1 Necessary and sufficient conditions

- Basic logic brush-up

- Important tool for semester to follow

1. The sentence “A is a necessary condition for B” means

(a) A must hold for B to hold, which means that

(b) whenever B is true, A must be true or, simply,

(c) A is implied by B (A ⇔ B)

2. The sentence “A is a sufficient condition for B” means that

(a) whenever A is true, B is true, or

(b) A is true only if B is true, or

(c) A implies B (A ⇒ B)

3. It is possible that both (A ⇒ B) and (A ⇔ B) hold. In that case we say that

(a) A is a necessary and sufficient condition for B, or

(b) A implies and is implied by B, or

(c) A and B are equivalent (A ⇔ B)

- Example: Let A be the statement “x is an integer ≤ 8” and let B be the statement “x is an integer ≤ 5.” In this case A is a necessary condition for B – whenever B
holds, $A$ must hold. However, $A$ is not a sufficient condition for $B$. If $x = 6$ then $A$ is true, but $B$ is not. Let the statement $B'$ be “$x$ is an integer $\not\equiv 8$”. In this case, $A$ is a necessary and sufficient condition for $B$, or $A \iff B$.

- Try-it-yourself: Describe the relations between the statements $A = \{X$ is a lemon\}, $B = \{X$ is a yellow fruit\} and $C = \{X$ is a yellow, sour fruit that begins with an L\}.

- Another useful thing is the contrapositive: Suppose $A$ is necessary for $B$, i.e. $B \Rightarrow A$. Then what if $A$ is false? Then, by 1.a above, $B$ cannot be true. Thus, $\neg A \Rightarrow \neg B$ but NOT $\neg B \Rightarrow \neg A$.

- Try-it-yourself: Relate the contrapositive to the example above.