1. Consider a bay which can be freely accessed by independent fishermen. The cost of sending out a boat is $c > 0$. When $b$ boats are sent out on the bay, the resulting catch is $f(b)$, such that each boat catches $f(b)/b$, where $f$ is continuous with $f'(b) > 0$, $f''(b) < 0$ at all $b \geq 0$, $f(0) = 0$, and $f'(b) \to \infty$ for $b \to 0$. The price of fish is $p > 0$, which is unaffected by the catch from the bay.

(a) Characterize the equilibrium number of boats sent out on the bay (don’t worry about integer problems). Hint: Consider when a fisherman is indifferent between sending out a boat or not.

(b) Characterize the optimal number of boats sent out onto the bay, and compare the result to that obtained in (a). Comment.

(c) Illustrate the problem graphically: Draw $f(b)$ and illustrate the solutions in (a) and (b).

(d) If necessary, what per-boat fishing tax could restore efficiency?

(e) Suppose that the government allocates the fishing rights to one fleet owner, rather than having independent fishermen fishing the bay. How many boats would this owner send out?

2. Adam and Eve are neighbours; Eve lives in the apartment above Adam. Adam likes to listen to AC/DC (loud, preferably at night), while Eve likes to practice her harp playing skills (preferably early in the morning). According to the rules of the ownership association, who follow local law, Adam has the right to play music whenever he wants, at the level he pleases, and similarly Eve has a right to practice harp playing whenever she pleases.

Let $a$ be the hours of music played by Adam and $e$ be the hours of harp practice enjoyed by Eve. Suppose that Adam incurs a monetary cost $c_A(a) = 2a$ for $a$ hours of music playing (for electricity, as he has a very potent amplifier), while Eve incurs a monetary cost $c_B(e) = 2e$ for $e$ hours of harp playing (to cover the cost of strings).

Although only Adam can produce $a$ and only Eve can produce $e$, each neighbour’s utility depends on the level of both activities, as well as on money $m$ (used for other purposes). Adam’s utility function is given by

$$u_A(a, e, m) = m_A + 20a - a^2 - 2e$$
while Eve’s utility function is given by

\[ u_E(a, e, m) = m_B + 10e - \frac{e^2}{2} - 6a. \]

Assume that initial wealth levels are given by \( w_A \) and \( w_E \) and that these are sufficiently large so as to ensure that all solutions to the consumers’ problems are interior.

(a) Suppose Adam and Eve independently maximize utility. Write up their maximization problems and find their optimal choices \( a^* \) and \( e^* \), respectively, as well as their utility levels in equilibrium.

(b) Knowing a bit of economics, Adam and Eve start to wonder whether this can really be optimal. To analyze this, they ask you, the outside consultant, to solve for the Pareto efficient quantities of \( a \) and \( e \), denoted \( a^P \) and \( e^P \). Write up the maximization problem for the total surplus, noting that – while not the case in general – in the present case it is actually permissible to add directly the utilities of Adam and Eve (you’ll see why later in the course). Solve for the optimal quantities and describe why the results differ from \( a^* \) and \( e^* \).

Adam and Eve agree to implement the Patero-efficient levels of AC/DC and harp playing. However, they soon revert back to their normal behaviour.

The chairperson of the ownership association is called in as an arbitor. She proposes the following procedure:

i. Eve makes a proposal to Adam. The proposal consists of maximum levels of AC/DC and harp playing and a transfer \( \tau > 0 \) to be paid from (or to) the proposer to (or from) the receiver of the proposal. We represent the proposal by a vector \((a_{EP}, e_{EP}, \tau_{EP})\) where subscript \( EP \) denotes that it is Eve’s proposal.

ii. Having heard the proposal, Adam can choose to accept it, in which case it is implemented, or to reject it, in which case they both behave as before.

(c) Set up the maximization problem for Eve, noting that she is constrained by the fact that Adam can always choose to opt out of the agreement and play AC/DC for as long as he likes, whenever he pleases.

(d) Solve for Eve’s optimal proposal, that is, \( a_{EP}^*, e_{EP}^*, \) and \( \tau_{EP}^* \) (hint: substitute the constraint into the objective function and solve for \( a_{EP} \) and \( e_{EP} \), then determine \( \tau_{EP} \)). Compare Eve’s optimal proposal with the results for \( a^* \) and \( e^* \), and \( a^P \) and \( e^P \), above. Explain, in words, why we get this result.

(e) Suppose now that Adam instead gets to make Eve a proposal, rather than the other way around. Does Adam’s proposal differ from Eve’s? If so, how?
(f) The chairperson from the ownership association announces that in the case
Adam accepts Eve’s proposal, the chairperson will personally make sure that
Eve’s proposed limits are obeyed. This service is available for the modest fee
of \( c \), to be divided equally among Adam and Eve \((\frac{c}{2})\) each. How does this
alter Eve’s optimal proposal from (d)? Can the Pareto-efficient solution be
implemented no matter the cost \( c \)? Explain.

3. Two other neighbours, Bonnie \((B)\) and Clyde \((C)\), living opposite each other on a
different street, want street lighting. Their utility functions are given by

\[
U_i(m_i, G) = m_i + \gamma_i G - G^2, \quad i = B, C
\]

where \( m_i \) is money used for private consumption and \( G = g_B + g_C \) is the aggregate
supply of street lighting. Bonnie’s and Clyde’s utility functions differ only by the
parameter \( \gamma_i \), the true values of which are \( \gamma_B = 1 \) and \( \gamma_C = 2 \).

(a) Suppose that \( i \)'s budget constraint is given by

\[
m_i + g_i = w_i, \quad i = B, C,
\]

where \( w_i \) is initial wealth. Find the Pareto-efficient level of street lighting \( G^P \),
as above using the fact that the form of the utility function allows for directly
adding the utilities.

(b) We now consider a special case of the problem of private provision of this
public good. Let \( i \)'s budget constraint be given by

\[
m_i + \theta_i G = w_i, \quad i = B, C
\]

where the personalized price \( \theta_i = g_i / (g_i + g_{-i}) \) is \( i \)'s cost share.

i. Show that this formulation is equivalent to the standard budget constraint
above.

ii. For each \( i \) maximize \((*)\) with respect to \( G \) subject to \((\circ)\) taking \( \theta_i \) as given
(i.e treat it as a parameter). Due to the set-up of the problem, in the
solution both Bonnie and Clyde will prefer the same level of public goods,
\( \bar{G} \). Combine the two first order conditions with the knowledge that, by
definition, \( \theta_B + \theta_C = 1 \), to find the cost shares and the amount of total
provision \( \bar{G} \). Comment and compare with the result from (a).

(c) Suppose now that while the form of the utility functions (equation \((*)\)) is
commonly known, the parameters \( \gamma_i \) are private information, known only by
the individuals themselves. In order, therefore, to determine the optimal pro-
vision of street lighting, some sort of procedure is needed to elicit the privately
known values of the \( \gamma_i \)'s.
We will consider a particularly simple procedure:

(A) First, Bonnie and Clyde each simply report a value which may or may not be the true value. These reports are called \( r_B \) and \( r_C \) and may be any real number.

(B) Second, based on these reports, cost shares \( \theta_B \) and \( \theta_C \) are calculated and the resulting level \( G \) is determined.

In order to solve this we proceed by backward induction (note that the second stage is not a game as such: given the reports, cost shares and the amount of public goods are calculated directly with no strategic interaction): First we solve stage \( B \) taking as given the reports from stage \( A \) and then we solve for the optimal reports knowing the effects on the cost shares at stage \( B \).

i. For each \( i \) maximize \((*)\) with respect to \( G \) subject to \((\circ)\) as functions of the reports \( r_B \) and \( r_C \), to show that the cost shares can be written as

\[
\begin{align*}
\theta_B &= \frac{r_B - r_C + 1}{2} \\
\theta_C &= \frac{r_C - r_B + 1}{2}
\end{align*}
\]

and that

\[
G = \frac{r_B + r_C - 1}{4}.
\]

(Again, recall that by definition \( \theta_B + \theta_C = 1 \).)

ii. We are now at stage \( A \). Now we want to find the optimal reports \( r^*_B \) and \( r^*_C \). Set up \( i \)’s maximization problem, noting that \( \theta_i \) and \( G \) are given above, and find the optimal report \( r^*_i \) as a function of the other player’s optimal report \( r^*_{-i} \) for \( i = B, C \). Comment.

iii. Solve for \( r^*_B \) and \( r^*_C \). Compare with \( \gamma_B \) and \( \gamma_C \). Is it optimal to be truthful? Comment.