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Investment Dynamics with Common and Private Values*

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Abstract

We consider a simple investment game, where each firm observes its idiosyncratic cost of investment, and a signal correlated with investment returns. In each round, firms that have not yet invested observe the history and choose whether to invest. Our analysis differs from most of the previous herding literature in that the timing of investment decisions is endogenous. We find that the flexibility to postpone investment can be a source of inefficiency, as firms with favorable signals delay their investment and free ride off the information provided by others. However, flexibility can promote efficiency, because firms with the most favorable information choose to invest first, allowing other firms to learn. For some large markets, there is an initial surge in investment, nearly revealing the state of the economy, and outcomes are (almost) efficient. For other large markets, there is a positive probability of no investment, even when the return is high and all firms would stand to profit.

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1. Introduction

An enduring debate in Economics is whether markets efficiently aggregate the information of individual agents, and whether efficient allocations are the result. The stakes are high. If markets are informationally efficient and lead to efficient allocations, the government should not interfere with the workings of markets. On the other hand, if markets are not informationally efficient, then active government intervention could improve welfare. For example, when the economy is in recession but the investment climate has improved, that information might be dispersed among many firms who observe only a small piece of the overall picture. Firms with favorable information might postpone investment until they are confident that other firms share their assessment, thereby prolonging a recession. Properly formulated investment subsidy programs would have the potential to improve economic welfare.

This paper addresses the issue of information aggregation and allocative efficiency in a dynamic setting, where firms receive two signals and then face a sequence of decisions about whether or not to invest. One signal is correlated with the aggregate state of the economy, which in our context is the unknown return on investment shared by all firms. We assume that this "common value" signal can take one of two values, and call a firm receiving the favorable signal a type-1 firm and a firm receiving the unfavorable signal a type-0 firm. The other signal is the cost of undertaking the investment, which is firm specific and independent of the costs faced by other firms. Observing the investment decisions of other firms could be used to improve inference about the aggregate state, but firms must disentangle whether another firm invests because it receives a favorable signal about investment returns or simply has a low cost.

The closest paper in the literature is the article by Chamley and Gale (1994), who analyze a model in which the investment cost is a fixed constant, so that the signal is one-dimensional. They find that there is a unique symmetric perfect Bayesian equilibrium, and that the equilibrium is inefficient. There is a positive probability that little or no investment occurs, even when the number of firms approaches infinity and investment is profitable for everyone. Indeed, firms are no better off than in the static game, in which firms must invest without learning anything about other firms' information. We will show that this strong inefficiency result disappears when investor heterogeneity is added to the model.

Our model has a richer informational environment than Chamley and Gale (1994). We introduce signals about a firm's private cost of investment, maintaining the signals about the common investment returns. In addition, we allow all firms the opportunity to invest, while only type-1 firms have the opportunity in Chamley and Gale (1994). Introducing heterogeneous costs is important for several reasons. It makes the inference problem more realistic and interesting. An investor can be a type-1 firm, whose signal about the aggregate state of the economy is favorable. Alternatively, an investor can be a type-0 firm, whose signal about the aggregate of the economy is unfavorable, but whose investment cost is low. Heterogeneous costs allow us to contribute to the debate about informational efficiency of markets. Gul and Lundholm (1995) attribute inefficiency to the fact that investment must be all or nothing, rather than varying with the strength of the signal. We show that equilibrium for the dynamic game typically yields higher surplus than for the static game, so the gains from learning are not fully dissipated by the incentive to delay investment and observe market activity. While introducing heterogeneous costs might impede the aggregation of information, due to the more difficult inference problem, it might aid the aggregation of information, if type-1 firms with a range of private cost realizations strictly desire to invest immediately. This initial surge of investment could be highly informative. In some large economies, equilibrium approximates the full-information first-best outcome. Finally, our formulation allows for pure strategy equilibrium, while a fixed and known investment cost model requires a mixed strategy equilibrium.

We want to distinguish our approach from the large literature on multiple equilibrium and coordination failure, based on the importance of self-fulfilling expectations of the *actions* that other firms are taking.[1] To emphasize the role of information and inference, we entirely eliminate the effect of one firm's actions on another firm's payoffs. We imagine that firms are not directly competing with each other, so that profitability depends on the aggregate shock and not the expansion decisions of other firms.[2] The interaction between firms is purely

[1]See Diamond (1982), Bryant (1983,1987), Milgrom and Roberts (1990), Cooper and John (1988), and Jones and Manuelli (1992).

[2]The investment decision in our model is related to the decision to enter a market. See Dixit and Shapiro (1986), Fudenberg and Tirole (1985), Vettas (2000), and in particular, papers that incorporate private information by Bolton and Farrell (1990) and Levin and Peck (2003). The present paper introduces common values, which allows us to interpret signals as information about demand. Another major difference is that a firm's revenues do not depend on the number of entrants.

informational.

Our model is related to some of the papers on herd behavior and information cascades. Bikhchandani, Hirshleifer, and Welch (1992) and Banerjee (1992) consider a model in which each investor, in an exogenously determined sequence, faces an investment decision after privately observing a signal related to investment yields. Our model differs in the crucial timing dimension. Instead of facing an exogenously given single opportunity to invest, our firms endogenously choose when to invest, if at all. It is this ability to delay investment and free ride on the information provided by others that leads to lower and later investment, reminiscent of the Keynesian story of self-fulfilling pessimism.[3] Chari and Kehoe (2003) endogenize the timing of investment, but in each period exactly one firm receives a signal, and in their equilibrium the exogenous sequence of firms receiving signals plays a prominent role.[4]

In section 2, we present the model, and show that equilibrium can be characterized by history-dependent cutoff values. In section 3, we characterize the equilibrium for the case of two firms. As a warmup for the general model, we analyze the special case of pure common values in section 4. Costs are known, as in Chamley and Gale (1994). Although type-1 firms are no better off than in the static game, type-0 firms could be better off. In section 5, we consider the general model. To illustrate the difficulty of working with two-dimensional signals, we provide examples of "reversals," in which more investment in round 0 is good news about investment returns, but more investment is bad news after some histories. We also show that, for a range of sufficiently high discount factors, the model is well behaved. A version of Chamley and Gale's "one-step property" is shown to hold, so that the equilibrium cutoffs can be characterized. Section 6 considers asymptotic results, as the number of firms approaches infinity.

[3]Keynes (General Theory, p. 210) argues that pessimism causes consumers to reduce their demand, a sort of inaction. The reduction in consumption demand is not combined with an order for future consumption. Thus, firms could be deterred from investing, justifying the pessimism. Similarly, in our model, when a firm with a strong signal does not invest, the fact that the firm might be willing to invest in the future is not revealed to the market.

[4]Caplin and Leahy (1994) consider a model in which firms receive signals in each period, and are allowed to suspend and restart investment. Their model is closer to a war of attrition than our investment game. In Jeitchko and Taylor (2001), investors privately observe the successes and failures of their investments, and update their beliefs about the unknown probability of success in order to decide whether or not to continue the investment. Investment returns depend on the overall level of investment as well as the success parameter, and at some point a coordination avalanche occurs. See also Morris and Shin (1998) and Baliga and Sjostrom (2002) for static games exhibiting this contagion effect.

Welfare and policy implications are discussed in section 7. Concluding remarks are offered in section 8.

2. The Model

There are n risk-neutral firms or potential investors, and each firm privately observes a signal correlated with the return on investment common to all investors. Letting Z denote the investment return and X_i denote the "common value" signal of firm i, we assume that $Z \in \{0,1\}$ and $X_i \in \{0,1\}$. We also assume that the unconditional expected return is 0.5, and that signals are independent, conditional on Z. The accuracy of the signal is given by the parameter, $\alpha \in [\frac{1}{2}, 1]$:

$$pr(Z = 0 \mid X_i = 0) = pr(Z = 1 \mid X_i = 1) = \alpha.$$

When we have $\alpha = \frac{1}{2}$, common-value signals have no information content at all, and when we have $\alpha = 1$, a common-value signal fully reveals the aggregate state. Thus, the parameter α effectively captures the informativeness of the common-value signal, X_i. We call a firm that has received the high signal, $X_i = 1$, a type-1 firm and a firm that has received the low signal, $X_i = 0$, a type-0 firm.

Each firm i also privately observes a second signal, representing the idiosyncratic cost of undertaking the investment, c_i. We assume that c_i is independent of all other variables, and distributed according to the distribution function F, defined over the support, $[\underline{c}, \overline{c}]$. For the most part, we will assume that F is continuous over a nondegenerate support. However, in section 4, we consider the special case in which the investment cost is constant. The structure of signals is assumed to be common knowledge.

Impatience is measured by the discount factor, $\delta < 1$. If firm i has cost c_i and the state is Z, its profits are zero if it does not invest, and $\delta^t(Z - c_i)$ if it invests in round t. We now describe the game. First, each firm observes its signals, (X_i, c_i). In each round, starting with round 0, each firm observes the history of play, and firms not yet invested simultaneously decide whether to invest. More formally, for $t = 0, 1, ...$, denote the action of firm i in round t as $e_i^t \in \{0,1\}$, where the action, 0, represents no change in status (either not yet invested or invested in a previous round) and the action, 1, represents investing in that round. We assume that once a firm invests it remains invested. Let k^t denote the number of firms who invest in round t,

$$k^t = \sum_{i=1}^{n} e_i^t,$$

and denote the history of length t as $h^{t-1} = (k^0, k^1, ..., k^{t-1})$. We will sometimes denote the history, h^t, as (h^{t-1}, k^t). Let h denote the set of histories of any length, including the null history observed in round 0. A strategy for firm i is a mapping from signal realizations and histories into a decision of whether to invest, satisfying the restriction that a firm can change its investment status at most once.[5]

Our solution concept is symmetric Bayesian Nash equilibrium. More sophisticated concepts are not needed, because beliefs off the equilibrium path play no role in the analysis. If, after some history, h^{t-1}, $k^t = 0$ is off the equilibrium path, then it must be the case that all remaining firms are investing with probability one. After a deviation by firm i not to invest, the beliefs of other firms are irrelevant, since they have invested and have nothing more to do. If, after some history, h^{t-1}, $k^t = k > 0$ is off the equilibrium path, then it must be the case that no firm is investing. After a deviation by firm i to invest, its payoff is determined independent of the future play of the game, so the beliefs of other firms are irrelevant to the decision to deviate. The following proposition greatly simplifies the analysis, by showing that equilibrium is characterized by cutoff investment costs, such that any firm with investment cost below the cutoff will invest, if it has not already done so.

Proposition 1: *Suppose that F is continuous over the nondegenerate support, $[\underline{c}, \overline{c}]$. Then any Bayesian Nash equilibrium has the interval property. For any history, h^{t-1}, that arises with positive probability in the equilibrium, there are functions, $\beta_0(h^{t-1})$ and $\beta_1(h^{t-1})$, such that a type-0 firm (not previously invested) invests in round t if and only if $c_i \leq \beta_0(h^{t-1})$ holds, and a type-1 firm (not previously invested) invests in round t if and only if $c_i \leq \beta_1(h^{t-1})$ holds.*

Proof. A type-1 firm that invests in round t receives expected profit,

$$pr(Z = 1 \mid h^{t-1}, X_i = 1) - c_i. \tag{2.1}$$

Suppose the firm does not invest, and instead chooses continuation strategy, s_i. The equilibrium strategies of the other firms and the history, h^{t-1}, determine the

[5] With a slight abuse of notation, this restriction can be written as $e_i^t(h^{t-1}; X_i, c_i) = 1$ implies $e_i^t(h^{t-1}, k^t, ..., k^{t+\tau}; X_i, c_i) = 0$ for all $k^t, ..., k^{t+\tau}$.

expected profit from the continuation strategy, which can be written as

$$R_1(h^{t-1}, s_i) - \varphi_1(h^{t-1}, s_i)c_i, \tag{2.2}$$

where $R_1(h^{t-1}, s_i)$ denotes expected discounted revenue of a type-1 firm and $\varphi_1(h^{t-1}, s_i)c_i$ denotes expected discounted investment cost of a type-1 firm, given h^{t-1} and s_i. If, given the history, h^{t-1}, a type-1 firm with cost c_i invests in round t, it follows from (2.1) and (2.2) that

$$pr(Z = 1 \mid h^{t-1}, X_i = 1) - R_1(h^{t-1}, s_i) \geq [1 - \varphi_1(h^{t-1}, s_i)]c_i \tag{2.3}$$

holds for all continuation strategies, s_i. From (2.3), and the fact that $\varphi_1(h^{t-1}, s_i) < 1$ holds, it follows that (2.3) holds as a strict inequality for all $c_i' < c_i$ and all continuation strategies, s_i. Therefore, if a type-1 firm with cost c_i invests in round t, a type-1 firm with a lower cost also invests in round t, unless it has already invested. An identical argument applies to type-0 firms. Because F is continuous and the support is nondegenerate, the probability that a firm's cost is exactly $\beta_0(h^{t-1})$ or $\beta_1(h^{t-1})$ is zero, so assuming that firms with these cutoff costs invest is without loss of generality. This establishes the interval property of equilibrium. ■

Consider the following welfare benchmarks. *Full-information first-best* occurs if firms observe the true state and invest whenever $Z - c_i > 0$ holds. *Complete information*, which occurs if firms observe the signals of all firms (rather than observing previous investment decisions, as in our model), is the most relevant upper bound for welfare. A lower bound on welfare is the outcome of the static game, in which firms decide whether to invest in round 0 or never invest. In Chamley and Gale (1994), welfare in the dynamic game and static game are the same. For our model, however, welfare can be higher in the dynamic game for two reasons. First, type-0 firms do not have the opportunity to invest in Chamley and Gale (1994). In our model, it is possible that a type-0 firm does not want to invest initially, but learns from market activity that investment is profitable. As shown in Proposition 2, type-0 firms are no worse off, and can be better off, in the dynamic model. Second, heterogeneous costs means that not all type-1 firms are indifferent between investing in round 0 and waiting for further developments. The increase in welfare of the allocation from the dynamic game over the allocation from the static game represents the social benefit of learning.

Another potentially interesting benchmark is the outcome of the "*rigid-timing*" game, in which firms face a once-and-for-all decision whether to invest in sequence, observing the decisions of firms ahead of it in the queue. This rigid timing model, with exogenous timing of the investment decision, is usually assumed in the herding literature. This benchmark allows us to measure the advantage or disadvantage of being able to choose when to invest. The dynamic game allows firms with the most favorable signals to invest earlier, which could help inform the market. On the other hand, there is a certain amount of free-riding that occurs when a type-1 firm decides to delay (profitable) investment, because that firm does not take into account the positive externality that early investment would provide to the market.

3. The Model with Two Firms, n=2

When we have two firms, and F is continuous and strictly increasing on $[0, 1]$, equilibrium has the following simple structure. There exist cutoffs, β_0 and β_1, such that a type-0 firm invests in round 0 if and only if its cost is below β_0, and a type-1 firm invests in round 0 if and only if its cost is below β_1. If no one invests in round 0, $k^0 = 0$, then there is no further investment. If one firm invests in round 0, $k^0 = 1$, then there are cutoffs, $\beta_0(1)$ and $\beta_1(1)$, such that a type-0 firm invests in round 1 if and only if its cost is below $\beta_0(1)$, and a type-1 firm invests in round 1 if and only if its cost is below $\beta_1(1)$. There is no investment after round 1.

The four conditions that determine the four equilibrium cutoffs are: (i) a type-0 firm with cost β_0 is indifferent between investing in round 0 and investing in round 1, if and only if the other firm invests in round 0, (ii) a type-1 firm with cost β_1 is indifferent between investing in round 0 and investing in round 1, if and only if the other firm invests in round 0, (iii) if a type-0 firm with cost $\beta_0(1)$ invests in round 1 after the other firm invests in round 0, its expected profit is zero, and (iv) if a type-1 firm with cost $\beta_1(1)$ invests in round 1 after the other firm invests in round 0, its expected profit is zero.[6]

[6] For continuous distributions with support, $[\underline{c}, \overline{c}]$, the possibility of corner solutions arises. If $\underline{c}$ is high enough, a type-0 firm, and possibly a type-1 firm, with cost $\underline{c}$ might strictly prefer not to invest in round 0. If $\overline{c}$ is low enough, a type-1 firm, and possibly a type-0 firm, with cost $\overline{c}$ might strictly prefer to invest in round 0. Characterizing these corner equilibria for $n = 2$ and arbitrary support, $[\underline{c}, \overline{c}]$, is straightforward but tedious.

Proposition 2: *When there are two firms and F is continuous and strictly increasing on $[0,1]$, any solution to the following equations is an equilibrium*

$$\frac{1-\alpha-\beta_0}{\delta} = \alpha(1-\alpha)F(\beta_1)+(1-\alpha)^2F(\beta_0)-2\beta_0\alpha(1-\alpha)F(\beta_1)- \beta_0(\alpha^2+(1-\alpha)^2)F(\beta_0), \tag{3.1}$$

$$\frac{\alpha-\beta_1}{\delta} = \alpha(1-\alpha)F(\beta_0)+\alpha^2F(\beta_1)-2\beta_1\alpha(1-\alpha)F(\beta_0)- \beta_1(\alpha^2+(1-\alpha)^2)F(\beta_1), \tag{3.2}$$

$$\beta_0(1) = \frac{(1-\alpha)[\alpha F(\beta_1)+(1-\alpha)F(\beta_0)]}{(1-\alpha)[\alpha F(\beta_1)+(1-\alpha)F(\beta_0)]+\alpha[\alpha F(\beta_0)+(1-\alpha)F(\beta_1)]} \tag{3.3}$$

$$\beta_1(1) = \frac{\alpha[\alpha F(\beta_1)+(1-\alpha)F(\beta_0)]}{\alpha[\alpha F(\beta_1)+(1-\alpha)F(\beta_0)]+(1-\alpha)[\alpha F(\beta_0)+(1-\alpha)F(\beta_1)]} \tag{3.4}$$

If $1>\alpha>\frac{1}{2}$ *holds, a solution must satisfy* $\beta_1(1)>\beta_1>\frac{1}{2}>\beta_0(1)>\beta_0>0$.

Proof. See the Appendix.

Example 1: Consider the following example, with parameters, $\alpha = .75$ and $\delta = 1$, and a cost distribution that is uniform on $[0,1]$. Table 1 shows the unique symmetric equilibrium of our "flexible timing" game. For the purpose of comparison, Table 1 also shows the unique equilibrium of the rigid timing game usually studied in the herding literature.

	flexible timing game	rigid timing game
β_0	0.17913	0.25
β_1	0.66261	0.75
$\beta_0(1)$	0.37576	0.35714
$\beta_1(1)$	0.84417	0.83333
$\beta_0(0)$	no further investment	0.16667
$\beta_1(0)$	no further investment	0.64286
profit (ex ante)	0.16054	0.15848

Table 1: $n=2,\ \delta=1,\ \alpha=.75,\ c_i\sim U[0,1]$

The fact that β_0 and β_1 are lower for the flexible timing game than for the rigid timing game illustrates the incentive to delay investment, due to the option value of not investing in round 1 if the other firm did not invest in round 0. A type-0 firm with cost $c_i \in (0.17913, 0.25)$ would receive positive profit by investing in round 0, but profit is higher by waiting until round 1. Similar reasoning applies to a type-1 firm with cost, $c_i \in (0.66261, 0.75)$. The cutoffs for investing in round 1, after observing the other firm invest in round 0, are higher for the flexible timing game than the rigid timing game. The reason is that there is a stronger inference that the other firm is a type-1 firm in the flexible timing game than in the rigid timing game, because β_1/β_0 is higher. Indeed, our simple example illustrates the role heterogeneous costs play in diluting the information gathered from another firm's investment. Suppose a firm observes its rival invest in round 0, and could infer that the rival is type-1. Then the hypothetical cutoffs for investment in round 1 would be $\widetilde{\beta}_0(1) = .50$ and $\widetilde{\beta}_1(1) = .90$. The actual values are significantly lower, reflecting the fact that investment by the rival is a noisy indicator that the rival is a type-1 firm. While the rival's investment is surely good news about the aggregate state, a firm must take into account the possibility that the rival is a type-0 firm with low investment cost.

Choosing the discount factor, $\delta = 1$, allows a clean comparison of welfare in the two games, since forcing one of the firms to delay investment in the rigid timing game is not itself a source of inefficiency. Rather, any inefficiency that arises when a type-1 firm delays investment is due to the fact that the other firm cannot benefit from that information. Ex ante profit is higher for the flexible timing game, so the gains from endogenous sorting outweigh the loss due to strategic delay for this example. To put these profit values into perspective, ex ante profit in the static game (where no learning is possible) is 0.15625, and ex ante profit would be 0.25 if signals were perfectly accurate ($\alpha = 1$). The market clearly benefits from the opportunity to learn.

For other examples, the welfare comparison is reversed. If we were to modify the example so that the cost of investing is known to be 0.6 for all firms, then, in the flexible timing game, type-0 firms never invest, and type-1 firms choose a mixed strategy. Since type-1 firms are indifferent between investing in round 0 and waiting, welfare must equal welfare in the static game. However, welfare is higher in the rigid timing game, because a type-1 firm benefits from not investing in round 1 after observing the other firm choose not to invest in round 0. A similar example can be constructed where the flexible timing game has a pure strategy equilibrium, for a continuous cost distribution with support $[0.599, 0.601]$.

4. The Model with Pure Common Values

For this section only, as a warmup to the general case, we eliminate the private value component by assuming that investment costs are constant, $c_i = c$ for all i. As a result, we will be forced to consider mixed strategy equilibria. The model with pure common values is similar to the model in Chamley and Gale (1994). However, Chamley and Gale assume that only firms with the favorable signal (type-1 firms in our terminology) have the opportunity to invest. Chamley and Gale find that there is a unique symmetric equilibrium, and that welfare is the same as in the static game, with only one round of investing. In our model, in which both type-0 and type-1 firms have the opportunity to invest, type-1 firms receive the same equilibrium expected profit as in the static game, but type-0 firms can be better off in the dynamic game. Also, multiplicity of symmetric equilibrium is possible.

The most efficient symmetric equilibrium can be described as follows. After any history, h^{t-1}, if expected profit for a type-1 firm is negative, then there is no further investment. If expected profit for a type-1 firm is positive, but a type-1 firm would prefer to delay the investment decision until round $t+1$ if it could observe all firms' signals, then type-0 firms do not invest, and type-1 firms choose a mixed action, investing with probability $q(h^{t-1})$. If a type-1 firm would prefer to invest in round t, rather than delaying the investment decision until round $t+1$ after observing all firms' signals, but a type-0 firm would prefer to delay, then all type-1 firms invest in round t, but type-0 firms wait. Finally, if a type-0 firm would prefer to invest in round t, rather than delaying the investment decision until round $t+1$ after observing all firms' signals, then all firms invest.

Before characterizing this equilibrium, we introduce some notation. Let the number of firms that have invested during rounds 0 through $t-1$ be denoted as $n(h^{t-1})$,

$$n(h^{t-1}) = \sum_{\tau=0}^{t-1} k^{\tau}.$$

Let H denote the following ratio of probabilities[7]

$$H = \frac{pr(h^{t-1} \mid Z = 0)}{pr(h^{t-1} \mid Z = 1)}.$$

[7] Implicit in these conditional probabilities are the mixing probabilities chosen in rounds 0 through $t-1$, and the fact that the firm considering this history has not yet invested. For simplicity, we suppress the dependence of H on the history, h^{t-1}.

Let $P_0(h^{t-1})$ denote the probability that we have $Z = 1$, given the history and given that a type-0 firm has not yet invested, and let $P_1(h^{t-1})$ denote the probability that we have $Z = 1$, given the history and given that a type-1 firm has not yet invested. Using Bayes' rule, we have

$$P_0(h^{t-1}) = \frac{1}{1 + \frac{H\alpha}{(1-\alpha)}} \quad \text{and} \tag{4.1}$$

$$P_1(h^{t-1}) = \frac{1}{1 + \frac{H(1-\alpha)}{\alpha}} \tag{4.2}$$

Given the history, h^{t-1}, and given that the probability that a type-1 firm invests in round t is q, define $k^*(h^{t-1}, q)$ to be the smallest value of k for which we have

$$pr(Z = 1 \mid h^{t-1}, k^t = k, X_i = 1, q(h^{t-1}) = q) \geq c. \tag{4.3}$$

From (4.3) we see that $k^*(h^{t-1}, q)$ denotes the minimum number of firms that must invest in round t in order to make investment profitable for a type-1 firm in round $t+1$. Because we are considering an equilibrium in which no type-0 firm invests before all type-1 firms have invested, it is straightforward to show that the left side of (4.3) is increasing in k and decreasing in q.[8] Thus, $k^*(h^{t-1}, q)$ is a nondecreasing step function of q, with the increments occurring at values of q such that (4.3) holds with equality for some k. Define the minimum number of firms that must invest in round t in order to make investment profitable for a type-0 firm in round $t+1$ to be $k^{**}(h^{t-1}, q)$, which is found by replacing (4.3) with

$$pr(Z = 1 \mid h^{t-1}, k^t = k, X_i = 0, q(h^{t-1}) = q) \geq c.$$

Finally, it will be useful to define notation for the expected profit that a firm saves in round t, due to the option of *not* investing in round $t+1$ when $k^t = k$ occurs. We denote this profit savings for type-1 firms as θ_k, defined by

$$\begin{aligned} \theta_k(h^{t-1}, q) &= pr(Z = 1, k^t = k \mid h^{t-1}, X_i = 1, q(h^{t-1}) = q)[c-1] \\ +pr(Z &= 0, k^t = k \mid h^{t-1}, X_i = 1, q(h^{t-1}) = q)[c]. \end{aligned} \tag{4.4}$$

[8] If inequality (4.3) is not satisfied for $k^t = n - n(h^{t-1}) - 1$, then investment cannot be profitable, even if all firms are type-1. In this case, no firm ever invests, and we can set $k^*(h^{t-1}) = n$ without loss of generality. Thus, $k^*(h^{t-1})$ is well defined.

Using Bayes' rule, equation (4.4) can be simplified to

$$\begin{aligned}\theta_k(h^{t-1}, q) &= \frac{pr(k^t = k \mid h^{t-1}, Z = 1, q(h^{t-1}) = q)}{1 + \frac{H(1-\alpha)}{\alpha}}[c-1] + \\ & \frac{pr(k^t = k \mid h^{t-1}, Z = 0, q(h^{t-1}) = q)}{1 + \frac{\alpha}{H(1-\alpha)}}[c].\end{aligned} \tag{4.5}$$

For a type-0 firm, denote the profit savings as $\eta_k(h^{t-1}, q)$, defined by

$$\begin{aligned}\eta_k(h^{t-1}, q) &= pr(Z = 1, k^t = k \mid h^{t-1}, X_i = 0, q(h^{t-1}) = q)[c-1] \\ +pr(Z &= 0, k^t = k \mid h^{t-1}, X_i = 0, q(h^{t-1}) = q)[c].\end{aligned} \tag{4.6}$$

Proposition 3: *There exists an equilibrium of the following form. Given the history, h^{t-1}, if $P_1(h^{t-1}) < c$ holds, investment ceases forever. If we have*

$$\left(P_1(h^{t-1}) - c\right)\left(\frac{1-\delta}{\delta}\right) > \sum_{k=0}^{k^*(h^{t-1},1)-1} \theta_k(h^{t-1}, 1) \quad and \tag{4.7}$$

$$\left(P_0(h^{t-1}) - c\right)\left(\frac{1-\delta}{\delta}\right) < \sum_{k=0}^{k^{**}(h^{t-1},1)-1} \eta_k(h^{t-1}, 1), \tag{4.8}$$

then all type-1 firms invest (if they have not previously invested). Type-0 firms invest in round $t+1$ if and only if we have

$$P_0(h^{t-1}, k^t) > c.$$

If we have

$$\left(P_0(h^{t-1}) - c\right)\left(\frac{1-\delta}{\delta}\right) > \sum_{k=0}^{k^{**}(h^{t-1},1)-1} \eta_k(h^{t-1}, 1), \tag{4.9}$$

then all firms invest (if they have not previously invested). Finally, for the remaining case in which investment is profitable for type-1 firms, but not as profitable as waiting and observing all firms' signals, the investment probability of type-1 firms, *q, solves*[9]

$$\left(P_1(h^{t-1}) - c\right)\left(\frac{1-\delta}{\delta}\right) = \sum_{k=0}^{k^*(h^{t-1},q)-1} \theta_k(h^{t-1}, q). \tag{4.10}$$

Proof. See the Appendix.

The equilibrium characterized in Proposition 3 is the most efficient symmetric equilibrium, but other equilibria are possible as well. Whenever we have a history for which investment is profitable for type-0 firms, it is rational for all firms to invest, because there would be nothing to be gained from waiting. For example, suppose we have $c < 1 - \alpha$, so that investment in round 0 is profitable for all firms. Then there is an equilibrium in which all firms invest in round 0, nothing is learned, and welfare is the same as in the static game. The belief that all firms invest in round 0 is self-fulfilling, and the investment of type-0 firms destroys the information content of market activity. However, if there is a potential benefit from learning, the equilibrium of Proposition 3 has type-0 firms delaying their investment and learning about the state from market activity. Welfare to type-0 firms is higher than in the static game.

Proposition 4: *For the equilibrium characterized in Proposition 3, type-1 firms receive the same expected profit as in the static game,* $\max[0, \alpha - c]$*. Suppose that we have either* $n > 2$ *or* $n = 2$ *and* $c < 1/2$*. Then type-0 firms receive expected profit strictly greater than their expected profit in the static game if and only if we have (i) type-0 firms prefer observing all firms' signals, and deciding whether to invest in round* 1*, to investing in round* 0*, and (ii) there is some possibility of learning from the activity of type-1 firms,* $c < \alpha$*.*

Proof. See the Appendix.

Sharp results are available for the probability that no investment occurs when firms are sufficiently patient. Surprisingly, this probability can be large, even when $Z = 1$ occurs and the number of firms approaches infinity. Consider the

[9] If the upper limit of summation on the right side of (4.7), (4.8), (4.9), or (4.10) is negative, then we define that sum to be zero. This occurs when profits from investing in round $t + 1$ are positive, even if $k^t = 0$ holds, so the option not to invest in round $t + 1$ has zero value.

investment decision of a type-1 firm in round 0. Equation (4.10) can be written as

$$(\alpha - c)\left(\frac{1-\delta}{\delta}\right) = \sum_{k=0}^{k^*(q)-1}\left[\binom{n-1}{k}\alpha^{k+1}q^k(1-\alpha q)^{n-1-k}(c-1)\right] + \cdot \sum_{k=0}^{k^*(q)-1}\left[\binom{n-1}{k}(1-\alpha)^{k+1}q^k(1-(1-\alpha)q)^{n-1-k}c\right]. \tag{4.11}$$

It can be shown that, for δ close enough to one, we have $k^*(q) = 1$. The equilibrium mixing probability, q, must be such that the expected profit of a type-1 firm is positive if only one firm invests in round 0. In fact, expected profit following no investment is converging to zero as $\delta \to 1$. The economic intuition behind this fact is that, when firms are perfectly patient, they will strictly prefer to delay investment whenever useful information can be learned. The option value of not investing when $k^0 = 0$ must be converging to zero. Rewriting (4.11) to reflect $k^*(q) = 1$, we have

$$(\alpha - c)\left(\frac{1-\delta}{\delta}\right) = (1-\alpha q)^{n-1}\alpha(c-1) + (1-(1-\alpha)q)^{n-1}(1-\alpha)c. \tag{4.12}$$

In the limit, when $\delta = 1$ holds, it follows from (4.12) that we have

$$\left[\frac{1-(1-\alpha)q}{1-\alpha q}\right]^{n-1} = \frac{(1-c)\alpha}{(1-\alpha)c} \tag{4.13}$$

The left side of (4.13) is the ratio of the probability of no investment in the bad state of nature, $Z = 0$, to the probability of no investment in the good state, $Z = 1$, from the perspective of a firm that does not invest in round 0. For large economies, as $n \to \infty$, it must be the case that the probability of investment by a single firm is negligible, $q \to 0$, but the probability that no firm invests is strictly between zero and one. From (4.13), we can solve for the limiting probability of no investment is the good and bad states, respectively,

$$\lim_{n\to\infty}(1-\alpha q)^{n-1} = \left[\frac{(1-c)\alpha}{(1-\alpha)c}\right]^{-\alpha/(2\alpha-1)} \tag{4.14}$$

$$\lim_{n\to\infty}(1-(1-\alpha)q)^{n-1} = \left[\frac{(1-c)(1-\alpha)}{\alpha c}\right]^{(1-\alpha)/(2\alpha-1)}. \tag{4.15}$$

The probability of no investment is significant. For example, if we have $\delta \to 1$, $n \to \infty$, $c = 1/2$, and $\alpha = 2/3$, the probability of no investment in the good and bad states, respectively, are 1/4 and 1/2.

5. Results for the General Model

It is difficult to characterize equilibrium for the general model, due to the two-dimensional nature of signals. One might conjecture that $P_0(h^{t-1}, k^t)$ and $P_1(h^{t-1}, k^t)$ are weakly increasing in k^t. We show, below, that this conjecture is false, in general. We provide an example exhibiting a "reversal," in which more investment in round 0 is good news, but after some histories, more investment in later rounds is actually bad news. The intuition is that, following a certain history, a type-0 firm is more likely to invest than a type-1 firm. However, the model is well behaved if firms are reasonably patient. Proposition 5 shows that, for δ sufficiently close to one, there exists an equilibrium in which $\beta_0(h^{t-1})$ and $\beta_1(h^{t-1})$ are characterized by indifference between investing in round t and investing in round $t+1$ if and only if investment is profitable.

We continue to use the notation of section 4, where H, $P_0(h^{t-1})$, and $P_1(h^{t-1})$ now depend on the investment cost cutoffs used in rounds 0 through $t-1$, and the fact that the firm considering this history has not yet invested. We will sometimes make explicit the dependence on the most recent investment cost cutoffs, hopefully without causing confusion. Define the probability that a type-0 (respectively, type-1) firm invests in round t, after the history h^{t-1}, by

$$q_0(h^{t-1}) = \frac{F(\beta_0(h^{t-1})) - F(\beta_0(h^{t-2}))}{1 - F(\beta_0(h^{t-2}))}, \tag{5.1}$$

$$q_1(h^{t-1}) = \frac{F(\beta_1(h^{t-1})) - F(\beta_1(h^{t-2}))}{1 - F(\beta_1(h^{t-2}))}. \tag{5.2}$$

From (5.1) and (5.2), we see that finding the investment cost cutoffs for investing in round t is equivalent to finding the probabilities that a firm will invest in round t (having not yet invested). Define $K^*(h^{t-1}, q_0, q_1)$ to be the set containing all values of k^t for which investment is unprofitable in round $t+1$ for the marginal type-1 firm. Thus, we have

$$K^*(h^{t-1}, q_0, q_1) = \{k : P_1(h^{t-1}, k^t = k; q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1) < \beta_1(h^{t-1})\}.$$

Similarly, for type-0, define

$$K^{**}(h^{t-1}, q_0, q_1) = \{k : P_0(h^{t-1}, k^t = k; q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1) < \beta_0(h^{t-1})\}.$$

For the model with pure common values in section 4, we characterize an equilibrium in which only type-1 firms consider investing during the stages of the game in which learning is important. Therefore, more investment is certainly good news. Here, however, a type-0 firm could decide to invest if its cost is low enough, so it is not necessarily true that K^* and K^{**} contain all values of k^t below a cutoff. However, Proposition 5 establishes that $P_0(h^{t-1}, k; q_0, q_1)$ and $P_1(h^{t-1}, k; q_0, q_1)$ are monotonic in k, and provides a simple characterization of whether more investment is good news or bad news. We first adopt the following notation:

$$Q(h^{t-1}, k, q_0, q_1) \equiv \frac{pr(k^t = k \mid h^{t-1}, Z = 0, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1)}{pr(k^t = k \mid h^{t-1}, Z = 1, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1)}. \quad (5.3)$$

The ratio, $Q(h^{t-1}, k, q_0, q_1)$, represents the likelihood that a firm that has not yet invested observes k firms invest in round t, given that the state is low, relative to the likelihood of observing k firms invest in round t, given that the state is high. This likelihood ratio depends on the history, h^{t-1}, and depends on the probabilities that a type-1 firm and a type-0 firm invest after h^{t-1}.

Application of Bayes' rule yields

$$P_0(h^{t-1}, k; q_0, q_1) = \frac{1}{1 + \frac{\alpha}{1-\alpha} H Q(h^{t-1}, k, q_0, q_1)} \text{ and} \quad (5.4)$$

$$P_1(h^{t-1}, k; q_0, q_1) = \frac{1}{1 + \frac{1-\alpha}{\alpha} H Q(h^{t-1}, k, q_0, q_1)} \quad (5.5)$$

Proposition 5: *If $q_1(h^{t-1}) > q_0(h^{t-1})$ holds, then $P_0(h^{t-1}, k; q_0, q_1)$ and $P_1(h^{t-1}, k; q_0, q_1)$ are strictly increasing in k. If $q_1(h^{t-1}) < q_0(h^{t-1})$ holds, then $P_0(h^{t-1}, k; q_0, q_1)$ and $P_1(h^{t-1}, k; q_0, q_1)$ are strictly decreasing in k. In other words, more investment in round t implies a higher (lower) posterior probability that the state is high, if and only if a type-1 firm invests in round t with a higher (lower) probability than a type-0 firm.*

Proof. See the Appendix.

The following example shows that reversals can occur, where more investment is good news in round 0, but after some histories, more investment is bad news. Example 2 has three firms, and if exactly one firm invests in round 1, expected revenues increase. However, after one firm invests in round 0, type-0 firms are more likely than type-1 firms to invest in round 1, so investment in round 1 is bad news. Choosing an example in which firms are infinitely impatient simplifies the calculations enormously, but the qualitative features of the example continue to hold for small positive δ.

Example 2: Consider the following example, with $\alpha = .75$, $\delta = 0$, $n = 3$, and the distribution function given by,

$$F(c_i) = \frac{c_i}{5} \qquad \text{for } c_i \leq 5$$

The distribution function is uniform over the interval, $[0, 5]$, which is about as well behaved as it gets. Of course, there is a positive probability that a firm has investment costs so high that it would not want to invest even if it know that the state was high.[10]

Because firms are infinitely impatient, they will invest at the first profitable opportunity. Therefore, the equilibrium cutoffs are given by

$$\begin{aligned} \beta_0(h^{t-1}) &= P_0(h^{t-1}) \text{ and} \\ \beta_1(h^{t-1}) &= P_1(h^{t-1}) \end{aligned}$$

whenever k^{t-1} represents good news, in the sense that $P_0(h^{t-1}) > P_0(h^{t-2})$ and $P_1(h^{t-1}) > P_1(h^{t-2})$ hold.[11] If k^{t-1} represents bad news, then there is no more investment. Table 2 presents our computations of the equilibrium cutoffs for the first few rounds.

[10] Example 2 can easily be altered to have $\overline{c} \leq 1$. Example 2 is equivalent to an example in which we have $F(c_i) = \frac{c_i}{5}$ for $c_i \leq 0.9$, and $F(c_i) = \frac{41c_i}{5} - \frac{36}{5}$ for $0.9 < c_i \leq 1$.

[11] Proposition 5 shows that good news for a type-0 firm is good news for a type-1 firm, and vice versa.

history	expected revenue and cutoff for investment $P_0(h^{t-1})$	$P_1(h^{t-1})$	probability of investing $q_0(h^{t-1})$	$q_1(h^{t-1})$
null	.25	.75	.05	.15
(0)	no more investment		0	0
(1)	.344488	.825472	.019892	.017758
(2)	.480769	.892857	.048583	.033613
$(1,0)$	.344733	.825628	.0000526	.0000374
$(1,1)$	no more investment		0	0

Table 2

From Table 2, we see why reversals can occur in equilibrium. More investment in round 0 is good news, because firms that invest are likely to be type 1. If no one invests in round 0, this is bad news, and there is no further investment. If one firm invests in round 0, this is good news, as indicated by the fact that expected revenue increases for both types. Notice, however, that after one firm invests in round 0, a type-0 firm is more likely to invest than a type-1 firm. Therefore, after the history $(1,1)$, the firm that invested in round 1 is likely to be a type-0 firm. While the investment in round 0 is good news, the investment in round 1 is bad news, and investment ceases. On the other hand, if no one invests in round 1, this is good news, and we have a reversal. Notice that, if two firms invest in round 0, a type-0 firm is also more likely to invest in round 1 than a type-1 firm. Expected revenue is higher after the history $(2,0)$ than the history $(2,1)$, but since there is only one remaining uninvested firm in round 1, obviously that firm does not learn anything from market activity in round 1.

In Example 2, following a history of the form, $(1,0,0,...)$, there is always a positive probability of investment by a second firm, although this probability quickly approaches zero. Each round in which there is no investment leads firms to slightly increase their posterior probability of the good state. This scenario is reminiscent of a war of attrition, although here there is no strategic interaction between firms.[12] If a second firm ever invests, this is bad news for the remaining firm, and investment ceases.

We now return to the general model. We will adapt the notation of section 4 to incorporate heterogeneous costs. The expected profit that a type-1 firm with

[12] Reversals, in which more investment switches from being good news to being bad news, is reminiscent of the phenomenon documented by Park and Smith (2003), in which timing games can switch from a “preemptive explosion” to a war of attrition.

cost c_i saves in round t, due to the option of *not* investing in round $t+1$ when $k^t = k$ occurs, is defined by

$$\begin{aligned} \theta_k(h^{t-1}, q_0, q_1, c_i) &= pr(Z = 1, k^t = k \mid h^{t-1}, X_i = 1, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1)[c_i - 1] \\ +pr(Z &= 0, k^t = k \mid h^{t-1}, X_i = 1, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1)[c_i]. \end{aligned}$$

Using Bayes' rule, this equation can be simplified to

$$\begin{aligned} \theta_k(h^{t-1}, q_0, q_1, c_i) &= [c_i - 1]\frac{pr(k^t = k \mid h^{t-1}, Z = 1, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1)}{1 + \frac{H(1-\alpha)}{\alpha}} + \\ &\quad [c_i]\frac{pr(k^t = k \mid h^{t-1}, Z = 0, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1)}{1 + \frac{\alpha}{H(1-\alpha)}}. \quad (5.6) \end{aligned}$$

For a type-0 firm, denote the profit savings as $\eta_k(h^{t-1}, q_1, q_2, c_i)$, defined by

$$\begin{aligned} \eta_k(h^{t-1}, q_0, q_1, c_i) &= pr(Z = 1, k^t = k \mid h^{t-1}, X_i = 0, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1)[c_i - 1] \\ +pr(Z &= 0, k^t = k \mid h^{t-1}, X_i = 0, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1)[c_i]. \end{aligned}$$

Using Bayes' rule, this equation can be simplified to

$$\begin{aligned} \eta_k(h^{t-1}, q_0, q_1, c_i) &= [c_i - 1]\frac{pr(k^t = k \mid h^{t-1}, Z = 1, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1)}{1 + \frac{H\alpha}{1-\alpha}} + \\ &\quad [c_i]\frac{pr(k^t = k \mid h^{t-1}, Z = 0, q_0(h^{t-1}) = q_0, q_1(h^{t-1}) = q_1)}{1 + \frac{1-\alpha}{H\alpha}}. \quad (5.7) \end{aligned}$$

If investment costs are continuously distributed over the unit interval, and if firms are sufficiently patient, Proposition 6 characterizes an equilibrium with the following features. The investment cutoffs are such that marginal firms are indifferent between investing in round t and investing in round $t+1$, if and only if investment in round $t+1$ is profitable. Thus, Chamley and Gale's one-step property holds.[13] After any history, h^{t-1}, in which investment is unprofitable for a type-1 firm with cost $\beta_1(h^{t-2})$, then investment ceases.

[13] The one-step property is far more difficult to establish in our context, with two dimensional signals. For example, it is conceivable that the marginal type-1 firm, rather than investing in round $t+1$ when it is profitable, can receive even higher profits by waiting until round $t+2$. The marginal type-1 firm would not consider waiting, for fear that investment will cease, unless a type-0 firm might invest. However, with known costs, a type-0 firm will never invest unless all type-1 firms are investing.

Proposition 6: *Suppose that F is continuous and strictly increasing on $[0,1]$. Then for δ sufficiently close to 1, there is an equilibrium characterized as follows. Following a history, h^{t-1}, in which investment is unprofitable (for a type-0 firm with cost $\beta_1(h^{t-2})$ or a type-1 firm with cost $\beta_1(h^{t-2})$), then investment ceases. If investment is profitable, then type-1 firms and type-0 firms invest with positive probability, and the investment cutoffs solve*

$$\left(P_1(h^{t-1}) - \beta_1(h^{t-1})\right)\left(\frac{1-\delta}{\delta}\right) = \sum_{k\in K^*(h^{t-1},q_0,q_1)} \theta_k(h^{t-1},q_0,q_1,\beta_1(h^{t-1})) \tag{5.8}$$

$$\left(P_0(h^{t-1}) - \beta_0(h^{t-1})\right)\left(\frac{1-\delta}{\delta}\right) = \sum_{k\in K^{**}(h^{t-1},q_0,q_1)} \eta_k(h^{t-1},q_0,q_1,\beta_0(h^{t-1})) \tag{5.9}$$

In proving Proposition 6, it is shown that after any history in which investment is profitable, both types of firm invest with positive probability. This fact is used to establish the one-step property. Suppose firm i is indifferent between investing in round t and investing in round $t+1$ whenever investment remains profitable, as characterized by (5.8) and (5.9). If firm i found it even more profitable to delay profitable investment beyond round $t+1$ for some k^{t+1}, then the marginal firm after history (h^{t-1},k^t) also has a profitable deviation, and we eventually reach a contradiction. However, if there is no marginal firm, because there is a zero probability that a firm of a given type will invest following the history (h^{t-1},k^t), then the argument breaks down. The full-support assumption is also important for Proposition 6. If we have $\underline{c} > 0$, then there may be equilibria in which all type-0 firms begin the game by staying on the sidelines, so (5.9) might not hold. Initially, there may be a phase in which only type-1 firms have a nontrivial cutoff for investment. The assumption, δ close to one, is only used to show that both types of firm invest with positive probability (after profitable histories). Our conjecture is that this assumption is not needed, but the analysis is simplified when firms are very patient.

6. The Model with Many Firms

In the limit, as $n \to \infty$, the law of large numbers implies that the aggregate state could be known with certainty if the firms were to pool their information. How efficient will the investment market be at aggregating this information? Because we have a large market, there will be many firms with costs near $\underline{c}$, so $\underline{c}$ plays an important role in characterizing the equilibrium. Proposition 7 characterizes the round 0 investment cutoffs for the limiting equilibria, as $n \to \infty$.

Proposition 7: *Fix α, δ, $\underline{c}$, $\overline{c}$, and the continuous and strictly increasing distribution function, F. Consider a sequence of economies, indexed by n, and let (β_0^n, β_1^n) be equilibrium investment cutoffs in round 0 for the economy with n firms. Consider a convergent subsequence, where $(\beta_0^n, \beta_1^n) \to (\beta_0^*, \beta_1^*)$. Then we have the following exhaustive possibilities:*

(1) Parameters satisfy $\underline{c} > \alpha$ [Region 1], and cutoffs satisfy $\beta_0^* = \beta_1^* = \underline{c}$,

(2) Parameters satisfy $\frac{\alpha(1-\delta)}{1-\alpha\delta} < \underline{c} < \alpha$ [Region 2], and cutoffs satisfy $\beta_0^* = \beta_1^* = \underline{c}$,

(3) Parameters satisfy $\frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta} < \underline{c} < \frac{\alpha(1-\delta)}{1-\alpha\delta}$ [Region 3], and cutoffs satisfy $\beta_0^* = \underline{c}$ and $\beta_1^* = \frac{\alpha(1-\delta)}{1-\alpha\delta}$,

(4) Parameters satisfy $\underline{c} < \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta}$ [Region 4], and cutoffs satisfy $\beta_0^* = \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta}$ and $\beta_1^* = \frac{\alpha(1-\delta)}{1-\alpha\delta}$,

(5) Parameters satisfy $\overline{c} \leq 1-\alpha$, and cutoffs satisfy $\beta_0^* = \beta_1^* = \overline{c}$.

Proof. See the Appendix.

We now discuss the equilibria characterized in Proposition 7, leaving the more complicated part (2) for last. In region 1, for all firms, the cost exceeds the expected return, so no one would be willing to be the first to invest. Since no one invests in round 0, no further inference is made, and investment never occurs. This equilibrium is inefficient, because when investment returns are high, $Z = 1$, investment is profitable for all firms (ex post), yet no investment takes place. For equilibria corresponding to part (3), type-0 firms do not invest in round 0, and type-1 firms invest with probability $F(\beta_1^n)$. By the law of large numbers, the limiting fraction of firms that invest in round 0 is $\alpha F(\beta_1^*)$ if the state is high, $Z = 1$, and the limiting fraction of firms that invest in round 0 is $(1-\alpha)F(\beta_1^*)$ if the state is low, $Z = 0$. Thus, activity in round 0 reveals the state, thereby justifying the hypothesized investment behavior in round 0. For equilibria corresponding to part (4), the limiting equilibrium cutoffs for round 0 are

$$\beta_0^* = \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta} \text{ and } \beta_1^* = \frac{\alpha(1-\delta)}{1-\alpha\delta}.$$

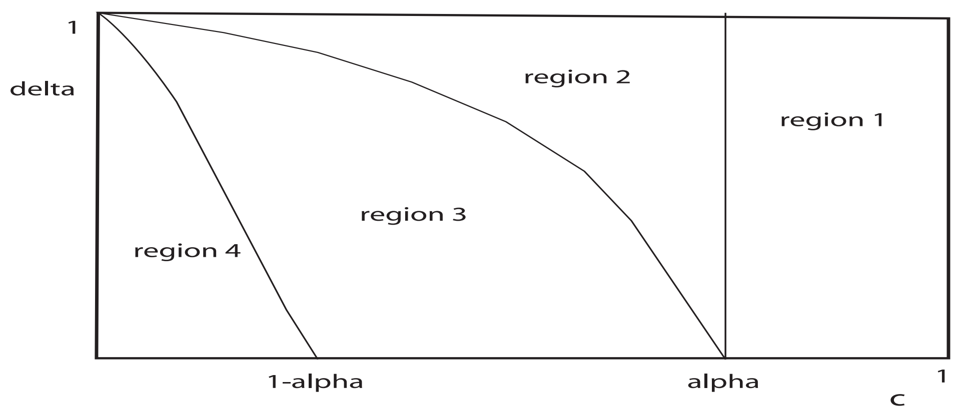

Figure 6.1:

By the law of large numbers, the limiting fraction of firms that invest in round 0 is $\alpha F(\beta_1^*) + (1-\alpha)F(\beta_0^*)$ if the state is high, $Z = 1$, and the limiting fraction of firms that invest in round 0 is $(1-\alpha)F(\beta_1^*) + \alpha F(\beta_0^*)$ if the state is low, $Z = 0$. Thus, activity in round 0 reveals the state, thereby justifying the hypothesized investment behavior in round 0.[14]

[14]This case applies whenever $\underline{c} \leq 0$ holds, which is not implausible, given our normalization that the lowest investment return is zero. In other words, we can interpret $\underline{c}$ to be the *difference* between the lowest investment cost and the lowest investment return. A plausible case, then, is that investment is profitable for firms with the lowest investment cost, even if the investment return is low.

For equilibria corresponding to part (5), which can occur in regions 2, 3, or 4 (see Figure 6.1), all firms invest in round 0. A coordination failure occurs, based on the self-fulfilling expectation that nothing will be learned by market activity. Given the parameters, this is the most inefficient equilibrium possible, because firms are held to the minimum payoff that they can guarantee simply by investing in round 0. This indiscriminate rush to invest is the opposite of the usual source of inefficiency in the model, where free riding firms delay their investment and risk a collapse. Notice that firms could be better off if the cost distribution were shifted to allow higher costs, thereby eliminating this type of equilibrium.

The more difficult and interesting case occurs when parameters are in region 2. Limiting investment cutoffs must satisfy $\beta_0^* = \beta_1^* = \underline{c}$, but if firms are certain that there will be no investment, a type-1 firm with cost close to $\underline{c}$ should invest in round 0. This paradox is resolved as follows. The logic behind the proof of Proposition 7 can be used to show that, for sufficiently large n, type-0 firms strictly prefer not to invest in round 0, $\beta_0^n = \underline{c}$. The cutoff for investment by type-1 firms is approaching $\underline{c}$, but the probability of $k^t = 0$ has a well-defined limit that is between zero and one. Thus, observing only one or two firms invest in round 0 could be good news, for all values of n. In equilibrium, there will be a positive probability that no investment occurs in the good state, so the equilibrium is inefficient. However, for all firms except the vanishing set of type-1 firms with cost below β_1^n, expected profits in the dynamic game are strictly higher than expected profits in the static game, so there are gains from learning.

Fix a particular firm i, and let $Q^{z,n-1}$ denote the probability that not one of the remaining $n-1$ firms invests in round 0, given that the state is z. For large n, we have $\beta_0^n = \underline{c}$, so that we can calculate

$$Q^{1,n-1} = (1 - \alpha q_1^n)^{n-1}, \tag{6.1}$$

$$Q^{0,n-1} = (1 - (1-\alpha) q_1^n)^{n-1}. \tag{6.2}$$

The limiting value of $Q^{z,n-1}$ is the probability that there is a complete lack of investment, given that the state is z. The procedure for calculating $Q^{z,n-1}$ starts with solving the analogue of (5.8),

$$(\alpha - \beta_1^n)\left(\frac{1-\delta}{\delta}\right) = \sum_{k \in K^*(0,q_1^n)} \theta_k(0, q_1^n, \beta_1^n) \tag{6.3}$$

for q_1^n, by guessing $K^*(0, q_1^n)$ and then verifying that the guess is correct. When firms are sufficiently patient so that $K^*(0, q_1^n) = \{0\}$ holds, equation (6.3) simplifies to

$$(\alpha - \beta_1^n)\left(\frac{1-\delta}{\delta}\right) = \alpha(\beta_1^n - 1)Q^{1,n-1} + (1-\alpha)\beta_1^n Q^{0,n-1}. \tag{6.4}$$

Unfortunately, closed-form solutions are not generally available, so we consider the special case in which $\delta \to 1$ holds. In that case, (6.4) reduces to

$$\frac{Q^{1,n-1}}{Q^{0,n-1}} = \frac{(1-\alpha)\beta_1^n}{\alpha(1-\beta_1^n)}. \tag{6.5}$$

From (6.1) and (6.2), we have

$$\frac{1-\alpha q_1^n}{1-(1-\alpha)q_1^n} = \left[\frac{Q^{1,n-1}}{Q^{0,n-1}}\right]^{1/(n-1)} \equiv \overline{Q}, \tag{6.6}$$

which yields

$$q_1^n = \frac{\overline{Q}-1}{(1-\alpha)\overline{Q}-\alpha}.$$

Therefore, we have

$$Q^{1,n-1} = (1-\alpha q_1^n)^{n-1} = \left[\frac{2\alpha-1}{\alpha(\frac{1}{\overline{Q}}+1)-1}\right]^{n-1}. \tag{6.7}$$

Combining (6.5) and (6.6), we have

$$\overline{Q} = \left[\frac{(1-\alpha)\beta_1^n}{\alpha(1-\beta_1^n)}\right]^{1/(n-1)}. \tag{6.8}$$

Substituting (6.8) into (6.7) yields

$$Q^{1,n-1} = \left[\frac{2\alpha-1}{\alpha(\left[\frac{(1-\alpha)\beta_1^n}{\alpha(1-\beta_1^n)}\right]^{-1/(n-1)}+1)-1}\right]^{n-1} \tag{6.9}$$

From (6.9), we can compute the limiting probability of no investment in the good state, as $n \to \infty$ and $\delta = 1$, by taking the logarithm of both sides and applying l'Hopital's rule, yielding

$$Q^{1,\infty} = \left(\frac{\alpha(1-\underline{c})}{(1-\alpha)\underline{c}}\right)^{-\alpha/(2\alpha-1)}. \tag{6.10}$$

Similar calculations yield

$$Q^{0,\infty} = \left(\frac{(1-\underline{c})(1-\alpha)}{\alpha \underline{c}} \right)^{(1-\alpha)/(2\alpha-1)}. \tag{6.11}$$

The above analysis started from the condition that investment by one firm was enough to maintain profitability in round 1 for a type-1 firm with cost β_1^n. This condition is satisfied with $\delta = 1$, because infinitely patient firms will wait until round 1 unless there is nothing valuable to learn by waiting; this can only occur if the marginal firm receives zero expected profits following $k^0 = 0$, so profits must be strictly positive following $k^0 = 1$. However, it can be shown that $K^*(0, q_1^n) = \{0\}$ is not a knife-edge phenomenon. By simultaneously solving (6.4), (6.7), $\beta_1^n = \underline{c}$, and the zero profit condition, $P_1(1) = \underline{c}$, we can find the boundary of parameter values for which $k^0 = 0$ is the only unprofitable news. The equation for the boundary is complicated, but in Figure 6.1, where we assume $\alpha = \frac{3}{4}$, this occurs when we are in region 2 and we have

$$\delta \geq \frac{9\sqrt{1-\underline{c}}}{9\sqrt{1-\underline{c}} + \sqrt{\underline{c}}}.$$

Comparing equations (6.10) and (6.11) with (4.14) and (4.15), we see that the probability of no investment is the same, in the model where all firms have investment cost, $\underline{c}$, and in the heterogeneous cost model with minimum cost, $\underline{c}$. The reason is that, in the heterogeneous cost model, the probability that we have $\beta_1 \to \underline{c}$, so $F(\beta_1)$ plays the same role in the analysis that the mixing probability, q, plays in the analysis of the game with homogeneous costs. Thus, a firm with cost $\underline{c}$ faces the same tradeoffs in both models.

When we observe enough investment in round 0 to keep the process moving, investment could cease at some point in the future. What can be said about investment beyond round 0, in the good state and in the bad state?

Proposition 8: *Fix α, δ, $\underline{c}$, $\overline{c}$, and the continuous and strictly increasing distribution function, F. Assume $\overline{c} = 1$. Consider a sequence of economies, indexed by n, and let $W^n(0, c_i)$ and $W^n(1, c_i)$ be equilibrium profits, conditional on being a type-0 or type-1 firm with investment cost c_i, for the economy with n firms. Consider a convergent subsequence, where $(W^n(0, c_i), W^n(1, c_i)) \to (W^*(0, c_i), W^*(1, c_i))$. Then we have:*

(1) In region 1, $W^*(0, c_i) = W^*(1, c_i) = 0$,

(2) In region 2,

$$\delta\left(\frac{\alpha-\underline{c}}{1-\underline{c}}\right)\left(\frac{1-\alpha}{\alpha}\right)(1-c_i) \leq W^*(0,c_i) \leq \frac{1}{\delta}\left(\frac{\alpha-\underline{c}}{1-\underline{c}}\right)\left(\frac{1-\alpha}{\alpha}\right)(1-c_i),$$

$$\delta\left(\frac{\alpha-\underline{c}}{1-\underline{c}}\right)(1-c_i) \leq W^*(1,c_i) \leq \frac{1}{\delta}\left(\frac{\alpha-\underline{c}}{1-\underline{c}}\right)(1-c_i),$$

(3) In region 3,

$$\begin{aligned}
W^*(0,c_i) &= \delta(1-\alpha)(1-c_i),\\
W^*(1,c_i) &= \delta\alpha(1-c_i) \quad \text{for } c_i \geq \frac{\alpha(1-\delta)}{1-\alpha\delta},\\
W^*(1,c_i) &= \alpha-c_i \quad \text{for } c_i < \frac{\alpha(1-\delta)}{1-\alpha\delta},
\end{aligned}$$

(4) In region 4,

$$\begin{aligned}
W^*(0,c_i) &= \delta(1-\alpha)(1-c_i), \quad \text{for } c_i \geq \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta},\\
W^*(0,c_i) &= 1-\alpha-c_i, \quad \text{for } c_i < \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta},\\
W^*(1,c_i) &= \delta\alpha(1-c_i) \quad \text{for } c_i \geq \frac{\alpha(1-\delta)}{1-\alpha\delta},\\
W^*(1,c_i) &= \alpha-c_i \quad \text{for } c_i < \frac{\alpha(1-\delta)}{1-\alpha\delta}.
\end{aligned}$$

Proof. See the Appendix.

The intuition for Proposition 8 is that, once the investment cutoffs are above $\underline{c}$, the fraction of firms that have invested will reveal the state. In region 1, we have no investment. In regions 3 and 4, firms that do not invest in round 0 invest in round 1 in the high state, and do not invest in the low state. In region 2, whether or not a positive fraction of firms invest depends delicately on the exact realizations of investment costs for the type-1 firms with the lowest costs. Basically, $P_1(h^{t-1})$ and $P_0(h^{t-1})$ replace α and $1-\alpha$ in determining which region the economy moves into. During the round in which the economy first moves into region 3 or region 4, a positive fraction of firms invest, but the state is not yet known. Therefore, there can be overinvestment as well as underinvestment. There is a positive probability that investment never takes off in the high state, and there is a positive probability that a positive fraction of firms invest in the low state. For this reason, it is difficult to characterize welfare when the economy begins in region 2. However, the welfare bounds we establish in part (2) are extremely accurate when δ is close to 1. For δ arbitrarily close to 1, equilibrium profits are exactly determined, and the probability of overinvestment is zero.

Suppose that we have $\delta \simeq 1$, and consider what happens to welfare as we vary $\underline{c}$. For $\underline{c} = 0$, we have $W^*(0, c_i) = (1-\alpha)(1-c_i)$ and $W^*(1, c_i) = \alpha(1-c_i)$, which implies that we achieve the first-best. Firms receive the profits they would receive if they acted with full knowledge of the state. For small $\underline{c}$, the outcome is nearly first-best efficient. There is no chance of overinvestment (a positive fraction investing in the bad state), and the chance of investment collapse (the fraction investing in the good state is zero) is small. As $\underline{c}$ rises, the chance of investment collapse rises, but if the fraction investing becomes positive, all firms will know that the state is high and invest.

Back to the general case, notice that the characterization in Proposition 8 does not assume anything about the distribution, F, except that it is continuous and strictly increasing! The only feature of F that affects equilibrium profits is the lower support, $\underline{c}$. If we start in region 3 or 4, and consider any firm, then changing the distribution in any way that lowers $\underline{c}$ will have no effect on the firm's expected profits. However, if we start in region 2, then lowering $\underline{c}$ will increase the firm's expected profits. If we start in region 1, then lowering $\underline{c}$ enough to move the economy into another region will increase the firm's expected profits; if we remain in region 1, then there is no effect.

Let us summarize the efficiency properties of large markets. Suppose all firms would delay investment if waiting allows firms to learn the state (case 1 or case 2). Then equilibrium is inefficient, free riding limits information flow and leads to underinvestment, much as in Chamley and Gale (1994). However, equilibrium yields higher welfare than the static game in case 2, unlike in Chamley and Gale (1994). When some firms receive favorable enough signals that they would not delay investment, even if waiting would allow them to learn the state (case 3 or case 4), then the market aggregates information efficiently. Even though a single firm has a blunt instrument for conveying information, invest or not, a market with a large number of firms can be highly informative.

7. Conclusions

A reasonable argument is that firms could credibly announce their signals to the market, thereby avoiding the inefficiencies arising in our model due to limited information flow. Similar criticisms could be made to much of the literatures on herding and coordination failures. We would respond to this argument on two levels. First, our analysis benefits from the useful abstraction of eliminating any strategic interaction between firms' investment decisions. Given that a firm invests, its profits depend only on the aggregate state and not on how many other firms invest. However, it remains to be seen whether firms would willingly reveal their favorable information to competing firms.[15] Second, it is costly to send and to receive communication, especially in large markets. In a model that includes a communication decision as well as an investment decision, there may be an incentive for firms to free-ride by not incurring the communication costs. As the size of the market increases, the communication cost might rise, and the information content of a single firm's communication might fall.

Our assumption of conditional independence implies that, for large economies, the state could be known for certain if firms pool their information. Future work will extend the model to information structures in which the state cannot be known with certainty. One idea is to replicate the economy, so that there are n classes of firms, with r identical firms in each class receiving identical signals. As r approaches infinity, the aggregate information possessed by all firms remains constant. A special feature of the replication economy is that firms know that there is a tie for who has the lowest cost, leading to complicated mixed-strategy equilibria, in spite of the continuous investment cost distribution. We are currently exploring this and other information structures.

A macroeconomic interpretation of the model is that the economy is in recession, but the investment climate might have improved. In equilibrium, firms with favorable signals might delay their investment, and there is a positive probability that no one invests, even if the investment climate has improved and the recession should be over. Although this is the implication of the theory, notice that firms with low investment costs are almost indifferent between investing in round 0 and waiting, and that common knowledge of rationality might be a strong assumption

[15] Although investment by competing firms might be harmful to investment returns, the opposite might be true. In line with the coordination failure literature, investment by other firms could stimulate economic activity and increase investment returns. Here, the question is whether a firm would want to admit that it has a weak signal.

in practice. Some type-1 firms might instead see profitable opportunities and invest, as in Keynes' notion of *animal spirits*. The fascinating point here is that this urge to invest can actually improve the informativeness of markets, thereby improving economic efficiency! This phenomenon would be interesting to test experimentally.

8. Appendix

Proof of Proposition 2. Suppose we have a solution to (3.1)-(3.4). Given β_0 and β_1, equations (3.3) and (3.4) imply that a type-0 firm with cost $\beta_0(1)$ and a type-1 firm with cost $\beta_1(1)$ receive zero expected profits by investing in round 1. If firms with lower costs invest in round 1 and firms with higher costs do not invest, there is no profitable deviation in round 1 or afterwards, following investment by the other firm in round 0.

Equations (3.1) and (3.2) imply that a firm with cost β_0 or $\beta_1$are indifferent between investing in round 0 or investing in round 1 after the other firm invested in round 0. If it were also profitable for a firm with cost β_0 or β_1 to invest in round 1 after the other firm did not invest in round 0, then *always* investing in round 1 is more profitable than investing in round 0, a contradiction. Therefore, after the other firm does not invest in round 0, a firm with cost β_0 or β_1 would receive negative profits from investing in round 1. This implies (i) a firm with cost below β_0 or β_1 is choosing a best response by investing in round 0, (ii) a firm with cost above β_0 or β_1 is better off not investing in round 0, and (iii) if the other firm does not invest in round 0, it cannot be profitable to invest in round 1 or afterwards. The reason for (iii) is that any firm that could profitably invest in round 1, after the other firm did not invest in round 0, must have a cost below β_0 or β_1, but then the firm would have invested in round 0.

The above analysis implies $\beta_0(1) > \beta_0 > 0$ and $\beta_1(1) > \beta_1$. We must have $\frac{1}{2} > \beta_0(1)$, because the expected return, given there is one type-1 firm and one type-0 firm, is $\frac{1}{2}$. However, the type-0 firm that observes the other firm invest is not sure that the other firm is type-1. We must have $\beta_1 > \frac{1}{2}$, because otherwise a type-1 firm would find it profitable to invest even if the other firm did not invest in round 0. ■

Proof of Proposition 3. If $P_1(h^{t-1}) < c$ holds, then investment is unprofitable for all firms, so no one invests in round t. Since nothing is learned from $k^t = 0$, investment remains unprofitable in round $t+1$, and so on.

If (4.7) and (4.8) hold, then a type-1 firm receives higher profits by investing than waiting and observing all firms' signals. Therefore, no deviation from investing can be profitable. Because, in equilibrium, all type-1 firms invest and all type-0 firms wait, type-0 firms are able to infer all firms' signals. By (4.8), a type-0 firm receives higher profits by waiting until round $t+1$, and investing in round $t+1$ if and only if we have $P_0(h^{t-1}, k^t) > c$. [Clearly, there is no incentive to delay investment beyond round $t+1$.]

If (4.9) holds, then (4.7) must hold as well. For both types of firms, investing in round t is more profitable than waiting, even if all firms' signals could be inferred by waiting. Therefore, no deviation from investing can be profitable.

Finally, consider the remaining case, where $P_1(h^{t-1}) > c$ holds and (4.7) does not hold. If there is a q satisfying (4.10), then a type-1 firm is indifferent between investing in round t and investing in round $t+1$ whenever profits remain nonnegative. For such a q, a type-1 firm is willing to randomize.[16] We must also show that there is no continuation strategy, in which a type-1 firm invests in round $t+2$ or later, yielding strictly higher profits. Fix k^t, and suppose that $P_1(h^{t-1}, k^t) < c$ holds. There will never be more investment by other firms, so nothing can be learned and investment can never be profitable. Now suppose that inequality (4.7) holds when we replace the history h^{t-1} with the history (h^{t-1}, k^t). Investing in round $t+1$ dominates not investing. Suppose instead that neither of the previous two cases applies, so that equation (10) holds when we replace the history h^{t-1} with the history (h^{t-1}, k^t). It follows that a type-1 firm is indifferent between investing in round $t+1$ and investing in round $t+2$ whenever profits remain nonnegative. Proceeding inductively, we see that there is no continuation strategy yielding higher profits than investing in round t.

To complete the proof, we must show that there is a q satisfying (4.10) following the history, h^{t-1}. Since (4.7) does not hold, it follows that the right side of (4.10) is greater than the left side, for $q = 1$. For $q = 0$, it is straightforward to show that $k^*(h^{t-1}, 0) = 0$ holds. Thus, the left side of (4.10), which is positive, exceeds the right side of (4.10), which is zero. If we can show that the right side of (4.10) is continuous in q, there must be a q solving (4.10), by the intermediate value theorem.

From (4.5), it is obvious that $\theta_k(h^{t-1}, q)$ is continuous in q for each k. Thus, the right side of (4.10) consists of the sum of $k^*(h^{t-1}, q)$ continuous functions,

[16] It is straightforward to show that, if a type-1 firm is indifferent, a type-0 firm strictly prefers not to invest in round t.

where the number of such functions is an integer that depends on q. The only possible source of discontinuity occurs when q is such that $k^*(h^{t-1}, q)$ jumps from κ to $\kappa+1$. From (4.3), this occurs when we have

$$pr(Z = 1 \mid h^{t-1}, k^t = \kappa + 1, X_i = 1, q(h^{t-1}) = q) = c. \tag{8.1}$$

From Bayes' rule, equation (8.1) is equivalent to

$$\frac{1}{1 + \frac{(1-\alpha)H}{\alpha} T_{\kappa+1}(h^{t-1}, q)} = c, \tag{8.2}$$

where $T_k(h^{t-1}, q)$ is defined by

$$T_k(h^{t-1}, q) = \frac{pr(k^t = k \mid Z = 0, h^{t-1}, q(h^{t-1}) = q)}{pr(k^t = k \mid Z = 1, h^{t-1}, q(h^{t-1}) = q)}. \tag{8.3}$$

From (4.5) and (8.3), we can write

$$\frac{\theta_{\kappa+1}(h^{t-1}, q)}{pr(k^t = \kappa + 1 \mid h^{t-1}, Z = 1, q(h^{t-1}) = q)} = \frac{c - 1}{1 + \frac{H(1-\alpha)}{\alpha}} + \frac{cT_{\kappa+1}(h^{t-1}, q)}{1 + \frac{\alpha}{H(1-\alpha)}}. \tag{8.4}$$

Substituting (8.2) into (8.4) and simplifying, it follows that $\theta_{\kappa+1}(h^{t-1}, q) = 0$ holds. Thus, as we transition from $k^*(h^{t-1}, q) = \kappa$ to $k^*(h^{t-1}, q) = \kappa + 1$, the last term in the summation is zero, so that the right side of (4.10) is continuous in q everywhere. ■

Proof of Proposition 4. If $c > \alpha$ holds, then there is no investment in the dynamic game or the static game. If $c < \alpha$ holds, then in the equilibrium of Proposition 1, type-1 firms invest in round 0 with positive probability. Therefore, either all type-1 firms invest in round 0, and receive expected profit, $\alpha - c$, or some type-1 firms are indifferent between investing in round 0 and waiting, so their expected profit is also $\alpha - c$.

If condition (i) is not satisfied, then all firms invest in round 0. If condition (ii) is not satisfied, then there is no investment in the dynamic game. Welfare in the dynamic and static games are the same. If conditions (i) and (ii) are satisfied, there are two cases to consider. Case 1: type-1 firms invest in round 0 with probability one. If profit is positive in the static game, then by condition (i), type-0 firms receive higher profit in the dynamic game by delaying investment and learning the signals of all firms. If profit is zero in the static game, type-0 firms receive positive expected profit in the dynamic game, because there is a positive probability that all other firms are type-1.[17] Case 2: type-1 firms are investing in round 0 with probability strictly between zero and one. Then type-0 firms strictly prefer to wait rather than invest in round 0. If profit is positive in the static game, then it is higher in the dynamic game. If profit is zero in the static game, type-0 firms receive positive expected profit in the dynamic game, because there is a positive probability that all other firms are type-1 and choose to invest in round 0. ■

Proof of Proposition 5. We will derive and simplify expressions for the numerator and denominator of $Q(h^{t-1}, k, q_0, q_1)$. These expressions are complicated by the fact that, given the state, we do not know the number of type-1 firms who have not yet invested at the beginning of round t. Let $\overline{\alpha}_0$ denote the probability that a firm is of type-0, given that the state is low, given the history, h^{t-1}, and given that the firm has not invested by the beginning of round t. Similarly, let $\overline{\alpha}_1$ denote the probability that a firm is of type-1, given that the state is high, given the history, h^{t-1}, and given that the firm has not invested by the beginning of round t. From Bayes' rule, we have

$$\overline{\alpha}_0 = pr(X_j = 0 \mid h^{t-1}, Z = 0, j \text{ not invested}) = \frac{1}{1 + \frac{1-\alpha}{\alpha}\left[\frac{1-F(\beta_1(h^{t-2})}{1-F(\beta_0(h^{t-2})}\right]}, \quad (8.5)$$

$$\overline{\alpha}_1 = pr(X_j = 1 \mid h^{t-1}, Z = 1, j \text{ not invested}) = \frac{1}{1 + \frac{1-\alpha}{\alpha}\left[\frac{1-F(\beta_0(h^{t-2})}{1-F(\beta_1(h^{t-2})}\right]}. \quad (8.6)$$

The following probabilities are conditional on firm i not having invested before round t. Let $\overline{n}$ denote the number of firms that have not yet invested before round t, not including firm i. Of these firms, let κ denote the number of type-1 firms, and let k_1 denote the number of these type-1 firms that invest in round t. Then we can write:

$$pr(k^t = k \mid h^{t-1}, Z = 0) = \sum_{k_1=0}^{k} \sum_{\kappa=k_1}^{\overline{n}} \left[\begin{array}{c} \binom{\overline{n}}{\kappa}\overline{\alpha}_0^{\overline{n}-\kappa}(1-\overline{\alpha}_0)^{\kappa}\left[\binom{\kappa}{k_1}q_1^{k_1}(1-q_1)^{\kappa-k_1}\right] \\ \left[\binom{\overline{n}-\kappa}{k-k_1}q_0^{k-k_1}(1-q_0)^{\overline{n}-\kappa-k+k_1}\right] \end{array} \right] \quad (8.7)$$

[17] If all $n-1$ other firms are type-1 and $n > 2$ holds, then expected profit is positive for the remaining type-0 firm. If there are only two firms, then we use the additional condition, $c < 1/2$.

and

$$pr(k^t = k \mid h^{t-1}, Z=1) = \sum_{k_1=0}^{k}\sum_{\kappa=k_1}^{\overline{n}} \left[\begin{array}{c} \binom{\overline{n}}{\kappa}\overline{\alpha}_1^{\kappa}(1-\overline{\alpha}_1)^{\overline{n}-\kappa}\left[\binom{\kappa}{k_1}q_1^{k_1}(1-q_1)^{\kappa-k_1}\right] \\ \left[\binom{\overline{n}-\kappa}{k-k_1}q_0^{k-k_1}(1-q_0)^{\overline{n}-\kappa-k+k_1}\right] \end{array} \right]. \tag{8.8}$$

Equations (8.7) and (8.8) can be simplified as follows:[18]

$$pr(k^t = k \mid h^{t-1}, Z=0) = \binom{\overline{n}}{k} A^k (1-A)^{n-k} \quad \text{and} \tag{8.9}$$

$$pr(k^t = k \mid h^{t-1}, Z=1) = \binom{\overline{n}}{k} B^k (1-B)^{n-k}, \tag{8.10}$$

where $A \equiv \overline{\alpha}_0 q_0 + (1-\overline{\alpha}_0)q_1$ and $B \equiv (1-\overline{\alpha}_1)q_0 + \overline{\alpha}_1 q_1$. From (5.3), (8.9), and (8.10), we have

$$Q(h^{t-1}, k, q_0, q_1) = \left(\frac{1-A}{1-B}\right)^{\overline{n}} \left(\frac{A(1-B)}{B(1-A)}\right)^{k}. \tag{8.11}$$

Equations (5.4) and (5.5) imply that $P_0(h^{t-1}, k; q_0, q_1)$ and $P_1(h^{t-1}, k; q_0, q_1)$ are increasing in k if and only if $Q(h^{t-1}, k, q_0, q_1)$ is decreasing in k. From (8.11), it follows that $Q(h^{t-1}, k, q_0, q_1)$ is decreasing in k if and only if $B > A$ holds, which is equivalent to the condition,

$$(\overline{\alpha}_0 + \overline{\alpha}_1 - 1)(q_1 - q_0) > 0. \tag{8.12}$$

Algebraic manipulation of equations (8.5) and (8.6) establishes that $\overline{\alpha}_0 + \overline{\alpha}_1 - 1$ must be positive, and the result follows. ■

Proof of Proposition 6. Define $G_0(h^{t-1}, q_0, q_1)$ and $G_1(h^{t-1}, q_0, q_1)$ by

$$\begin{aligned} G_1(h^{t-1}, q_0, q_1) &= P_1(h^{t-1}) - \beta_1(h^{t-1}) \\ &\quad + \delta \sum_{k \notin K^*(h^{t-1}, q_0, q_1)} \theta_k(h^{t-1}, q_0, q_1, \beta_1(h^{t-1})) \quad \textit{and} \end{aligned} \tag{8.13}$$

$$\begin{aligned} G_0(h^{t-1}, q_0, q_1) &= P_0(h^{t-1}) - \beta_0(h^{t-1}) \\ &\quad + \delta \sum_{k \notin K^{**}(h^{t-1}, q_0, q_1)} \eta_k(h^{t-1}, q_0, q_1, \beta_0(h^{t-1})). \end{aligned} \tag{8.14}$$

[18] These expressions were simplified with the help of Maple 7.00.

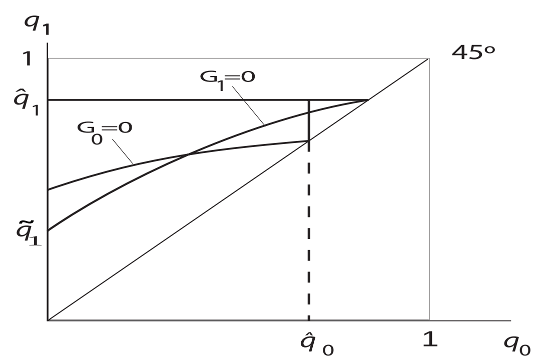

Figure 8.1:

In (8.13) and (8.14), we treat $\beta_0(h^{t-1})$ as a function of q_0 and $\beta_1(h^{t-1})$ as a function of q_1, based on (5.1) and (5.2). Notice that, since we are summing over realizations of k^t for which profits are nonnegative for the marginal firm, the summations in (8.13) and (8.14) are (weakly) negative. It is easy to see that $\theta_k(h^{t-1}, q_0, q_1, \beta_1(h^{t-1}))$ and $\eta_k(h^{t-1}, q_0, q_1, \beta_1(h^{t-1}))$ are continuous functions of q_0 and q_1. Suppose (q_0, q_1) is a point of discontinuity in $K^*(h^{t-1}, q_0, q_1)$, in which k is dropped from the set. This can only occur when $\theta_k(h^{t-1}, q_0, q_1, \beta_1(h^{t-1})) = 0$ holds. Therefore, $G_1(h^{t-1}, q_0, q_1)$ is continuous in (q_0, q_1). An identical argument holds for $G_0(h^{t-1}, q_0, q_1)$. We now show that, in $q_0 - q_1$ space, the manifold satisfying $G_1(h^{t-1}, q_0, q_1) = 0$ and the manifold satisfying $G_0(h^{t-1}, q_0, q_1) = 0$ intersect (see Figure 8.1).

Suppose that $\beta_1(h^{t-2})$ and $\beta_0(h^{t-2})$ have been characterized by $G_1(h^{t-2}, q_0, q_1) = 0$ and $G_0(h^{t-2}, q_0, q_1) = 0$, and suppose that k^{t-1} is good news for both types, in the sense that a type-1 firm with cost $\beta_1(h^{t-2})$ and a type-0 firm with cost

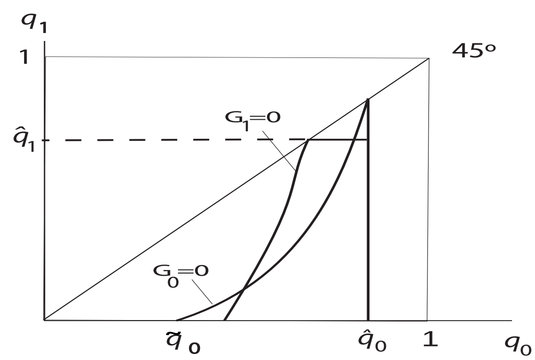

Figure 8.2:

$\beta_0(h^{t-2})$ would receive strictly positive expected profits by investing in round t. Let $\widehat{q}_1$ denote the value of q_1 for which a type-1 firm with the corresponding cost, $\beta_1(h^{t-1})$ based on (5.2), receives zero expected profits by investing in round t, $P_1(h^{t-1}) - \beta_1(h^{t-1}) = 0$. Similarly, let $\widehat{q}_0$ denote the value of q_0 for which a type-0 firm with the corresponding cost, $\beta_0(h^{t-1})$ based on (5.1), receives zero expected profits by investing in round t, $P_0(h^{t-1}) - \beta_0(h^{t-1}) = 0$.

Claim 1: There exists $\delta^* < 1$ such that $\delta^* < \delta < 1$ and $\widehat{q}_1 > \widehat{q}_0$ implies there exists (q_0, q_1), with $0 < q_0 < q_1$, such that $G_0(h^{t-1}, q_0, q_1) = 0$ and $G_1(h^{t-1}, q_0, q_1) = 0$ hold.

Proof of Claim 1: Because more investment increases expected revenues when $q_0 < q_1$ holds, it follows that we have $P_1(h^{t-1}, \overline{n}) > P_1(h^{t-1})$. A firm with zero expected profits from investing in round t would have positive expected profits by investing in round $t+1$ following $k^t = \overline{n}$, so we have $\theta_{\overline{n}}(h^{t-1}, q_0, \widehat{q}_1, \beta_1(h^{t-1})) < 0$ for all $q_0 < \widehat{q}_1$. It follows that $G_1(h^{t-1}, q_0, \widehat{q}_1) < 0$ holds for all $q_0 < \widehat{q}_1$. In Figure 8.1, $G_1 < 0$ holds along the entire segment with height $\widehat{q}_1$, between the vertical axis and the 45° line. We also know that $G_1(h^{t-1}, q, q) > 0$ holds for all $q < \widehat{q}_1$, because of the following argument. A lower probability of investing corresponds to a lower investment cost, $\beta_1(h^{t-1})$ based on (5.2), which implies positive expected revenue. Since we are restricting ourselves to $q_0 = q_1$, nothing can be learned from activity in round t, so all realizations of k^t are profitable and the sum in (8.13) equals $-[P_1(h^{t-1}) - \beta_1(h^{t-1})]$. Since we have $\delta < 1$, $G_1(h^{t-1}, q, q) > 0$ holds for all $q < \widehat{q}_1$. In Figure 8.1, $G_1 > 0$ holds along the 45° line below the point $(\widehat{q}_1, \widehat{q}_1)$. By continuity, the $G_1 = 0$ manifold must go from the point, $(\widehat{q}_1, \widehat{q}_1)$ to somewhere on the vertical axis, lying below the segment, $q_1 = \widehat{q}_1$ and above the 45° line. Denote the vertical intercept of the $G_1 = 0$ manifold by $\widetilde{q}_1$, which satisfies $G_1(h^{t-1}, 0, \widetilde{q}_1) = 0$.[19]

By the same reasoning, it follows that $G_0(h^{t-1}, \widehat{q}_0, q_1) < 0$ holds for all $q_1 > \widehat{q}_0$. In Figure 8.1, $G_0 < 0$ holds along the entire vertical segment with $q_0 = \widehat{q}_0$, between the points, $(\widehat{q}_1, \widehat{q}_1)$ and $(\widehat{q}_0, \widehat{q}_1)$. We also know that $G_0(h^{t-1}, q, q) > 0$ holds for all $q < \widehat{q}_0$, because of the following argument. A lower probability of investing corresponds to a lower investment cost, $\beta_0(h^{t-1})$ based on (5.1), which implies positive expected revenue. Since we are restricting ourselves to $q_0 = q_1$, nothing can be learned from activity in round t, so all realizations of k^t are profitable and the sum in (8.14) equals $-[P_0(h^{t-1}) - \beta_0(h^{t-1})]$. Since we have $\delta < 1$, $G_0(h^{t-1}, q, q) > 0$ holds for all $q < \widehat{q}_0$. In Figure 8.1, $G_0 > 0$ holds along

[19] Our conjecture is that the vertical intercept is unique, but otherwise let $\widetilde{q}_1$ be the intercept of the $G_1 = 0$ manifold that first reached along the path from the point $(\widehat{q}_1, \widehat{q}_1)$.

the 45° line below the point $(\widehat{q}_0, \widehat{q}_0)$. To complete claim 1, we will show that $G_0(h^{t-1}, 0, \widetilde{q}_1) > 0$ holds, because then as we move along the $G_1 = 0$ manifold from $(0, \widetilde{q}_1)$ to $(\widehat{q}_1, \widehat{q}_1)$, we would go from $G_0 < 0$ to $G_0 > 0$; by continuity there would be a point for which we have $G_1 = 0$ and $G_0 = 0$.

We now show that $G_0(h^{t-1}, 0, \widetilde{q}_1) > 0$ holds, first by making the argument for $\delta = 1$ and then by extending the argument to $\delta^* < \delta < 1$. From (5.7), $\delta = 1$, and $G_0(h^{t-2}, q_0(h^{t-2}), q_1(h^{t-2})) = 0$, (8.14) can be rewritten as

$$0 = \sum_{k \in K^{**}(h^{t-2}, q_0, q_1)} \eta_k(h^{t-2}, q_0(h^{t-2}), q_1(h^{t-2}), \beta_0(h^{t-2})),$$

which can be simplified to

$$0 = [\beta_0(h^{t-2}) - 1](1-\alpha)S_1 + \alpha H^{t-2} \beta_0(h^{t-2}) S_0, \tag{8.15}$$

where we have

$$\begin{aligned}
H^{t-2} &\equiv \frac{pr(h^{t-2} \mid Z = 0)}{pr(h^{t-2} \mid Z = 1)}, \\
S_0 &\equiv \sum_{k \in K^{**}(h^{t-2}, q_0, q_1)} pr(k^{t-1} = k \mid h^{t-2}, Z = 0, q_0(h^{t-2}) = q_0, q_1(h^{t-2}) = q_1), \\
S_1 &\equiv \sum_{k \in K^{**}(h^{t-2}, q_0, q_1)} pr(k^{t-1} = k \mid h^{t-2}, Z = 1, q_0(h^{t-2}) = q_0, q_1(h^{t-2}) = q_1).
\end{aligned}$$

From (5.6), $\delta = 1$, and $G_1(h^{t-2}, q_0(h^{t-2}), q_1(h^{t-2})) = 0$, (8.13) can be rewritten as

$$0 = \sum_{k \in K^{*}(h^{t-2}, q_0, q_1)} \theta_k(h^{t-2}, q_0(h^{t-2}), q_1(h^{t-2}), \beta_1(h^{t-2})),$$

which can be simplified to

$$0 = [\beta_1(h^{t-2}) - 1]\alpha S_1 + (1-\alpha) H^{t-2} \beta_1(h^{t-2}) S_0. \tag{8.16}$$

Combining (8.15) and (8.16), we have[20]

$$\frac{1 - \beta_0(h^{t-2})}{\beta_0(h^{t-2})} = \frac{\alpha^2}{(1-\alpha)^2} \frac{1 - \beta_1(h^{t-2})}{\beta_1(h^{t-2})}. \tag{8.17}$$

[20]For δ sufficiently close to 1, $G_0(h^{t-2}, q_0, q_1) = 0$ and $G_1(h^{t-2}, q_0, q_1) = 0$ imply $K^*(h^{t-2}, q_0, q_1) = K^{**}(h^{t-2}, q_0, q_1)$. Based on Proposition 5, we must have either $K^* = K^{**} = \{0\}$, or $K^* = K^{**} = \{\overline{n}\}$. Profit savings from not investing in round $t-1$ must be near zero for the marginal firm, so profit savings from not investing after the worst news are near zero. All other events represent favorable news.

From $\delta = 1$ and $G_1(h^{t-1}, 0, \widetilde{q}_1) = 0$, we have $K^*(h^{t-1}, 0, \widetilde{q}_1) = \{0\}$. The profit savings from not investing in round $t+1$ are zero, so it follows from Proposition 5 that the only (weakly) unprofitable event is $k^t = 0$. Manipulating (5.6) and $\theta_0(h^{t-1}, 0, \widetilde{q}_1, \beta_1(h^{t-1})) = 0$ yields

$$\frac{1-\beta_1(h^{t-1})}{\beta_1(h^{t-1})} = \frac{(1-\alpha)HQ(h^{t-1}, 0, 0, \widetilde{q}_1)}{\alpha}. \tag{8.18}$$

Also, $G_0(h^{t-1}, 0, \widetilde{q}_1) > 0$ holds if and only if $\eta_0(h^{t-1}, 0, \widetilde{q}_1, \beta_0(h^{t-2})) < 0$ holds. Manipulating (5.7), the condition becomes

$$\frac{1-\beta_0(h^{t-2})}{\beta_0(h^{t-2})} > \frac{\alpha HQ(h^{t-1}, 0, 0, \widetilde{q}_1)}{1-\alpha}. \tag{8.19}$$

Combining (8.17)-(8.19), $G_0(h^{t-1}, 0, \widetilde{q}_1) > 0$ holds if and only if

$$\frac{\alpha^2}{(1-\alpha)^2}\frac{1-\beta_1(h^{t-2})}{\beta_1(h^{t-2})} = \frac{1-\beta_0(h^{t-2})}{\beta_0(h^{t-2})} > \frac{\alpha^2}{(1-\alpha)^2}\frac{1-\beta_1(h^{t-1})}{\beta_1(h^{t-1})}. \tag{8.20}$$

Since we have $\widetilde{q}_1 > 0$, it must be the case that $\beta_1(h^{t-1}) > \beta_1(h^{t-2})$ holds, so inequality (8.20) is always satisfied.

We now show that the argument extends to δ close to one. For all $\varepsilon > 0$, there must exist δ^* such that, for $\delta^* < \delta < 1$, the right sides of (8.15) and (8.16) are sufficiently close to zero that the difference between the left side and the right side of (8.17) is less than ε. The difference between the left side and the right side of equation (8.18) can be made less than ε as well. However, the difference between the left side and the right side of (8.19) is bounded above zero, because $\widetilde{q}_1$ is bounded above zero.[21] Therefore, for small enough ε, the first term must exceed the third term in (8.20). This completes claim 1.

Claim 2: There exists $\delta^* < 1$ such that $\delta^* < \delta < 1$ and $\widehat{q}_1 < \widehat{q}_0$ implies there exists (q_0, q_1), with $0 < q_1 < q_0$, such that $G_0(h^{t-1}, q_0, q_1) = 0$ and $G_1(h^{t-1}, q_0, q_1) = 0$ hold.[22]

Proof of Claim 2: An argument identical to the proof of claim 1 establishes that the $G_0 = 0$ manifold below the 45° line connects the points, $(\widetilde{q}_0, 0)$ to $(\widehat{q}_0, \widehat{q}_0)$.

[21] This is because, if $q_0 = 0$ and q_1 is positive but close to zero, $k^{t-1} = 0$ will not be sufficiently bad news to make investment unprofitable for a type-0 firm with cost $\beta_0(h^{t-2})$.

[22] For the remaining knife-edge possibility, $\widehat{q}_1 = \widehat{q}_0$, then $G_0(h^{t-1}, \widehat{q}_0, \widehat{q}_1) = 0$ and $G_1(h^{t-1}, \widehat{q}_0, \widehat{q}_1) = 0$ hold.

See Figure 8.2. Since $G_1(h^{t-1}, \widehat{q}_0, \widehat{q}_0) < 0$ must hold, we are done if we can show $G_1(h^{t-1}, \widetilde{q}_0, 0) > 0$, because G_1 would have to become zero somewhere along the $G_0 = 0$ manifold. As shown in the proof of claim 1, we have

$$\frac{1-\beta_0(h^{t-2})}{\beta_0(h^{t-2})} = \frac{\alpha^2}{(1-\alpha)^2}\frac{1-\beta_1(h^{t-2})}{\beta_1(h^{t-2})}. \tag{8.21}$$

From $\delta = 1$ and $G_0(h^{t-1}, \widetilde{q}_0, 0) = 0$, we have $K^*(h^{t-1}, \widetilde{q}_0, 0) = \{\overline{n}\}$. The profit savings from not investing in round $t+1$ are zero, so it follows from Proposition 5 that the only (weakly) unprofitable event is $k^t = \overline{n}$. Manipulating (5.7) and $\eta_{\overline{n}}(h^{t-1}, \widetilde{q}_0, 0, \beta_0(h^{t-1})) = 0$ yields

$$\frac{1-\beta_0(h^{t-1})}{\beta_0(h^{t-1})} = \frac{\alpha HQ(h^{t-1}, \overline{n}, \widetilde{q}_0, 0)}{1-\alpha}. \tag{8.22}$$

Also, $G_1(h^{t-1}, \widetilde{q}_0, 0) > 0$ holds if and only if $\theta_{\overline{n}}(h^{t-1}, \widetilde{q}_0, 0, \beta_1(h^{t-2})) < 0$ holds. Manipulating (5.6), the condition becomes

$$\frac{1-\beta_1(h^{t-2})}{\beta_1(h^{t-2})} > \frac{(1-\alpha)HQ(h^{t-1}, \overline{n}, \widetilde{q}_0, 0)}{\alpha}. \tag{8.23}$$

Combining (8.21)-(8.23), $G_1(h^{t-1}, \widetilde{q}_0, 0) > 0$ holds if and only if

$$\frac{(1-\alpha)^2}{\alpha^2}\frac{1-\beta_0(h^{t-2})}{\beta_0(h^{t-2})} = \frac{1-\beta_1(h^{t-2})}{\beta_1(h^{t-2})} > \frac{(1-\alpha)^2}{\alpha^2}\frac{1-\beta_0(h^{t-1})}{\beta_0(h^{t-1})}. \tag{8.24}$$

Since we have $\widetilde{q}_0 > 0$, it must be the case that $\beta_0(h^{t-1}) > \beta_0(h^{t-2})$ holds, so inequality (8.24) is always satisfied. An argument identical to that in the proof of claim 1 extends inequality (8.24) to $\delta^* < \delta < 1$, which completes claim 2.

Claim 3: Construct $q_0(h^{t-1})$ and $q_1(h^{t-1})$ according to the above procedure when a type-1 firm with cost $\beta_1(h^{t-2})$ and a type-0 firm with cost $\beta_0(h^{t-2})$ would receive strictly positive expected profits by investing in round t. Otherwise, let $q_0(h^{t-1})$ and $q_1(h^{t-1})$ equal zero. The corresponding investment cost cutoffs, based on (5.1) and (5.2), determine an equilibrium.

Proof of Claim 3: Obviously, once investment becomes unprofitable for firms with the lowest possible cost, then a deviation to invest cannot be beneficial. If a type-1 firm with cost $\beta_1(h^{t-2})$ and a type-0 firm with cost $\beta_0(h^{t-2})$ would receive strictly positive expected profits by investing in round t, simple algebra establishes that $G_0(h^{t-1}, q_0, q_1) = 0$ and $G_1(h^{t-1}, q_0, q_1) = 0$ and (8.13) and (8.14) imply (5.8) and (5.9). Equations (5.8) and (5.9) equate the profits from investing in round t to the profits from waiting until round $t+1$, and investing if and only if investment remains profitable.

Suppose there is a history, h^{t-1}, and an uninvested type-1 firm whose most profitable deviation is to invest in round t. Then the firm's investment cost, c_i, satisfies $c_i > \beta_1(h^{t-1})$. However, a type-1 firm with cost $\beta_1(h^{t-1})$ is indifferent between investing in round t and investing in round $t+1$ if and only if investment remains profitable. Therefore, firm i must strictly prefer investing in round $t+1$ if and only if investment remains profitable, rather than investing in round t, a contradiction. An identical argument applies to type-0 firms.

Let J denote the set of type-1 firms with a profitable deviation **not** to invest after some history in which they are supposed to invest, according to the candidate equilibrium. Suppose that J is nonempty, and let $c_i = \limsup_{c \in J} c$. Either a type-1 firm with cost c_i has a profitable deviation not to invest after some history, h^{t-1}, or a type-1 firm with cost below c_i has a profitable deviation not to invest after some history, h^{t-1}. In either case, no type-1 firm with cost above $\beta_1(h^{t-1})$ has a profitable deviation not to invest. After the history, h^{t-1}, a type-1 firm with cost $\beta_1(h^{t-1})$ is indifferent between investing in round t and investing in round $t+1$ if and only if investment remains profitable. Therefore, for firm i, a type-1 firm with cost below $\beta_1(h^{t-1})$ to strictly prefer not to invest in round t, it must plan *not* to invest in round $t+1$ following some profitable history, (h^{t-1}, k^t). It follows that a type-1 firm with cost $\beta_1(h^{t-1}, k^t)$ receives strictly higher profits by following firm i's continuation strategy than by investing in round $t+1$. This contradicts $c_i = \limsup_{c \in J} c$. We conclude that J is empty, so that no firm has a profitable deviation not to invest after any history in which they are supposed to invest. An identical argument holds for type-0 firms. This establishes that the candidate is an equilibrium. ■

Proof of Proposition 7. Consider parameters in region 1. Even a type-1 firm with cost $\underline{c}$ would receive negative expected profits by investing, so no firm invests in round 0. Therefore, nothing is learned from past behavior, so there is no investment in subsequent rounds.

Consider parameters in region 2. Suppose we have $\beta_1^* > \beta_0^* \geq \underline{c}$. For sufficiently large n, the law of large numbers implies the following. If we have $Z = 1$, with probability arbitrarily close to one, the fraction of firms investing in round 0 is arbitrarily close to $\alpha F(\beta_1^*) + (1-\alpha)F(\beta_0^*)$. If we have $Z = 0$, with probability arbitrarily close to one, the fraction of firms investing in round 0 is arbitrarily close to $\alpha F(\beta_0^*) + (1-\alpha)F(\beta_1^*)$. Because these fractions are different, firms can infer the true state from round 0 activity, with probability arbitrarily close to one. A type-1 firm with cost c receives expected profits of $\alpha - c$ by investing in round 0, but would receive arbitrarily close to $\delta\alpha(1-c)$ by waiting until round 1. Because $\frac{\alpha(1-\delta)}{1-\alpha\delta} < \underline{c}$ holds, it follows that all type-1 firms should wait until round 1, contradicting $\beta_1^* > \underline{c}$. If we have $\beta_0^* > \beta_1^* \geq \underline{c}$, once again firms can infer the state from round 0 activity, and we reach the same contradiction.

Suppose we have $\beta_0^* = \beta_1^* > \underline{c}$. For all $\varepsilon > 0$, there exists N such that $n > N$ implies $\beta_1^n < \beta_1^* + \varepsilon$ and $\beta_0^n > \beta_0^* - \varepsilon$. Therefore, if firm i is type-1 with cost $c_i = \beta_1^* + \varepsilon$, it prefers not to invest in round 0, choosing instead some other strategy, s_i^n. Thus, we have

$$\alpha - \beta_1^* - \varepsilon < \alpha\pi(s_i^n, \beta_1^* + \varepsilon \mid Z = 1) + (1-\alpha)\pi(s_i^n, \beta_1^* + \varepsilon \mid Z = 0),$$

which implies

$$\alpha - \beta_1^* - \varepsilon < \alpha\pi(s_i^n, \beta_1^* \mid Z = 1) + (1-\alpha)\pi(s_i^n, \beta_1^* \mid Z = 0) + \varepsilon, \tag{8.25}$$

where $\pi(s, c \mid Z = z)$ denotes the discounted expected profits for a firm with investment cost c, playing the strategy s, given that the state is z.[23] If firm i is type-0 with cost $c_i = \beta_0^* - \varepsilon$, it prefers to invest in round 0, rather than choosing the strategy, s_i^n. Thus, we have

$$1 - \alpha - \beta_0^* + \varepsilon > (1-\alpha)\pi(s_i^n, \beta_0^* - \varepsilon \mid Z = 1) + \alpha\pi(s_i^n, \beta_0^* - \varepsilon \mid Z = 0),$$

which implies

$$1 - \alpha - \beta_0^* + \varepsilon > (1-\alpha)\pi(s_i^n, \beta_0^* \mid Z = 1) + \alpha\pi(s_i^n, \beta_0^* \mid Z = 0) - \varepsilon. \tag{8.26}$$

From (8.25), (8.26), and $\beta_0^* = \beta_1^*$, we have

$$0 < (2\alpha - 1)\left[\pi(s_i^n, \beta_1^* \mid Z = 1) - \pi(s_i^n, \beta_1^* \mid Z = 0) - 1\right] + 4\varepsilon. \tag{8.27}$$

Because $\pi(s_i^n, \beta_1^* \mid Z = 1) < 1 - \beta_1^*$ and $\pi(s_i^n, \beta_1^* \mid Z = 0) > -\beta_1^*$ must hold, it follows that the term in brackets in (8.27) is negative. Therefore, for sufficiently small ε, we have a contradiction. The only possibilities are: (i) $c_i = \beta_1^* + \varepsilon$ is

[23] Since there are no histories to observe and we are conditioning on the true state, firm i's common value signal, X_i, provides no additional information about expected revenues. Therefore, $\pi(s, c \mid Z = z)$ is independent of a firm's common value signal.

impossible, so we have $\beta_1^* = \overline{c}$ (which is accounted for in part 5 of the proposition), or $\beta_0^* = \beta_1^* = \underline{c}$.

Consider parameters in region 3. Suppose we have $\beta_1^* \neq \beta_0^*$. By the law-of-large-numbers argument given above for region 2, for sufficiently large n, a firm not investing in round 0 will be able to infer the true state arbitrarily precisely. Based on the parameter range for region 3, all type-0 firms would rather learn the state in round 1 than invest in round 0. A type-1 firm with cost c_i would rather invest in round 0 than learn the state in round 1 if and only if we have $c_i < \frac{\alpha(1-\delta)}{1-\alpha\delta}$. Therefore, we must have $\beta_0^* = \underline{c}$ and $\beta_1^* = \frac{\alpha(1-\delta)}{1-\alpha\delta}$.

Suppose we have $\beta_0^* = \beta_1^* = \underline{c}$. For a type-1 firm with cost, $c_i < \frac{\alpha(1-\delta)}{1-\alpha\delta}$, investing in round 0 strictly dominates any other strategy, so we cannot have $\beta_1^* = \underline{c}$.

Suppose we have $\overline{c} > \beta_0^* = \beta_1^* > \underline{c}$. By the identical argument given above for region 2, we reach a contradiction. The remaining possibility for region 3 is $\beta_0^* = \beta_1^* = \overline{c}$, which is accounted for in part 5 of the proposition.

Consider parameters in region 4. Suppose we have $\beta_1^* \neq \beta_0^*$. By the law-of-large-numbers argument given above for region 2, for sufficiently large n, a firm not investing in round 0 will be able to infer the true state arbitrarily precisely. Based on the parameter range for region 4, a type-0 firm with cost c_i would rather invest in round 0 than learn the state in round 1 if and only if we have $c_i < \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta}$. A type-1 firm with cost c_i would rather invest in round 0 than learn the state in round 1 if and only if we have $c_i < \frac{\alpha(1-\delta)}{1-\alpha\delta}$. Therefore, we must have $\beta_0^* = \frac{(1-\alpha)(1-\delta)}{1-(1-\alpha)\delta}$ and $\beta_1^* = \frac{\alpha(1-\delta)}{1-\alpha\delta}$.

Suppose we have $\beta_0^* = \beta_1^* = \underline{c}$. For a type-1 firm with cost, $c_i < \frac{\alpha(1-\delta)}{1-\alpha\delta}$, investing in round 0 strictly dominates any other strategy, so we cannot have $\beta_1^* = \underline{c}$.

Suppose we have $\overline{c} > \beta_0^* = \beta_1^* > \underline{c}$. By the identical argument given above for region 2, we reach a contradiction. The remaining possibility for region 4 is $\beta_0^* = \beta_1^* = \overline{c}$, which is accounted for in part 5 of the proposition.

The remaining possibility, is $\beta_0^* = \beta_1^* = \overline{c}$. However, if we have $\overline{c} > 1 - \alpha$, then a type-0 firm with cost above $1 - \alpha$ receives negative expected profits by investing in round 0, so we cannot have $\beta_0^* = \overline{c}$. ∎

Proof of Proposition 8. Part (1) is obvious. Suppose the parameters are in region 3, and consider a type-1 firm with investment cost below β_1^*. Then Proposition 7 implies that for sufficiently large n, this firm will invest in round

0, in which case $W^*(1, c_i) = \alpha - c_i$ holds. All other firms will not invest in round 0, for sufficiently large n, due to the assumption, $\overline{c} \geq 1$. If other firms invest in round 1 if and only if we have $k^0/n \geq F(\beta_1^*)/2$, then by the law of large numbers, the probability of investing in the low state converges to zero, and the probability of investing in the high state converges to one. Since the probability of the high state is α for a type-1 firm and $1 - \alpha$ for a type-0 firm, part (3) of Proposition 8 follows. The same argument applies to region 4, except that the firms that do not invest in round 0 should invest in round 1 if and only if we have $k^0/n \geq [F(\beta_0^*) + (F(\beta_1^*)]/2$.

Suppose the parameters are in region 2. For the equilibrium of the economy with n firms, let $E^{t,\varepsilon}$ be the event that a type-1 firm with investment cost $\underline{c} + \varepsilon$ invests in round t, and define

$$\begin{aligned} T_1^n(1, \varepsilon) &= \sum_{t=1}^{\infty} \delta^t pr(Z = 1 \text{ and } E^{t,\varepsilon} \mid X_i = 1), \\ T_1^n(0, \varepsilon) &= \sum_{t=1}^{\infty} \delta^t pr(Z = 0 \text{ and } E^{t,\varepsilon} \mid X_i = 1), \\ T_0^n(1, \varepsilon) &= \sum_{t=1}^{\infty} \delta^t pr(Z = 1 \text{ and } E^{t,\varepsilon} \mid X_i = 0), \\ T_0^n(0, \varepsilon) &= \sum_{t=1}^{\infty} \delta^t pr(Z = 0 \text{ and } E^{t,\varepsilon} \mid X_i = 0). \end{aligned}$$

Clearly, $W^n(1, c_i)$ must be continuous and decreasing in c_i, because otherwise some type-1 firm would have a profitable deviation to imitate the strategy chosen by a type-1 firm with nearby investment cost. For all $\varepsilon_1 > 0$, there exists $\varepsilon_2 > 0$, such that $\varepsilon < \varepsilon_2$ implies

$$W^n(1, \underline{c}) - W^n(1, \underline{c} + \varepsilon) < \varepsilon_1. \tag{8.28}$$

We have $W^n(1, \underline{c}) = \alpha - \underline{c}$ (they invest in round 0) and $W^n(1, \underline{c} + \varepsilon) = T_1^n(1, \varepsilon)(1 - \underline{c} - \varepsilon) - T_1^n(0, \varepsilon)(\underline{c} + \varepsilon)$. Imposing $\varepsilon < \min[\varepsilon_1, \varepsilon_2]$, it follows from (8.28) that we have

$$\alpha - \underline{c} - T_1^n(1, \varepsilon)(1 - \underline{c}) + T_1^n(0, \varepsilon)(\underline{c}) < 2\varepsilon_1. \tag{8.29}$$

Claim: For all c_i and all $\varepsilon > 0$, there exists N such that $n > N$ implies

$$W^n(1, c_i) \geq \delta T_1^n(1, \varepsilon)(1 - c_i) - \varepsilon_1. \tag{8.30}$$

Proof of Claim: Consider a type-1 firm with investment cost c_i, which waits until the round after a type-1 firm with cost $\underline{c}+\varepsilon$ would invest. That is, we are considering histories such that $\beta_1^n(h^{t-1}) \geq \underline{c}+\varepsilon$. By the law of large numbers, for sufficiently large n, with probability arbitrarily close to one, the cumulated investment in state 1 is arbitrarily close to $\alpha F(\beta_1^n(h^{t-1})) + (1-\alpha)F(\beta_0^n(h^{t-1}))$, and the cumulated investment in state 0 is arbitrarily close to $(1-\alpha)F(\beta_1^n(h^{t-1}))+ \alpha F(\beta_0^n(h^{t-1}))$. A type-1 firm with cost $\beta_1^n(h^{t-1})$ is indifferent between investing in round t and some continuation strategy, s, in which the probability of eventual investment is less than one. (This fact follows from $\delta < 1$ and our assumption, $\overline{c} \geq 1$, which guarantees that there are type-1 firms with costs above $\beta_1^n(h^{t-1})$ that do not invest.) However, a type-0 firm with the same cost, $\beta_1^n(h^{t-1})$, must strictly prefer the continuation strategy, s, because $P_1(h^{t-1}) > P_0(h^{t-1})$. Thus, we conclude that $\beta_0^n(h^{t-1}) < \beta_1^n(h^{t-1})$ must hold. By investing in round $t+1$ if and only if we have $\beta_1^n(h^{t-1}) \geq \underline{c}+\varepsilon$ and

$$\frac{\sum_{\tau=1}^{t} k^\tau}{n} \geq \frac{F(\beta_0^n(h^{t-1})) + F(\beta_1^n(h^{t-1}))}{2},$$

the probability of investing in the high state (when we have $\beta_1^n(h^{t-1}) \geq \underline{c}+\varepsilon$) is arbitrarily close to one, and the probability of investing in the low state (when we have $\beta_1^n(h^{t-1}) \geq \underline{c}+\varepsilon$) is arbitrarily close to zero. Thus, adopting this strategy yields expected profit arbitrarily close to $\delta T_1^n(1,\varepsilon)(1-c_i)$, thereby establishing the Claim.

From (8.29), and the fact that $T_1^n(0,\varepsilon) \geq 0$, we have

$$T_1^n(1,\varepsilon) \geq \frac{\alpha-\underline{c}}{1-\underline{c}} - \frac{2\varepsilon_1}{1-\underline{c}}. \tag{8.31}$$

From (8.30) and (8.31), we have

$$W^n(1,c_i) \geq \delta\left[\frac{\alpha-\underline{c}}{1-\underline{c}}\right](1-c_i) - \delta\left[\frac{2\varepsilon_1}{1-\underline{c}}\right](1-c_i) - \varepsilon_1. \tag{8.32}$$

Letting $c_i = \underline{c}$ hold in (8.30), we have

$$\delta T_1^n(1,\varepsilon)(1-\underline{c}) - \varepsilon_1 \leq \alpha - \underline{c},$$

and using (8.29), this becomes

$$\delta T_1^n(1,\varepsilon)(1-\underline{c}) - \varepsilon_1 < T_1^n(1,\varepsilon)(1-\underline{c}) - T_1^n(0,\varepsilon)(\underline{c}) + 2\varepsilon_1,$$

implying

$$T_1^n(0,\varepsilon)(\underline{c}) < T_1^n(1,\varepsilon)(1-\underline{c})(1-\delta) + 3\varepsilon_1. \tag{8.33}$$

A type-1 firm with cost c_i cannot possibly do better than to decide whether or not to invest during the same round that firm $(1, \underline{c}+\varepsilon)$ invests, but with full knowledge of the state. Therefore, we have

$$W^n(1,c_i) \leq T_1^n(1,\varepsilon)(1-c_i). \tag{8.34}$$

Since firm $(1,\underline{c})$ weakly prefers to invest in round 0, rather than mimic the strategy of firm $(1, \underline{c}+\varepsilon)$, we have

$$\alpha - \underline{c} \geq T_1^n(1,\varepsilon)(1-\underline{c}) - T_1^n(0,\varepsilon)(\underline{c}).$$

Thus, we have

$$T_1^n(1,\varepsilon)(1-\underline{c}) \leq \alpha - \underline{c} + T_1^n(0,\varepsilon)(\underline{c}). \tag{8.35}$$

Using (8.33) and (8.35), we have

$$T_1^n(1,\varepsilon)(1-\underline{c}) \leq \alpha - \underline{c} + T_1^n(1,\varepsilon)(1-\underline{c})(1-\delta) + 3\varepsilon_1,$$

from which we have

$$T_1^n(1,\varepsilon) \leq \frac{\alpha-\underline{c}}{\delta(1-\underline{c})} + \frac{3\varepsilon_1}{\delta(1-\underline{c})}. \tag{8.36}$$

From (8.34) and (8.36), we conclude

$$W^n(1,c_i) \leq \left[\frac{\alpha-\underline{c}}{\delta(1-\underline{c})}\right](1-c_i) + \frac{3\varepsilon_1(1-c_i)}{\delta(1-\underline{c})}. \tag{8.37}$$

For type-0 firms, a simple calculation yields

$$\begin{aligned} T_0^n(1,\varepsilon) &= \frac{1-\alpha}{\alpha} T_1^n(1,\varepsilon) \quad \text{and} \\ T_0^n(0,\varepsilon) &= \frac{\alpha}{1-\alpha} T_1^n(0,\varepsilon). \end{aligned}$$

The law-of-large-numbers argument given in the Claim also establishes

$$\begin{aligned} W^n(0,c_i) &\geq \delta T_0^n(1,\varepsilon)(1-c_i) - \varepsilon_1, \text{ implying} \\ W^n(0,c_i) &\geq \delta \frac{1-\alpha}{\alpha} T_1^n(1,\varepsilon)(1-c_i) - \varepsilon_1. \end{aligned} \tag{8.38}$$

From (8.31) and (8.38), it follows that

$$W^n(0,c_i) \geq \delta \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{\alpha-\underline{c}}{1-\underline{c}}\right](1-c_i) - \delta\left[\frac{1-\alpha}{\alpha}\right]\left[\frac{2\varepsilon_1}{1-\underline{c}}\right](1-c_i) \tag{8.39}$$

holds. A type-0 firm with cost c_i cannot possibly do better than to decide whether or not to invest during the same round that firm $(1, \underline{c}+\varepsilon)$ invests, but with full knowledge of the state. Therefore, we have

$$W^n(0,c_i) \leq T_0^n(1,\varepsilon)(1-c_i) = .\frac{1-\alpha}{\alpha}T_1^n(1,\varepsilon)(1-c_i). \tag{8.40}$$

Combining (8.36) and (8.40), we have

$$W^n(0,c_i) \leq \left[\frac{1-\alpha}{\alpha}\right]\left[\frac{\alpha-\underline{c}}{\delta(1-\underline{c})}\right](1-c_i) + \left[\frac{1-\alpha}{\alpha}\right]\frac{3\varepsilon_1(1-c_i)}{\delta(1-\underline{c})}. \tag{8.41}$$

Because inequalities (8.34), (8.37), (8.39), and (8.41) hold for all ε_1, the results for region 2 follow. ■

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