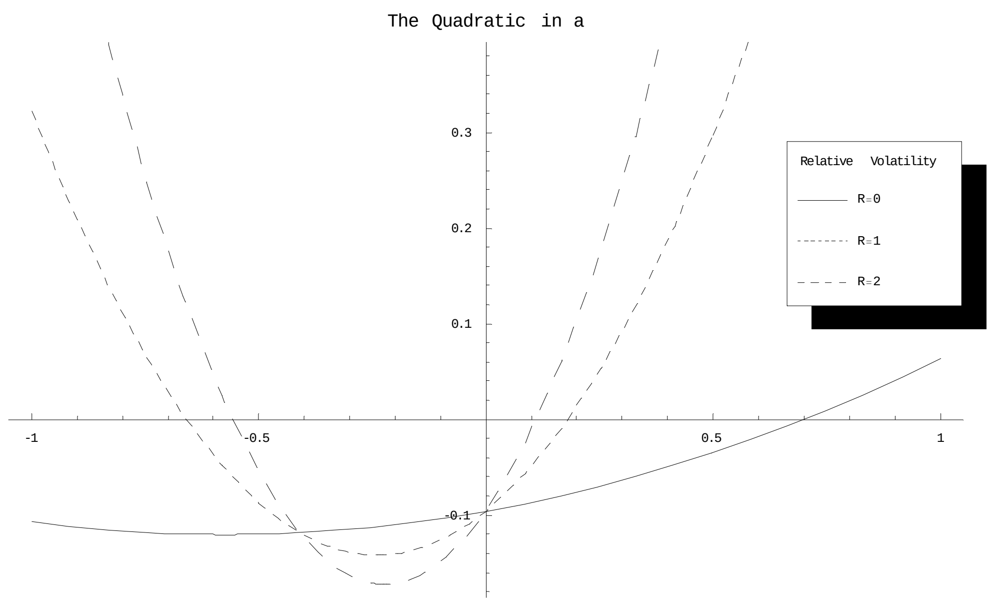

Figure 1: Roots of the quadratic $f(a)$. Here $R = \sigma_\eta^2/\sigma_u^2$ with $\beta = .9$ and $\rho = .7$.

Figure 2. Including Judgement

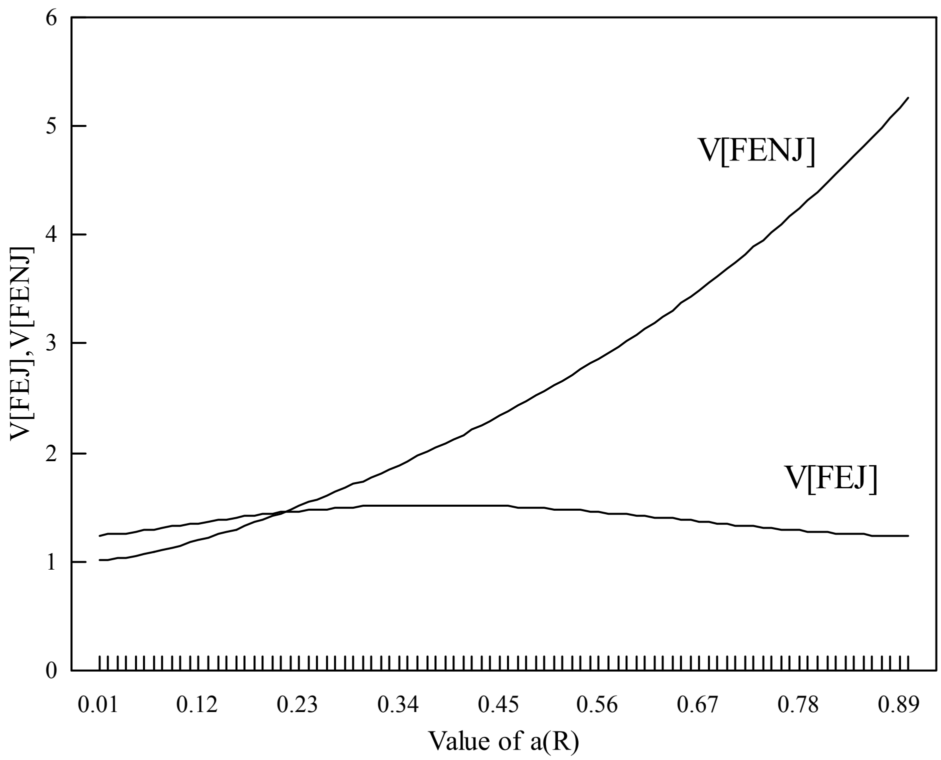

Figure drawn for ß=.9, rho=.9.

Figure 2: The forecast error variance using judgement is smaller provided a is larger than about .21, which is far from the value of ρ used to draw the figure (.9).

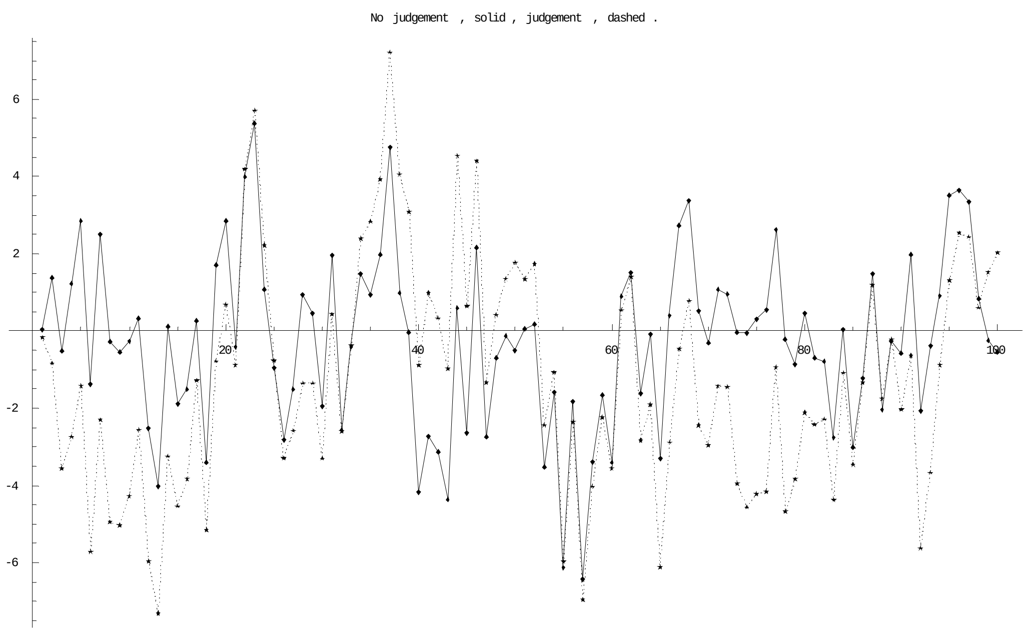

Figure 3: The time series of the fundamental equilibrium of an economy, the solid line, versus the exuberance equilibrium of the same economy.