Preliminary notes in Microeconomics

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This note is intended as a summing up of the material which has been taught through the first two years and of which the students should have some knowledge. The note will introduce some straightforward generalizations of the concepts introduced in the first two years.

Generally we consider two sides of a market: the consumers and the firms. These are joined together through the market in which a price on each market is common to both sides, i.e. they behave competitively. We first consider the two sides separately, then we consider exchange economies and finally we bring together consumers and producers in production economies.

First some notation: We will be concerned with subsets of finite dimensional vector spaces denoted by $\mathbb{R}^n$, such that the inner product is the dot product $x \cdot y = \sum_{i=1}^{n} x_i y_i$. A convex set $C \subset \mathbb{R}^n$ is such that if $x, y \in C$ and $\lambda \in [0, 1]$ then $\lambda x + (1 - \lambda)y \in C$. If $x, y \in \mathbb{R}^n$ we write $x \geq y$ if $x_i \geq y_i$ for every $i = 1, \ldots, n$; $x > y$ if $x \geq y$ and for some $j$ we have $x_j > y_j$; and $x >> y$ if $x_i > y_i$ for every $i = 1, \ldots, n$.

If $f : \mathbb{R}^n \to \mathbb{R}$ then the gradient at the point $x_0$ is denoted by $\nabla f(x_0) = (\frac{\partial f(x_0)}{\partial x_1}, \ldots, \frac{\partial f(x_0)}{\partial x_n})$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous differentiable if $\nabla f(x)$ exists and is continuous; the function is $m$-times continuous differentiable if $\frac{\partial^m f(x)}{\partial x_1 \ldots \partial x_m}$ exists and are continuous differentiable; the function is smooth if for all $m \in \mathbb{N}$ the function is $m$-times continuous differentiable.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is concave if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ it is the case that $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$; the function is quasi-concave if $f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$.

We write $\arg \max \{f(t) | t \in I\}$ when considering the set $\{t \in I | f(t) \geq f(s) \forall s \in I\}$ where $\forall$ means 'for all'.
1 The consumer

We consider an economy with $\ell \in \mathbb{N}$ (finite) commodities. A consumer is characterized by a consumption set $X \subset \mathbb{R}^\ell$ and a preference relation $\succsim$ on $X$. We shall denote a consumer by $(X, \succsim)$. $x = (x_1, \ldots, x_\ell) \in X$ denotes a consumption bundle where $x_i$, $i \in \{1, \ldots, \ell\}$, is the consumption of commodity $i$. For a pair of commodity bundles $x, y \in X$ we say that the consumer (weakly) prefers bundle $x$ to bundle $y$ if $x \succsim y$ and that the consumer is indifferent between $x$ and $y$ if $x \succsim y$ and $y \succsim x$, in which case we write $x \sim y$. Generally we have that preferences of a consumer satisfy the following axioms:

**Axiom 1** Let $\succsim$ be a preference relation on $X$ then $\succsim$ is said to be:

1. **Complete**: if for all $x, y \in X$ either $x \succsim y$ or $y \succsim x$ or both
2. **Transitive**: if for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$

Further we will assume that $\succsim$ is 'continuous', thus as we make small adjustments in consumption bundles the preferences will not change very much. Given a consumption bundle $x \in X$ we consider the bundles which the consumer is indifferent to, and we call these the indifference set of $x$ and denote it by $I(x) = \{y \in X | x \sim y\}$. Similarily we define the weakly preferred bundles set or the upper contour set of $x \in X$ by $P(x) = \{y \in X | y \succsim x\}$.

**Definition 1** Let $\succsim$ be a preference relation on $X$ then $\succsim$ is said to be:

1. **Monotone**: if for all $x, y \in X$ if $x \gg y$ then $x \succ y$
2. **Strictly monotone**: if for all $x, y \in X$ if $x > y$ then $x \succ y$
3. **Convex**: if for all $x, y \in X$ and $\lambda \in [0, 1]$ where $y \succsim x$ then $\lambda x + (1 - \lambda)y \succsim x$
4. **Strictly convex**: if for all $x, y \in X$ and $\lambda \in [0, 1]$ where $y \succsim x$ then $\lambda x + (1 - \lambda)y > y$

Convexity of preferences are equivalent to convex upper contour sets, while strictly convex preferences have strictly convex upper contour sets. Note that strictly monotone preferences are also monotone preferences, and that strictly convex preferences are also convex preferences.

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1 This statement can be made formally in the following way: $\succsim$ is continuous if for every sequence $(x_n)$, $(y_n)$ such that $x_n \in X$, $y_n \in X$ and $x_n \succsim y_n$ for every $n \in \mathbb{N}$ and $x_n \to x, y_n \to y$ where $x \in X$ and $y \in X$ then $x \succsim y$. 
Given preferences \(\succ\) on \(X\) we define a **utility function** \(u : X \to \mathbb{R}\) to represent the preferences \(\succ\) if it satisfy that for every \(x, y \in X\): \(x \succ y \iff u(x) \geq u(y)\). The mentioned properties of a preference relation is directly applicable to utility function assuming that preferences are smooth:

**Lemma 1** Let \(\succ\) be a preference relation on \(X\) which is smooth then

1. if \(\nabla u(x) > 0\) for every \(x \in X\) then \(\succ\) is strictly monotone
2. if \(u()\) is quasi-concave then \(\succ\) is convex
3. if \(u()\) is strictly quasi-concave then \(\succ\) is strictly convex

Given prices \(p \in \mathbb{R}^\ell\), where \(p_k\) is the price on commodity \(k\), and an income \(w > 0\) the **budget set** is given by \(B(p, w) = \{x \in X | p \cdot x \leq w\}\) (where \(p \cdot x\) denotes the 'dot'-product given by \(p \cdot x = \sum_{i=1}^{\ell} p_i x_i = p_1 x_1 + \ldots + p_\ell x_\ell\)); thus the budget set is the consumption bundles which the consumer can afford given the prices \(p\) and the income \(w\). The consumption bundles which exhaust the income of the consumer we denote the **budgethyperplane** and is the set \(\bar{B} = \{x \in X | p \cdot x = w\}\). Thus we can state the primary choice concept:

**Definition 2** Given a consumer \((X, \succ)\) we denote the **market demand** given the prices \(p\) and income \(w\) by \(x(p, w)\) which is the set:

\[
x(p, w) = \{x \in B(p, w) | x \succ y \forall y \in B(p, w)\}
\]

This we also call the demand function and the problem which is solved by the demand function is called the utility maximization problem. \(x_i(p, w)\) is the amount of commodity \(i\) the consumer wishes to consume at prices \(p\) and income \(w\), and demand of any commodity generally depends on the prices of all the other commodities. If \(u()\) is a utility function representing \(\succ\) then we can write the demand bundle as:

\[
x(p, w) = \arg \max \{u(x) | x \in B(p, w)\}
\]

We denote the maximum utility given prices \(p \in \mathbb{R}^\ell\) and income \(w\) by \(v(p, w) = u(x(p, w))\) which is called the **indirect utility** function, thus we define:

\[
v(p, w) = \max \{u(x) | x \in B(p, w)\}
\]

By the duality we can define the minimum expenditure given some utility level \(\bar{u}\) by

\[
\min \{p \cdot x | u(x) \geq \bar{u}\}
\]
The solution to the expenditure minimization problem is called the **Hicksian demand** and the value is called the **expenditure function** and they are thus defined as:

\[
h(p, \bar{u}) = \arg \min \{p \cdot x | u(x) \geq \bar{u} \}
\]

\[
e(p, \bar{u}) = \min \{p \cdot x | u(x) \geq \bar{u} \}
\]

We say that the demand function satisfy **Walras law** if \( p \cdot x(p, w) = w \) for all \( p \in \mathbb{R}^\ell \).

If the utility function has some 'nice' properties we can characterize the consumer choice in a convenient way: the budget hyperplane will be tangent to the indifference curve through the optimal bundle:

**Lemma 2** Let \( u() \) be a quasi-concave utility function such that \( \nabla u(x) > 0 \) for all \( x \in X \). Then there exists \( \lambda \in \mathbb{R} \) such that \( \lambda \neq 0 \) and:

\[
\nabla u(x) = \lambda p
\]

\[
p \cdot x = w
\]

This also implies the well-known(?) condition of the equality between the price-ratios and the marginale rates of substitution, since if we consider the system of equations above we note that the \( i \)'th equation is

\[
\frac{\partial u(x)}{\partial x_i} = \lambda p_i
\]

and thus for each \( i, j \in \{1, ..., \ell\} \) where \( i \neq j \) we have that

\[
\frac{\partial u(x)}{\partial x_i} \frac{\partial u(x)}{\partial x_j} = \frac{p_i}{p_j}
\]

Which is the well-known (first-order) condition of the utility maximization problem.

Up to now we have considered the income to be independent of the prices; but the generalization is straight forward; we assume the presence of private property and now characterize the consumer by the triple \((X, \succeq, \omega)\) where we have added an endowment \( \omega \in \mathbb{R}^\ell \) which is the consumption bundle which the consumer possesses; thus without going to the market the consumer is guaranteed a minimum of consumption (if \( \omega \in X \) of course), and the consumer can decide to sell some of the initial endowment creating income which he can use to buy other commodities. Thus we can now define the budget set as \( B(p, \omega) = \{x \in X | p \cdot x \leq p \cdot \omega \} = \{x \in X | p \cdot (x - \omega) \leq 0 \} \) and thus setting \( w = p \cdot \omega \) in any of the definitions above.
This ‘endogenisation’ clearly stirs up the comparable statics; while with exogen income we had the substitution effect and the income effect\(^2\) when considering the demand-response to a price change\(^3\) - now we have added jet another effect: the endowment effect; if the consumer is a net seller of a commodity (ie. if \(x_i(p) - \omega_i < 0\)), a price-incrementation of that commodity increase the income of the consumer, and thus may increase the demand - even though the commodity is a normal commodity and the substitution effect is negative.

When there is private property and \(\omega\) is the initial endowment, then we define the net-demand of the consumer as \(z(p) = x(p) - \omega = (x_1(p) - \omega_1, ..., x_\ell(p) - \omega_\ell)\), which i called the **excess demand** of the consumer

## 2 The Firm

While still assuming an economy in which there are \(\ell\) different commodities; a (neoclassical) firm is characterized by the **production possibilities set** \(Y \subset \ell\) and a **production plan** \(y \in \mathbb{R}^\ell\) is feasible if \(y \in Y\). The \(i\)'th coordinate \(y_i\) represents the net output of the firm of commodity \(i\), thus the entry is negative if the commodity is an input and positive if the commodity is an output.

Consider the situation in which the firm only produce 1 commodity, denote it by \(q\), and uses \(J\) commodities as input, denoted by \(z = (z_1, ..., z_J)\), and consider the production function \(f : \mathbb{R}^J \rightarrow \mathbb{R}\); which gives the maximal output possible using the inputbundle \(z\); Then the production possibility set is \(Y = \{(q, (-z_1, ..., -z_J))| f(z) \geq q\}\).

Generally we prefer the general notion of a production possibility set to the notion of a production function.

The production possibilities set can have several properties:\(^4\)

**Definition 3** Let \(Y \subset \mathbb{R}^\ell\) be a production possibility set then we can have:

1. **No free lunch**: \(Y \cap \mathbb{R}^\ell_{++} = \emptyset\)

2. **Possibility of no-action**: \(0 \in Y\)

\(^2\)Remember that the substitution effect is the change in demand due to the change in the price-ratio keeping the (utility level/real income level) fixed, while the income effect is the change in demand due to the change in the purchasing power of the consumer

\(^3\)As an exercise: proof that the substitution effect is always negative (hint: use the definition of the expenditure function and the relation with the Hicksian demand function)

\(^4\)In each case try to draw an example where the production possibilities set satisfy and one where it does not satisfy the property.
3. **Free disposal**: if \( y \in Y \) and \( y' \leq y \) then \( y' \in Y \)

4. **Irrversible**: if \( y \in Y \setminus \{0\} \) then \(-y \notin Y\)

5. **Nondecreasing return to scale**: if \( y \in Y \) and \( \alpha > 1 \) then \( \alpha y \in Y \)

6. **Nonincreasing return to scale**: if for any \( y \in Y \setminus \{0\} \) we have \( \alpha y \in Y \) then we must have \( \alpha \in [0,1] \)

7. **Constant return to scale**: if \( y \in Y \) and \( \alpha \geq 0 \) then \( \alpha y \in Y \)

8. **Convex**: if \( y, y' \in Y \) and \( \alpha \in [0,1] \) then \( \alpha y + (1 - \alpha)y' \in Y \)

9. **Additivity**: if \( y, y' \in Y \) then \( y + y' \in Y \)

Given that the firm is small relatively to the market we consider the competitive firm which takes the prices as given, i.e. has no market-power. Given the prices \( p \in \mathbb{R}^\ell \) we consider the profit given a production plan \( y \in Y \) to be \( p \cdot y = \sum_{i=1}^\ell p_i y_i \). The firm will generally want to maximize the profit thus the problem of the firm is

\[
\max \{ p \cdot y | y \in Y \}
\]

The solution to this problem is called the firm’s **supply function** and denoted by \( y(p) \) and the profit of the firm the **profit function** denoted by \( \pi(p) \); thus

\[
y(p) = \arg \max \{ p \cdot y | y \in Y \}
\]

and

\[
\pi(p) = \max \{ p \cdot y | y \in Y \} = p \cdot y(p)
\]

Thus we have that given prices \( p \) and if \( y(p) \) is welldefined, then \( y^* \in y(p) \) means that \( p \cdot y^* \geq p \cdot y \) for every \( y \in Y \). Remember the following properties of the firms profit function\(^5\)

**Lemma 3** If the firm has possibility of no-action then the firm can not have negative profit

**Lemma 4** If the firm has constant returns to scale then the firm must have either 0 or \( \infty \) profit.

\(^5\)As an exercise prove the lemmatas.
A necessary condition when profit-maximizing is cost-minimizing, i.e. producing a given output in the cheapest possible way. If we have a firm with production possibilities set $Y = Q \times \prod_{i=1}^{M} Z_i$ i.e. one output and $M$ inputs, a production function $f : \prod_{i=1}^{M} Z_i \rightarrow Q$, and factor prices $w \in \mathbb{R}^M$, then the cost minimization problem of producing output $q$ is

$$\min_z w \cdot z \\
\text{s.t. } q \leq f(z) \\
z \in \prod_{i=1}^{M} Z_i$$

The solution is called the conditional factor demand functions $z(w, q)$ and the lowest cost is the cost function, $c(w, q)$. Thus:

$$z(w, q) = \arg \min \{w \cdot x | f(z) \geq q\}$$

is the conditional factor demand and

$$c(w, q) = \min \{w \cdot x | f(z) \geq q\}$$

is the cost function. Note that cost-minimization may have a solution even though profit-maximization does not have a solution.

3 The Edgeworth-economy

The Edgeworth economy is an exchange economy in which there are 2 consumers, 2 commodities and in which consumers have private property of their initial endowment. Thus we can write this economy as a tuple $(I, (X_A, u_A(\cdot), \omega_A), (X_B, u_B(\cdot), \omega_B))$ where $I = \{A, B\}$ is the set of consumers, $X_i \subset \mathbb{R}^2$, $u_i : X_i \rightarrow \mathbb{R}$ and $\omega_i = (\omega_{i,1}, \omega_{i,2})$ is the consumption set, utility function and the initial endowment of consumer $i = A, B$. Denote this economy as $\mathcal{E}$.\footnote{The set $I$ is superfluous above, since it is obvious that there are two consumer, but generally an economy is given as: \((I, (X_i, u_i(\cdot), \omega_i))\)}

We can now define the following concepts of an edgeworth economy:

**Definition 4** An allocation of an economy $\mathcal{E}$ is a pair $(x_A, x_B)$ such that $x_A, x_B \in \mathbb{R}^2$.

**Definition 5** An allocation $(x_A, x_B)$ of an economy $\mathcal{E}$ is feasible if we have that $x_A + x_B = \omega_A + \omega_B$, $x_A \in X_A$ and $x_B \in X_B$.\footnote{The set $I$ is superfluous above, since it is obvious that there are two consumer, but generally an economy is given as: \((I, (X_i, u_i(\cdot), \omega_i))\)}
We could have allowed the feasible allocations to have a weak inequality \( \leq \) such that the condition would be that \( x_A + x_B \leq \omega_A + \omega_B \); if this is the case we call the economy a free-disposal economy. We will, however, take the stated definition as our notion.

We now present the notion of a Pareto-optimal allocation:

**Definition 6** Let \( E \) be an economy and \((x_A^*, x_B^*)\) an allocation; then \((x_A^*, x_B^*)\) is **Pareto-optimal** if

1. \((x_A^*, x_B^*)\) is feasible
2. for all feasible \((x_A, x_B)\) we have that \( u_i(x_i^*) \geq u_i(x_i) \) for all \( i = A, B \)

We say that an allocation \((x_A, x_B)\) is **pareto-dominated** if there exists an allocation \((x_A', x_B')\) such that \( u_i(x_i') \geq u_i(x_i) \) for all \( i = A, B \) and either \( u_A(x_A') > u_A(x_A) \) or \( u_B(x_B') > u_B(x_B) \) or both. Thus we can define a pareto-optimal allocation as an allocation which is not pareto-dominated.

Further we can define the notion of a market or walrasian equilibrium:

**Definition 7** Let \( E \) be an economy; A **Walrasian equilibrium** of \( E \) is an allocation \((x_A^*, x_B^*)\) and a price system \( p^* \in \mathbb{R}^\ell \) such that:

1. \((x_A^*, x_B^*)\) is feasible
2. \( x_i^* \in B_i(p, \omega_i) \) and if \( x_i \in B_i(p, \omega_i) \) then \( u_i(x_i^*) \geq u_i(x_i) \) for all \( i = A, B \)

Note that a price system \( p \) is an equilibrium price system if

\[
x_A(p) + x_B(p) = \omega_A + \omega_B
\]

or stated in terms of excess demand functions

\[
z_A(p) + z_B(p) = 0
\]

The advantage of an Edgeworth-economy is the possibility of graphical presentation, using the edgeworth-box. This tool can graph the concepts just presented. While the main results generalizes into environments with several commodities and consumers, the edgeworth-box has its limitations.
4 Production economy

This is an economy $E^P$ in which there are both consumers and firms; First we consider the most simple production economy: The Robinson-Crusoe-economy: We now have an economy described by the tuple $(\mathcal{I}, \mathcal{J}, (X_i, u_i(\cdot), \omega_i), (Y_j))$ where $\mathcal{I} = \{1\}$ is the set of consumers, $\mathcal{J} = \{1\}$ is the set of firms (production units), for every consumer $i \in \mathcal{I}$ there are specified a consumption set $X_i \subset \mathbb{R}^\ell$, a utility function $u_i : X_i \to \mathbb{R}$ and a initial endowment $\omega_i$ and for each firm $j \in \mathcal{J}$ a production possibilities set $Y_j \subset \mathbb{R}^\ell$. Note that the dimension of the vectorspace, in which the consumption possibility set and the productions possibilities set is imbedded in, is the same, $\ell$. In the Robinson-Crusoe economy we have that $\ell = 2$.

Thus we can now define the concepts of an allocation, the feasible allocations, the pareto optimal allocations and the walrasian equilibrium just as with the Edgeworth economy:

Definition 8 Let $E^P$ be a (Robinson-Crusoe) production economy, then an allocation is a pair $(x, y)$, $x, y \in \mathbb{R}^\ell$ where $x$ is an consumption bundle and $y$ is a production plan

Definition 9 Let $E^P$ be a (Robinson-Crusoe) production economy, then an allocation $(x, y)$ is feasible if $x = \omega + y$, $x \in X$ and $y \in Y$

Definition 10 Let $E^P$ be a (Robinson-Crusoe) production economy, then an allocation $(x^*, y^*)$ is pareto-optimal if

1. $(x^*, y^*)$ is feasible
2. for every $(x', y')$ which is feasible we must have that $u(x^*) > u(x')$

Before we can define a walrasian equilibrium, there is problem which needs to be handled: who owns the firms and is entitled to the profit given the production plan carried out? When there is only one consumer this is quite trivial, since he gets all the profit, but generally we will assume a given distribution of shares $(\mu_{ij})$ such that $\sum_{i \in \mathcal{I}} \mu_{ij} = 1$; thus $\mu_{ij}$ is consumer $i$’s share of stocks in firm $j$. If we consider the 1 consumer - 1 firm case this gives us the budget set of the consumer as:

$$B(p, \omega) = \{x \in X | p \cdot x \leq p \cdot \omega + \pi(p)\}$$

Now we are ready to define the walrasian equilibrium of an production economy:

$$B(p, \omega, (\pi_{ij})_{j=1}^J) = \{x_i \in X | p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^J \mu_{ij} \pi_j (p)\}$$

since the consumer receives a share (possibly 0) profit from each firm. Of course if the production possibility sets have constant return to scale there are no profit, and thus no distribution problem.
Definition 11 Let $\mathcal{E}^P$ be a (Robinson-Crusoe) production economy, then an allocation $(x^*, y^*)$ and a pricesystem $p^*$ is an **Walrasian equilibrium** if:

1. $(x^*, y^*)$ is feasible
2. $x^* \in B(p^*, \omega)$ and for every $x' \in B(p^*, \omega)$ we have that $u(x^*) > u(x')$
3. for every $y' \in Y$ then $p^* \cdot y' \leq p^* \cdot y^*$

As in the edgeworth-economy the equilibrium can be described in terms of excess demand; let $z$ be the excess demand of the economy defined by

$$z(p) \equiv x(p) - \omega - y(p)$$

where $x(p)$ is the demand function of the consumer and $y(p)$ is the firms supply function. Thus prices $p^*$ which are equilibrium prices must thus solve the equations:

$$z(p^*) = x(p^*) - \omega - y(p^*) = 0$$

**Example 1** 2 commodities, 1 consumer, 1 firm economy

Consider the consumer $(\mathbb{R} \times [0, 1], u(x, l) = x^a l^{1-a}, (0, 1))^9$ and the firm where the production possibilities set is $Y = \{(y, -L) \in \mathbb{R}^2 | y \leq \sqrt{L}, L \geq 0\}$. Then given the pricesystem $(p, w)$ where $p$ is the price on the consumption good and $w$ is the price on labour (the wage), the consumer demand function is $x(p, w) = \alpha \frac{w + \pi(p, w)}{p}$ and $l(p, w) = (1 - \alpha) \frac{w + \pi(p, w)}{w}$ where $\pi(p, w)$ is the profit of the firm. The firm profitmaximization gives a supply function which is $y(p, w) = \frac{p}{2w}$ and thus a profit function of $\pi(p, w) = \frac{p^2}{4w}$. The feasibility condition gives us that $x = y$ and $L = \bar{l} - l$, and walras law and homogenity of demand and supply functions of degree 0 allows us to normalize $p = 1$ and only consider one market. Thus we get that $x(p, w) = y(p, w)$ or $\alpha(w + \frac{1}{4w}) = \frac{p}{2w}$ and thus we have an equilibrium pricesystem $(p^*, w^*) = (1, \sqrt{\frac{2-\alpha}{2-\alpha}})$, where the equilibrium allocation is $((x^*, l^*), (y^*, -L^*)) = ((\sqrt{\frac{\alpha}{2-\alpha}}, \frac{2(1-\alpha)}{2-\alpha}), (\sqrt{\frac{\alpha}{2-\alpha}}, -\frac{\alpha}{2-\alpha})).$

Given that there is only one consumer the problem of pareto optimal allocations is reduced to solve the problem:

$$\max_{x, l, y, L} x^a l^{1-a} \quad \text{s.t.} \quad x = y, L = l - l, y = \sqrt{L}$$

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8 generally we would have that $z(p) \equiv \sum_{i=1}^{I} x_i(p) - \sum_{i=1}^{I} \omega_i - \sum_{j=1}^{J} y_j(p)$ thus the sum of each consumers consumption plan, endowment and firms production plan

9 where $\alpha \in [0, 1]$
which can be solved by using the Lagrange method such that the Lagrangian is

\[ L(x, y, l, L) = x^\alpha l^{1-\alpha} - \lambda_1(x - y) - \lambda_2(L - l + 1) - \lambda_3(y - \sqrt{L}) \]

and the first order conditions are

\[
\begin{align*}
\alpha x^{\alpha-1} l^{1-\alpha} - \lambda_1 &= 0 \\
(1-\alpha)x^{\alpha-1} l^{1-\alpha} - \lambda_2 &= 0 \\
\lambda_1 - \lambda_3 &= 0 \\
-\lambda_2 + \frac{1}{2\sqrt{L}} &= 0
\end{align*}
\]

and of course the equations \( x = y \), \( L = l - 1 \) and \( y = \sqrt{L} \). This reduces to the well-known condition of tangency of marginal substitution rate and marginal rate of transformation:

\[ MRS = \frac{\alpha x^{\alpha-1} l^{1-\alpha}}{(1-\alpha)x^{\alpha-1} l^{1-\alpha}} = \frac{\lambda_1}{\lambda_2} = 2\sqrt{L} = MRT \]

and now we can check that the Walrasian equilibrium is in fact Pareto-optimal:

\[ MRS = \frac{\alpha}{1-\alpha} x^* = \frac{\alpha}{1-\alpha} \frac{2(1-\alpha)}{2-\alpha} = \sqrt{\frac{4\alpha}{2-\alpha}} = 2\sqrt{\frac{\alpha}{2-\alpha}} = 2\sqrt{L^*} = MRT \]

Example 2: 2 goods, 4 commodities, 2 consumers, 2 firms

Consider the consumers \( i = A, B \) where \((\mathbb{R}^2_+, u_i(x_1^i, x_2^i), (0, 0, L_i, K_i))\) thus the consumers are endowed with 'labour' \( L_i \) and 'capital' \( K_i \) but the utility only depends on the amount of consumption, thus the consumer always supply the entire amount of labour and capital. The firms production possibilities set are given by

\[ Y_1 = \{(y_1, 0, -L_1, -K_1) | y_1 \leq f(L_1, K_1), L_1, K_1 \geq 0\} \]

and

\[ Y_2 = \{(0, y_2, -L_2, -K_2) | y_2 \leq g(L_2, K_2), L_2, K_2 \geq 0\} \]

such that \( f \) (resp. \( g \)) is the production function of firm 1 (resp. 2). Then the feasible allocations is the set \( ((x_1^A, x_2^A, L_A, K_A), (x_1^B, x_2^B, L_B, K_B), (y_1, 0, -L_1, -K_1), (0, y_2, -L_2, -K_2)) \) such that

\[
\begin{align*}
x_1^A + x_1^B &= y_1 \\
x_2^A + x_2^B &= y_2 \\
L_A + L_B &= L_1 + L_2 \\
K_A + K_B &= K_1 + K_2
\end{align*}
\]
and \((x_1^A, x_2^A) \in \mathbb{R}^2_{++}, (x_1^B, x_2^B) \in \mathbb{R}^2_{++}, (y_1, 0, -L_1, -K_1) \in Y_1\) and \((0, y_2, -L_2, -K_2) \in Y_2\). Consider a pricesystem \(p = (p_1, p_2, w, r)\) where \(p_1\) is the price of good 1, \(p_2\) is the price of good 2, \(w\) is the wage rate and \(r\) is the rental rate on capital. Given the distribution of share of the firms \(\mu = ((\mu_1^A, \mu_2^A), (\mu_1^B, \mu_2^B))\) such that \(\mu_1^A - \mu_1^B = 1\) and \(\mu_2^A - \mu_2^B = 1\), we have that the budget set of each agents are

\[
B_A(p, \omega_A, \mu_A) = \{(x_1^A, x_2^A) | p_1 x_1^A + p_2 x_2^A \leq wL_A + rK_A + \mu_1^A \pi_1(p) + \mu_2^A \pi_2(p)\}
\]

\[
B_B(p, \omega_B, \mu_B) = \{(x_1^B, x_2^B) | p_1 x_1^B + p_2 x_2^B \leq wL_B + rK_B + \mu_1^B \pi_1(p) + \mu_2^B \pi_2(p)\}
\]

where \(\pi_i(p) = p_i y_i(p) - wL_i(p) - rK_i(p)\) is the profit function of the firms \(i = 1, 2\).