

Buy-Out Prices in Auctions: Seller Competition and Multi-Unit Demand*

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Abstract

On many *online* auction sites it is now possible for a seller to augment his auction with a maximum or *buy-out* price. The use of this instrument has been justified in “one-shot” auctions by appeal to impatience or risk aversion. Here we offer additional justification by observing that trading on internet auctions is not of a “one-shot” nature, but that market participants (sellers as well as buyers) expect more transactions in the future. This has important implications when bidders desire multiple objects. Specifically, it is shown that an early seller has an incentive to introduce a buy-out price, if similar products are offered later on by other sellers. The buy-out price will increase revenue in the current auction, but revenue in future auctions will decrease, as will the sum of revenues. When the first seller sets his buy-out price optimally, the sequence of auctions is inefficient, in the sense that the first item may be awarded to a bidder who would have received none in an efficient selling mechanism.

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1 Introduction

The presence of buy-out prices¹ in online auctions has thus far been explained by focusing on a *single* auction and assuming that individuals exhibit either *risk aversion* or *impatience*.² In this paper we take a somewhat broader view of auction markets, realizing, in particular, that buyers and sellers alike are aware of the fact that new products will be offered on the market in the future. This will tend to depress revenue in today's auctions, as buyers know that close substitutes will be offered tomorrow. In this *dynamic* environment we will show that there is good reason for an *early* seller to introduce a buy-out price, even if agents are patient and risk neutral.³

Buy-out prices or maximum prices in online auctions were noted by Lucking-Reiley (2000) in his empirical overview of auction activities on the Internet. Since (sell) auctions are ostensibly staged to illicit high prices in situations where markets are thin and sellers are short on information about the willingness-to-pay of potential buyers, such buy-out prices may appear surprising. In fact, Lucking-Reiley explicitly posed this as a challenge to theorists. In addition, he quoted evidence to suggest that the *exercise* of posted buy-out options is not uncommon in online auctions.⁴

Reynolds and Wooders (2003) provide some additional information on the *frequency of buy-out prices* in *Yahoo!* and *eBay* auctions, though, *not* on the frequency with which the option was *exercised* by some bidder. The categories sampled on March 27, 2002, were automobiles, clothing, DVD players, VCR's, digital cameras and TV sets. A total of 1.248 auctioned items were sampled from *Yahoo!*, of which 842 had a buy-out price posted by the seller (roughly, 66%). In similar fashion, 31.142 auctioned items were sampled from *eBay*, of which 12.480 had a buy-out price posted by the seller (roughly, 40%). There is some variation across the categories of goods sampled, but the frequency of buy-out prices never drops below 25% in the sample. Hence, in these categories, at least, the appearance of buy-out prices is very frequent.

For *eBay*, Mathews (2002) presents some numbers on the *frequency with which buy-out options are exercised* when offered.⁵ For two categories of games (racing and sports) for *Sony PS2*, Mathews reports that on January 29 - 30, 2001, 210 items were on offer. A buy-out option was available on 124 items (59%), and it was exercised 34 times (27% of the times it was offered). So, at least in these categories, the exercise frequency is high.

Formally, we analyze *eBay*'s version of a buy-out price, termed the *Buy It Now* price. Here is how the *Buy It Now* price roughly works from the seller's viewpoint:⁶ "If a buyer is willing to meet your *Buy It Now* price before the first bid comes in, your item sells instantly and your auction ends. Or, if a bid comes in first, the *Buy It Now* option disappears. Then your auction proceeds normally." Hence, in *eBay* auctions, the buy-out price is temporary.⁷

Throughout this paper we assume that potential buyers or bidders have *multi-unit demands*, with diminishing marginal utility. With two objects for sale and at least two bidders, it has been shown by

¹Alternatively, this is often referred to as *buy prices* or *maximum prices*. In offline settings, this phenomenon also has a certain affinity with "\$*xx* or best offer", where it is, presumably, implicit that, if someone makes an offer of \$*xx*, then the trade is finalized immediately, while if someone makes a lower offer initially, then the seller will wait a while to see if a better offer comes along. Also, a buy-out price has a certain similarity with a massive *jump bid* intended to end an auction quickly.

²See, Budish and Takeyama (2001), Mathews (2002), Reynolds and Wooders (2003) and Hidvégi, Wang and Whinston (2003).

³Throughout this paper potential buyers bid *non-cooperatively*. In future work we hope to return to the use of buy-out prices in auctions where sellers try to respond to possible *bidder collusion*.

⁴He quotes the case of *LabX* (a lab equipment auction site), where buy-out options are exercised by some bidder in 10% of the cases where they appear. Hence, buy-out prices do more than just attract attention.

⁵He also presents aggregate numbers on the frequency with which buy-out prices are offered at *eBay*. The reported range around 40% is roughly in line with the numbers reported for specific categories by Reynolds and Wooders (2003).

⁶For more details on the *eBay* version and other versions of a buy-out price, see e.g. Lucking-Reiley (2000), Budish and Takeyama (2001), Mathews (2002) and Reynolds and Wooders (2003).

⁷For more details on the *Buy It Now* feature in *eBay* auctions the reader should consult pages.ebay.com/help/sell/bin.html. *eBay* introduced this feature in January 2001.

Black and de Meza (1992) that auction revenue will increase over time and that the auction outcome is efficient under these assumptions. In particular, in a sequence of second-price or English auctions, the seller offering his good today will not earn as much as a competing seller offering a similar good tomorrow, that is, prices are increasing.⁸

However, for the case with *two competing sellers*, we show that the first (i.e., the *early*) seller can always increase his revenue by introducing a buy-out price. The revenue to the second seller is adversely affected, as is overall revenue. An optimally chosen buy-out price in the first auction also introduces *inefficiency*, in the sense that *a bidder who should have won no object wins one*. Our analysis is partial in the following sense. We consider a sequence of two second-price (or English) auctions, allowing the first seller the possibility of introducing a buy-out price without giving the second seller the opportunity to respond in kind. Thus, we essentially show that an auction market without buy-out prices is unstable, in the sense that current sellers will try to force the auction site to (at least temporarily) allow buy-out prices.⁹

The rest of the paper is organized as follows. In Section 2 we set up a simple model and present the results for the bench-mark case where a sequence of two second-price auctions is staged. Then, Section 3 shows that the first seller among a pack of competing sellers can increase his lot by offering a buy-out price and presents results on the path of revenues, the optimal buy-out price and the overall efficiency of the string of auctions. In Section 4 we comment further on the relationship between the buy-out price, total revenue and efficiency. In addition, this section remarks on the robustness of our main result to changes in the auction format (second-price vs. English auction) and the nature of the buy-out option (temporary vs. permanent). Section 5 provides a detailed discussion of an example with uniformly distributed valuations, while Section 6 contains a few concluding remarks. A selection of proofs is in the Appendix.

2 Model and Bench-Mark

In this section we first set up the model and then derive results for the bench-mark case where a sequence of two second-price auctions is staged.

We assume that two objects are offered for sale sequentially,¹⁰ and that there are n potential buyers on the market. Each buyer i , $i = 1, 2, \dots, n$, is characterized by a type, v_i , drawn from a continuously differentiable distribution function, $F(v_i)$, without mass points. The associated density is referred to as $f(v_i) = \frac{dF(v_i)}{dv_i}$.¹¹ We assume that $v_i \in [0, \bar{v}]$. The value to bidder i of the first unit purchased is v_i , while the value of the second unit is kv_i , $0 < k < 1$. Hence, each bidder desires both units, but individual demands are downward sloping.

Below, we shall occasionally take the perspective of a particular bidder, i , and label his rivals j , $j = 1, 2, \dots, n - 1$. Now, i 's competitors have random valuations of the first item denoted y_j which we

⁸In fact, Black and de Meza (1992) were interested in what some have referred to as *The Declining Price Anomaly*. Therefore, they went on to consider an option of the following kind: the winner of the first item is given the option of buying the second item at the same price. This, apparently, is observed in certain multi-unit auctions, and it is enough to lead to a declining price path.

⁹In a previous paper (see Kirkegaard and Overgaard (2003)), we considered the consequences of buy-out prices for a *single seller* intending to sell two objects when there are only two potential buyers. We showed that this seller can increase his total expected revenue by augmenting the *second* auction with a buy-out price, which depends on the outcome of the first auction. The buy-out price should be set fairly low, thus allowing the winner of the first auction a disproportionately large chance of winning the second auction as well. Hence, in that case too, the sequence of auctions is *inefficient*, in the sense that *one buyer may win two objects when efficiency dictates he should only win one*. This will cause overall revenue to increase. The reason is the same as that which induces a monopolist to offer quantity discounts that are detrimental to efficiency: buyers with high demand contribute with higher marginal revenue on two objects than buyers with low demand do on one object.

¹⁰The two objects are considered homogenous by the bidders, or they are simply two units of the same good.

¹¹In order to derive expressions for expected payoffs below, we shall assume that F is three times continuously differentiable.

order as $y_1 \geq y_2 \geq \dots \geq y_{n-1}$. This allows us to refer to bidder j as bidder i 's j th-strongest rival. When relevant, we shall let i be male and all the rivals female. Also, when appealing to the order-statistics, we shall generally refer to $F_{m,n}(x)$ as the distribution function of the m 'th-highest of n draws, with associated density $f_{m,n}(x) = \frac{dF_{m,n}(x)}{dx}$.

Throughout, we assume that two different sellers each own one object initially. The two objects are offered sequentially, and we allow the first seller to stipulate a buy-out price of the *eBay*-variety (*Buy It Now*). Thus, in the general case we consider the following augmented game:

- 1 Seller 1 announces a buy-out price, B . At this stage bidders can submit a bid of B or refrain from bidding. The object is sold at the price B if at least one bidder bids B . If several bidders bid B , one bidder is picked at random to win. If no one bids B , a normal second-price auction is staged. The price can exceed B in this event.
- 2 Seller 2 auctions off the second item, using a second-price auction.

In the first stage of this game, the bidders *first* have to decide whether to take the buy-out price B or leave it. If one or more bidders take the buy-out price, the first auction ends, and the winner pays B . If no one takes the buy-out price, the first stage continues to a standard second-price auction. The second stage simply consists of a standard second-price auction.

First, though, we summarize the results of the bench-mark case, where no buy-out price can be stipulated by the first seller (or, that it is set so high as to be irrelevant for the play of the game).

2.1 The bench-mark: Two straight second-price auctions

To keep the analysis simple, we ignore the use of reserve prices in the following.¹² In this setting, Black and de Meza (1992) were the first¹³ to solve for equilibrium strategies in a sequence of two second-price, sealed bid auctions, under more general assumptions than those considered here.¹⁴ Applied to our set of assumptions, they find the following.

Proposition 1 (Black and de Meza (1992)) *When there are n a priori symmetric bidders in the game, the unique symmetric equilibrium is for agent i to bid $E(\max\{ky_1, y_2\} \mid y_1 = v_i)$ in stage one, and to bid v_i in stage two if stage one was lost, and kv_i otherwise. The equilibrium outcome is efficient, and expected revenues are increasing from the first to the second auction.*

Thus, in the last round, a bidder simply bids his valuation of the remaining object. This, however, depends on whether the bidder won or lost the first object. In the first round, each bidder essentially bids what he expects it to take to win the second if he just loses the first. This, however, is the expectation of the maximum of k times his valuation for the first item won and the valuation of the first unit won by his second-strongest rival.

To see what is going on here, we first note that in case of *symmetric, increasing* bidding strategies, the fine details of any bidder's bid function are only consequential if there happens to be a rival bidder who has a valuation very close to that of the bidder in question. Hence, in equilibrium a bidder's strategy

¹²Reserve prices are generally useful because they allow sellers to ration output by excluding potential buyers with low valuations. Reserve prices, thus, affect the probability that objects are sold. The effect of buy-out prices is different. In particular, buy-out prices do not influence the probability that objects are sold, but they may change the identity of the winners. It follows that a buy-out price is not a substitute for a reserve price, and that it may have a role to play, even when a reserve price is present.

¹³See also Katzman (1999).

¹⁴Black and de Meza explicitly consider sealed-bid auctions, while they also have an informal discussion of English auctions. F evrier, Roos and Visser (2003), on the other hand, focus on English auctions. See Section 4 below for some further comments.

is pinned down by an indifference relation: the bidder should be indifferent between winning and losing, if his strongest rival is identical to himself.

In a *one-shot*, second-price auction bidder i essentially bids what he expects it to take to win the item, *if* he is the “top dog” - the high-valuation bidder - *and* there is someone like him among the rivals. The relevant indifference relation can be written as

$$\overbrace{v_i - b(E(y_1 | y_1 = v_i))}^{\text{just winning}} = \overbrace{0}^{\text{just losing}}$$

However, $E(y_1 | y_1 = v_i) = v_i$, and the optimal bid of i is given by

$$b(v_i) = E(y_1 | y_1 = v_i) = v_i$$

Thus, we obtain the familiar result that it is optimal for bidder i to bid his valuation.

In a *sequence of two* second-price auctions things are a little more complicated. Consider the last round first. If i won the first item, his valuation of the second item is kv_i . Then, in the last round, bidder i 's indifference relation is predicated on $y_1 = kv_i$ (the strongest rival is like him *at this stage*). Thus, we can write

$$\overbrace{kv_i - b^2(E(y_1 | y_1 = kv_i))}^{\text{just winning second}} = \overbrace{0}^{\text{just losing second}}$$

where $b^2(\cdot)$ denotes the second-round bid. Noting that $E(y_1 | y_1 = kv_i) = kv_i$, we obtain

$$b^2(E(y_1 | y_1 = kv_i)) = b^2(kv_i) = kv_i = E(y_1 | y_1 = kv_i)$$

Similarly, if i lost the first item, his valuation of the second item is v_i . Then, in the last round, bidder i 's indifference relation is predicated on $\max\{ky_1, y_2\} = v_i$ (the strongest rival is like him *at this stage*). We can write this as

$$\overbrace{v_i - b^2(E(\max\{ky_1, y_2\} | \max\{ky_1, y_2\} = v_i))}^{\text{just winning second}} = \overbrace{0}^{\text{just losing second}}$$

and we obtain

$$\begin{aligned} b^2(E(\max\{ky_1, y_2\} | \max\{ky_1, y_2\} = v_i)) &= b^2(v_i) \\ &= v_i = E(\max\{ky_1, y_2\} | \max\{ky_1, y_2\} = v_i) \end{aligned}$$

The upshot is that bidder i should bid kv_i in the last round if he won the first and v_i if he lost. This is just bidding one's value in the last round.

More interestingly, consider the first round. We note that if i is the “top dog” and there is someone like i in the pack of rivals, then they each win one item in equilibrium.¹⁵ Hence, optimal bidding by i in the first round is derived from an indifference between winning the first and the second item, which (using the results already derived) we can write as

$$\overbrace{[v_i - \underbrace{b^1(v_i)}_{b^1(y_1) \text{ with } y_1 = v_i}] + 0}^{\text{just winning first and losing second}} = \overbrace{0 + [v_i - E(\max\{ky_1, y_2\} | y_1 = v_i)]}^{\text{just losing first and winning second}}$$

¹⁵When strategies are symmetric and increasing, the first auction is won if the toughest rival has a lower valuation, and lost if the toughest rival has a higher valuation. If the toughest rival has the same valuation as the agent himself, there is a tie, and the winner of the first auction is determined by chance. We argue that the agent must be indifferent between winning and losing the first auction in this case. Assume, to the contrary, that the agent prefers to win (lose) against an identical, strongest rival. Then, the agent should bid more (less) aggressively at the outset to win (lose) with probability one (rather than one half). This implies that the original strategies are not in equilibrium, unless the indifference condition is satisfied.

Thus, in the first auction, bidder i should bid what he expects to have to pay to win the second, if he just loses the first. That is, optimal bidding in the first round is captured by

$$b^1(v_i) = E(\max\{ky_1, y_2\} \mid y_1 = v_i) = E(\max\{kv_i, y_2\} \mid y_1 = v_i)$$

In the general case with n bidders, we conclude that bidder i should bid the expectation of the *maximum* of k times his strongest rival's valuation of the first item and his second-strongest rival's valuation of the first item *predicated on the strongest rival being identical to himself*.

Essentially, the reasons why the expected revenue in the second auction is higher than in the first are as follows. Since auctions are both "second price", their prices (hence, revenues) are determined by the runners-up, that is, the bidders with the second-highest marginal valuations. Furthermore, any bidder bases his bid in the first auction on the assumption that his strongest rival has the same valuation. Note that if the runner-up and the winner of the first auction indeed have the same valuations, expected prices (revenues) will be constant, which is just another way of stating the indifference condition. However, the probability that the valuations of two bidders coincide is zero. In a sense, the runner-up of the first auction underestimates the valuation of the winner or the price in the second auction. Hence, expected prices (revenues) are increasing.¹⁶

Finally, for later reference, let us remark on the special, two-bidder case. When $n = 2$, y_2 is zero by construction, and the optimal bid of i in the first auction reduces to

$$b^1(v_i) = E(\max\{ky_1, y_2\} \mid y_1 = v_i) = E(\max\{ky_1, 0\} \mid y_1 = v_i) = kv_i$$

Hence, in the first round a bidder simply bids as he would in a second round after winning the first.

The next two results further characterize the first-round bidding and the associated expected revenues.

Lemma 1 *The optimal bidding strategy in the first auction can be written as*

$$b^1(v) = \frac{kvF_{1,n-2}(kv) + \int_{kv}^v xf_{1,n-2}(x)dx}{F_{1,n-2}(v)} \quad (1)$$

where $F_{1,n-2}(x) = F^{n-2}(x)$ is the distribution function of the first-order statistic in a sample of $n - 2$, while $f_{1,n-2}(x) = \frac{dF_{1,n-2}(x)}{dx} = (n - 2)F^{n-3}(x)f(x)$ is the density.

Proof. First note that $b^1(v) = E(\max\{ky_1, y_2\} \mid y_1 = v)$ can be rewritten as $b^1(v) = E(\max\{kv, y_2\} \mid y_1 = v) = E(\max\{kv, y_2\} \mid y_2 < v)$. But, $E(\max\{kv, y_2\} \mid y_2 < v)$ is just

$$\begin{aligned} E(\max\{kv, y_2\} \mid y_2 < v) &= \frac{\int_0^{kv} kvf_{1,n-2}(y_2)dy_2 + \int_{kv}^v y_2f_{1,n-2}(y_2)dy_2}{F_{1,n-2}(v)} \\ &= \frac{kvF_{1,n-2}(kv) + \int_{kv}^v y_2f_{1,n-2}(y_2)dy_2}{F_{1,n-2}(v)} \end{aligned}$$

as stated in the lemma. ■

In the following, we let

$$m(v) = \min\left\{\frac{v}{k}, \bar{v}\right\}$$

Notice that if the runner-up in the first auction has type v , he will win the second auction if the highest rival type (that belonging to the winner of the first auction) is between v and $m(v)$, since in that case the winner of stage one will bid at most $km(v) \leq v$. However, if the highest rival type is between $m(v)$ and \bar{v} , the bidder will be the runner-up in both auctions. We can then state the following result.

¹⁶Observe that if $k = 0$ (unit demands), the fact that the runner-up underestimated the valuation of the winner is irrelevant, because the winner does not compete in the second auction.

Lemma 2 *In two straight second-price auctions with n bidders, the expected revenues in the first and second auctions are, respectively,*

$$ER_1^{SSP} = \int_0^{\bar{v}} kv \frac{F^{n-2}(kv)}{F^{n-2}(v)} f_{2,n}(v) dv + \int_0^{\bar{v}} v \cdot \frac{(1-F(v))^2 - (1-F(m(v)))^2}{(1-F(v))^2} \cdot f_{3,n}(v) dv \quad (2)$$

and

$$ER_2^{SSP} = \int_0^{\bar{v}} v \cdot \frac{1-F(m(v))}{1-F(v)} \cdot f_{2,n}(v) dv + \int_0^{\bar{v}} kv \cdot \frac{F(v) - F(kv)}{1-F(v)} \cdot \frac{F^{n-2}(kv)}{F^{n-2}(v)} f_{2,n}(v) dv + \int_0^{\bar{v}} v \frac{(F(m(v)) - F(v))^2}{(1-F(v))^2} f_{3,n}(v) dv \quad (3)$$

where $f_{2,n}(x) = n(n-1)(1-F(x))F^{n-2}(x)f(x)$ is the density of the second-order statistic in a sample of n , while $f_{3,n}(x) = \frac{n(n-1)(n-2)}{2}(1-F(x))^2F^{n-3}(x)f(x)$ is the density of the third-order statistic.

Proof. See Appendix. ■

To see what is going on here, note that we can write (1) as

$$b^1(v) = \frac{kvF_{1,n-2}(kv) + [xF_{1,n-2}(x)]_{kv}^v - \int_{kv}^v F_{1,n-2}(x)dx}{F_{1,n-2}(v)}$$

or

$$b^1(v) = v - \frac{\int_{kv}^v F_{1,n-2}(x)dx}{F_{1,n-2}(v)}$$

from which is more clear how bids in the first auction are *shaded* below the valuation of the first item won. If bidders have horizontal demands, $k = 1$, then bids are not shaded, that is, $b^1(v) = v$. In contrast, if bidders have unit demands, $k = 0$, then bids are shaded to $b^1(v) = v - \frac{\int_0^v F_{1,n-2}(x)dx}{F_{1,n-2}(v)} = \frac{\int_0^v x f_{1,n-2}(x)dx}{F_{1,n-2}(v)}$, which is essentially just the expected value of the strongest remaining rival in the second auction once an equal rival has been eliminated by winning the first auction.

To intuitively understand the second-round revenue in (3), recall that bidding strategies are simple in the second auction, in the sense that all bidders will simply bid their *relevant* valuations. Thus, bidder i will bid $b^2(v_i) = kv_i$, if he won the first auction, and $b^2(v_i) = v_i$ otherwise. This will allow us to write ER_2^{SSP} as the sum of three terms following from three types of events: (1) the winner of the first auction wins again, (2) the winner of the first auction does not win again but is the runner-up and (3) the winner of the first auction does not win again nor is the runner up. Let x denote the valuation of the winner of the first auction, while y_1 and y_2 denote the valuations of his strongest and second-strongest rivals, respectively. Thus, $x \geq y_1 \geq y_2$.

The winner of the first auction also win the second auction if $x \geq kv_1 \geq y_1 \geq y_2$, in particular, if $x \geq \frac{y_1}{k}$. Writing the first term in (3) as

$$\int_0^{k\bar{v}} y_1 \cdot \frac{1-F(\frac{y_1}{k})}{1-F(y_1)} \cdot f_{2,n}(y_1) dy_1$$

captures the revenue contribution when the winner of the first auction also wins the second auction at a price of y_1 . The winner of the first auction is the runner-up in the second auction if $x \geq y_1 \geq kx \geq y_2$, in which case the bidder with valuation y_1 wins the second auction at a price of kx . The second term in (3) written as

$$\int_0^{\bar{v}} kx \cdot \frac{F(x) - F(kx)}{1 - F(x)} \cdot \frac{F^{n-2}(kx)}{F^{n-2}(x)} f_{2,n}(x) dx$$

captures the revenue contribution in this event. Finally, the winner of the first auction is neither the winner nor the runner-up of the second auction if $x \geq y_1 \geq y_2 \geq kx$. In this event, the bidder with valuation y_1 wins the second auction at a price of y_2 . Then, the third term in (3) written as

$$\int_0^{\bar{v}} y_2 \frac{(F(m(y_2)) - F(y_2))^2}{(1 - F(y_2))^2} f_{3,n}(y_2) dy_2$$

captures the revenue.

We further note that $ER_1^{SSP} \rightarrow \int_0^{\bar{v}} v f_{3,n}(v) dv = E(v_{[3]})$ and $ER_2^{SSP} \rightarrow \int_0^{\bar{v}} v f_{3,n}(v) dv = E(v_{[3]})$ as $k \rightarrow 0$, where $E(v_{[3]})$ is the expectation of the third-highest of n independent random draws from $F(v)$. This, however, is just a special version of Weber's (1983) result that a sequence of second-price (or English) auctions where bidders have *unit demands* yields the same expected revenue to all sellers. Notice further that with only *two* bidders and two items for sale, the equilibrium revenue is zero to both sellers. It is impossible to extract rent from buyers when there is no excess demand, recalling our assumption of no reserve prices.¹⁷

Similarly, we note that $ER_1^{SSP} \rightarrow \int_0^{\bar{v}} v f_{2,n}(v) dv = E(v_{[2]})$ and $ER_2^{SSP} \rightarrow \int_0^{\bar{v}} v f_{2,n}(v) dv = E(v_{[2]})$ as $k \rightarrow 1$. $E(v_{[2]})$ is just the expectation of the second-highest of n independent randoms draws from $F(v)$. When $k = 1$, *individual demands are horizontal*, and the behavior in the second auction is independent of the outcome of the first auction. The high-valuation bidder will win both objects at a price of $v_{[2]}$, and revenue is the same in both auctions.

Given the increasing path of revenues over two straight second-price auctions, it is clear that the first of *two competitive sellers* has an incentive to change the auction format.¹⁸ In this paper we shall restrict attention to the possible role of a buy-out price in the first auction when two competing sellers are selling identical objects. The first seller is interested in shifting revenues from the second to the first auction, while we shall also be interested in the consequences for efficiency and total revenue when the buy-out price is set optimally by the first seller.¹⁹ A mechanism may be inefficient in the present case, in the sense that it may (probabilistically) allocate an object to a bidder who would have received no object in an efficient mechanism. As we shall see this will be a feature of the mechanism for the case with two competing sellers where the first seller sets an optimal buy-out price.²⁰

¹⁷Our general argument above for the n bidder case captures further aspects of Weber's results. With $k = 0$ (unit demands) bidding in both the first and the second auction is ultimately based purely on the expected *second*-highest value among a bidder's rivals, thus, on the *third-order* statistic $v_{[3]}$ of the n random valuations. From this it follows that expected revenue is the same in the two auctions when $k = 0$ (cf. the observation in the previous footnote).

¹⁸That is, short of moving to the last spot if possible. If selling-time is an endogenous variable, the two symmetric sellers might conceivably be involved in a war of attrition to become the last seller. This, however, is not the topic of this paper, and seller positions in the auction sequence are assumed to be exogenous.

¹⁹In Kirkegaard and Overgaard (2003) we also considered the case where there is a *single seller* who attempts to sell two identical objects in a sequence of auctions when $n = 2$. Absent discounting (impatience), this seller is only interested in total expected revenue from the two auctions, while he is indifferent as to whether revenues are increasing or decreasing over the sequence. We showed, however, that a suitably chosen buy-out price in the *second* auction, depending on the outcome of the first auction, increases the total expected revenue of a single seller at the potential expense of efficiency.

²⁰Likewise, a mechanism may allocate both objects to a bidder who would only have received one object in an efficient mechanism. This will arise in the case considered in Kirkegaard and Overgaard (2003), where a single seller sets a buy-out price in the second auction which depends on the outcome of the first auction.

3 Buy-Out Prices

Next, we allow the first seller to stipulate a buy-out price and consider the augmented game described above:

- 1 Seller 1 announces a buy-out price, B . At this stage bidders can submit a bid of B or refrain from bidding. The object is sold at the price B if at least one bidder bids B . If several bidders bid B , one bidder is picked at random to win. If no one bids B , a normal second-price auction is staged. The price can exceed B in this event.
- 2 Seller 2 auctions off the second item, using a second-price auction.

We first derive the relationship between the level of B and the valuations of bidders who will take this buy-out price. Then we look at the relationship between the buy-out price and the expected revenues to the two sellers, including how they are ranked. Finally, we determine the optimal buy-out price for the first seller and discuss the efficiency properties of the resulting sequence of auctions.

3.1 Equilibrium bidding strategies and expected revenues

We first look for a symmetric equilibrium in this augmented game in which bidders with valuations above some level \hat{v} take the buy-out price B in stage 1, while bidders with valuations below \hat{v} do not. In the augmented game, it is clear that if no bidder takes B , then it is common knowledge in equilibrium that both bidders have a type below \hat{v} . That is, beliefs are symmetric, and the logic of Proposition 1 (Black and de Meza (1992)) applies to the remainder of stage 1 and to stage 2. Hence, in stage 1 bids will be given by $b^1(v_i)$ in (1) with $v_i < \hat{v}$, $\forall i = 1, 2, \dots, n$. Further, regardless of how the good is sold in stage 1, it is well known that the bid in stage 2 will be kv_i if bidder i won the first auction, and v_i otherwise. In the following we suppress the subscript when this can be done without confusion.

In the equilibrium of the augmented game, a given value of B will induce a set $[\hat{v}, \bar{v}]$ of bidder types to take the buy-out price B in stage 1. Changing B will change \hat{v} . Hence, we can determine which \hat{v} to target, and chose B accordingly. Thus, we write $B(\hat{v})$ as the value of B that induces bidder types above \hat{v} to take B in a symmetric equilibrium.

Recall the assumption that if several bidders indicate their willingness to take the buy-out price, the winner of the first round is determined by lottery. We first state the following lemma.

Lemma 3 *In equilibrium, a bidder with valuation above \hat{v} wins the first stage with probability*

$$\Pr(W|\hat{v}) = \frac{1 - F_{1,n}(\hat{v})}{n(1 - F(\hat{v}))}.$$

Proof. See Appendix. ■

This allows us to state the following result, where $F_{1,m}(x) = F^m(x)$ is the distribution function of the first-order statistic in a sample of m , and $f_{1,m}(x) = \frac{dF^m(x)}{dx} = mF^{m-1}(x)f(x)$ is the density.

Proposition 2 *Let $B(\hat{v})$ be defined by*

$$\begin{aligned} B(\hat{v})(1 - F_{1,n}(\hat{v})) &= \hat{v}[1 - F_{1,n}(\hat{v}) - n(1 - F(\hat{v}))F_{1,n-1}(\hat{v}) \\ &\quad - \frac{1}{2}n(n-1)(1 - F(\hat{v}))(F(m(\hat{v})) - F(\hat{v}))F_{1,n-2}(\hat{v})] \\ &\quad + n(n-1)(1 - F(\hat{v})) \int_0^{\hat{v}} \left(kx F_{1,n-2}(kx) + \int_{kx}^x y f_{1,n-2}(y) dy \right) f(x) dx \\ &\quad + \frac{1}{2}n(n-1)(1 - F(\hat{v})) \int_{\hat{v}}^{m(\hat{v})} \left(kx F_{1,n-2}(kx) + \int_{kx}^{\hat{v}} y f_{1,n-2}(y) dy \right) f(x) dx \end{aligned} \quad (4)$$

Then, it is an equilibrium for bidders with $v \in [\hat{v}, \bar{v}]$ to take the buy-out price $B(\hat{v})$ in stage 1 and for bidders with $v \in [\underline{v}, \hat{v})$ not to and instead bid $b^1(v)$.

Proof. Recall that the bid in the first auction, if the buy-out price is not accepted, is equal to the expected value of the bid in the second auction, given the first auction is lost to a bidder with the same valuation. That is, bidding proceeds according to

$$b^1(x) = kx \frac{F_{1,n-2}(kx)}{F_{1,n-2}(x)} + \int_{kx}^x y \frac{f_{1,n-2}(y)}{F_{1,n-2}(x)} dy$$

which implies that

$$b^1(x)F_{1,n-2}(x) = kxF_{1,n-2}(kx) + \int_{kx}^x yf_{1,n-2}(y)dy$$

We are now ready to derive the payoff from bidding or not bidding $B(\hat{v})$, given the bidder has valuation \hat{v} .

By accepting the buy-out price, expected payoff is

$$\begin{aligned} EU(B, \hat{v}) &= (\hat{v} - B(\hat{v})) \Pr(W|\hat{v}) + \int_0^{k\hat{v}} (k\hat{v} - x)f_{1,n-1}(x)dx \\ &\quad + \frac{1}{2}(n-1) \int_{\hat{v}}^{m(\hat{v})} \left((\hat{v} - kx)F_{1,n-2}(kx) + \int_{kx}^{\hat{v}} (\hat{v} - y)f_{1,n-2}(y)dy \right) f(x)dx. \end{aligned}$$

The first term is the expected payoff from the first auction. The second term captures the possibility that the bidder wins both auctions. Finally, the third term comes from the fact that the bidder may lose the first auction, but win the second. If he loses the first auction, the winner must have a valuation above \hat{v} . If there are several rivals with valuation above \hat{v} , the bidder will also lose the second auction (since he, by assumption, has valuation \hat{v}). Thus, only if there is precisely one bidder among the $n-1$ rivals, is it possible to lose stage one (which then happens with probability .5) and subsequently win stage two. This, however, requires that the rival has type below $m(\hat{v})$, such that his bid in stage 2 will be below \hat{v} . The price in stage two is clearly the maximum of k times the winner's valuation and the highest of the other rivals' valuations.

If the bidder with valuation \hat{v} decides not to accept $B(\hat{v})$, it is optimal for him to outbid everybody else in the first auction, if the buy-out price is not accepted by any of the rivals. By doing so, he ends up paying the bid of his strongest rival, $b^1(v)$ say, with $v \leq \hat{v}$, and possibly also winning stage two. If he does not outbid his rivals, he will win stage two, at an expected price of

$$b^1(v) = kv \frac{F_{1,n-2}(kv)}{F_{1,n-2}(v)} + \int_{kv}^v y \frac{f_{1,n-2}(y)}{F_{1,n-2}(v)} dy$$

Hence, if the bidder with valuation \hat{v} does not bid $B(\hat{v})$, his expected payoff is

$$\begin{aligned} EU(NB, \hat{v}) &= \int_0^{\hat{v}} (\hat{v} - b^1(x))f_{1,n-1}(x)dx + \int_0^{k\hat{v}} (k\hat{v} - x)f_{1,n-1}(x)dx \\ &\quad + (n-1) \int_{\hat{v}}^{m(\hat{v})} \left((\hat{v} - kx)F_{1,n-2}(kx) + \int_{kx}^{\hat{v}} (\hat{v} - y)f_{1,n-2}(y)dy \right) f(x)dx \end{aligned}$$

The first term comes from the fact that the bidder outbids everybody else in the first auction, if no one accepted $B(\hat{v})$. The second and third terms are similar to those in $EU(B, \hat{v})$, with the exception that the first stage is now lost with probability one if there is a rival with valuation above \hat{v} .

As we are looking for an equilibrium in which $B(\hat{v})$ is accepted if, and only if, the bidder has valuation above \hat{v} , it must be the case that the bidder with valuation \hat{v} is indifferent between accepting and not accepting. This can be written as

$$\begin{aligned}
& B(\hat{v})(1 - F_{1,n}(\hat{v})) \\
&= \hat{v}(1 - F_{1,n}(\hat{v})) - n(1 - F(\hat{v})) \int_0^{\hat{v}} (\hat{v} - b^1(x)) f_{1,n-1}(x) dx \\
&\quad - \frac{1}{2}n(n-1)(1 - F(\hat{v})) \int_{\hat{v}}^{m(\hat{v})} [(\hat{v} - kx)F_{1,n-2}(kx) + \int_{kx}^{\hat{v}} (\hat{v} - y)f_{1,n-2}(y)dy] f(x) dx \\
&= \hat{v}[1 - F_{1,n}(\hat{v}) - n(1 - F(\hat{v}))F_{1,n-1}(\hat{v}) - \frac{1}{2}n(n-1)(1 - F(\hat{v}))F_{1,n-2}(\hat{v})(F(m(\hat{v})) - F(\hat{v}))] \\
&\quad + \int_0^{\hat{v}} b^1(x)n(n-1)(1 - F(\hat{v}))F_{1,n-2}(x)f(x) dx \\
&\quad + \int_{\hat{v}}^{m(\hat{v})} \left(kx F_{1,n-2}(kx) + \int_{kx}^{\hat{v}} y f_{1,n-2}(y) dy \right) \frac{1}{2}n(1 - F(\hat{v}))(n-1)f(x) dx
\end{aligned}$$

Substituting for $b^1(x)$, we get

$$\begin{aligned}
& B(\hat{v})(1 - F_{1,n}(\hat{v})) \\
&= \hat{v}[1 - F_{1,n}(\hat{v}) - n(1 - F(\hat{v}))F_{1,n-1}(\hat{v}) - \frac{1}{2}n(n-1)(1 - F(\hat{v}))F_{1,n-2}(\hat{v})(F(m(\hat{v})) - F(\hat{v}))] \\
&\quad + n(n-1)(1 - F(\hat{v})) \int_0^{\hat{v}} \left(kx F_{1,n-2}(kx) + \int_{kx}^x y f_{1,n-2}(y) dy \right) f(x) dx \\
&\quad + \frac{1}{2}n(n-1)(1 - F(\hat{v})) \int_{\hat{v}}^{m(\hat{v})} \left(kx F_{1,n-2}(kx) + \int_{kx}^{\hat{v}} y f_{1,n-2}(y) dy \right) f(x) dx
\end{aligned}$$

as stated in the proposition.

It remains to verify that buyers with type different from \hat{v} have no incentive to deviate. This is done in the Appendix. ■

For later reference, we note that with $n = 2$, (4) reduces to

$$\begin{aligned}
B(\hat{v})(1 - F^2(\hat{v})) &= \hat{v}(1 - F(\hat{v})^2 - 2(1 - F(\hat{v}))F(\hat{v}) - (1 - F(\hat{v}))(F(m(\hat{v})) - F(\hat{v}))) \\
&\quad + \int_0^{\hat{v}} kx 2(1 - F(\hat{v}))f(x) dx + \int_{\hat{v}}^{m(\hat{v})} kx(1 - F(\hat{v}))f(x) dx,
\end{aligned}$$

or

$$\begin{aligned}
B(\hat{v})(1 + F(\hat{v})) &= \hat{v}(1 + F(\hat{v}) - 2F(\hat{v}) - (F(m(\hat{v})) - F(\hat{v}))) \\
&\quad + \int_0^{\hat{v}} 2kx f(x) dx + \int_{\hat{v}}^{m(\hat{v})} kx f(x) dx \\
&= \hat{v}(1 - F(m(\hat{v}))) + \int_0^{\hat{v}} kx f(x) dx + \int_0^{m(\hat{v})} kx f(x) dx
\end{aligned}$$

where we see that $B(\hat{v}) \rightarrow E(kv)$ for $\hat{v} \rightarrow \bar{v}$. However, this is just $E(b^1(v)|v = y_1)$. In the general case

with $n > 2$, applying L'Hôpital's rule to (4) for $\hat{v} \rightarrow \bar{v}$ yields

$$\begin{aligned} B(\bar{v}) &= (n-1) \int_0^{\bar{v}} \left(kx F_{1,n-2}(kx) + \int_{kx}^x y f_{1,n-2}(y) dy \right) f(x) dx \\ &= \int_0^{\bar{v}} \left(\frac{kx F_{1,n-2}(kx) + \int_{kx}^x y f_{1,n-2}(y) dy}{F_{1,n-2}(x)} \right) (n-1) F_{1,n-2} f(x) dx \\ &= \int_0^{\bar{v}} b^1(x) f_{1,n-1}(x) dx = E(b^1(v) | v = y_1). \end{aligned}$$

Hence, for $\hat{v} \rightarrow \bar{v}$, $B(\hat{v})$ approaches the expected value of the highest bid among the $n-1$ rivals.

Generally, $B(\cdot)$ may not be monotonic, implying that for a given value of B , there could be multiple symmetric equilibria. However, as shown below, for any distribution and $k \in (0, 1)$, the first seller can strictly increase his revenue by offering a buy-out price that will be accepted with positive probability.

For given \hat{v} , equilibrium bidding strategies give rise to expected revenues stated as follows, where $K_{n,m} = \binom{n}{m} = \frac{n!}{m!(n-m)!}$ and $P_{n,m} = \frac{n!}{(n-m)!} = K_{n,m} m!$

Proposition 3 *The expected revenue in the first auction is*

$$\begin{aligned} ER_1(\hat{v}) &= \hat{v} [1 - F_{1,n}(\hat{v}) - n(1 - F(\hat{v})) F_{1,n-1}(\hat{v}) - \frac{1}{2} n(n-1)(1 - F(\hat{v})) F_{1,n-2}(\hat{v}) (F(m(\hat{v})) - F(\hat{v}))] \\ &\quad + n(n-1) \int_0^{\hat{v}} \left(kv F_{1,n-2}(kv) + \int_{kv}^v y f_{1,n-2}(y) dy \right) (1 - F(v)) f(v) dv \\ &\quad + \frac{1}{2} n(n-1)(1 - F(\hat{v})) \int_{\hat{v}}^{m(\hat{v})} \left(kv F_{1,n-2}(kv) + \int_{kv}^{\hat{v}} y f_{1,n-2}(y) dy \right) f(v) dv \end{aligned} \quad (5)$$

while the expected revenue in the second auction is

$$ER_2(\hat{v}) = E_1 + E_2 + E_3 + E_4 \quad (6)$$

where

$$\begin{aligned} E_1 &= P_{n,2} \cdot \int_0^{\hat{v}} kv F(kv)^{n-2} (F(v) - F(kv)) f(v) dv \\ &\quad + P_{n,2} \cdot \int_{\hat{v}}^{m(\hat{v})} kv F(kv)^{n-2} \left(F(\hat{v}) - F(kv) + \frac{1}{2} (F(v) - F(\hat{v})) \right) f(v) dv \\ &\quad + P_{n,2} \cdot \sum_{i=0}^{n-2} K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv F(\hat{v})^{n-2-i} (F(kv) - F(\hat{v}))^i (F(v) - F(kv)) f(v) dv \end{aligned}$$

$$\begin{aligned} E_2 &= P_{n,2} \cdot \int_0^{\hat{v}} v F(v)^{n-2} (1 - F(m(v))) f(v) dv \\ &\quad + P_{n,2} \cdot \sum_{i=0}^{n-2} K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{\hat{v}}^{\bar{v}} v F(\hat{v})^{n-2-i} (F(v) - F(\hat{v}))^i (1 - F(m(v))) f(v) dv \\ &\quad + P_{n,2} \cdot \sum_{i=1}^{n-2} K_{n-2,i} \cdot \frac{i}{i+2} \cdot \int_{\hat{v}}^{\bar{v}} v F(\hat{v})^{n-2-i} (F(v) - F(\hat{v}))^i (1 - F(v)) f(v) dv \end{aligned}$$

$$\begin{aligned}
E_3 &= P_{n,2} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{m(\hat{v})} kvF^{n-2}(kv)(1-F(v))f(v)dv \\
&\quad + P_{n,2} \cdot \sum_{i=0}^{n-2} K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{m(\hat{v})}^{\bar{v}} kvF^{n-2-i}(\hat{v})(F(kv)-F(\hat{v}))^i(1-F(v))f(v)dv
\end{aligned}$$

and

$$\begin{aligned}
E_4 &= P_{n,3} \cdot \frac{1}{2} \cdot \int_0^{\hat{v}} vF^{n-3}(v)(F(m(v))-F(v))^2f(v)dv \\
&\quad + P_{n,3} \sum_{i=0}^{n-3} K_{n-3,i} \cdot \frac{1}{i+3} \cdot \int_{\hat{v}}^{\bar{v}} vF^{n-3-i}(\hat{v})(F(v)-F(\hat{v}))^i(F(m(v))-F(v))^2f(v)dv \\
&\quad + P_{n,3} \cdot \frac{1}{2} \cdot \int_{k\hat{v}}^{\hat{v}} vF^{n-3}(v)(F(m(v))-F(\hat{v}))(1-F(m(v)))f(v)dv \\
&\quad + P_{n,3} \sum_{i=0}^{n-3} K_{n-3,i} \cdot \frac{1}{i+3} \cdot \int_{\hat{v}}^{\bar{v}} vF^{n-3-i}(\hat{v})(F(v)-F(\hat{v}))^i(1-F(m(v)))(F(m(v))-F(v))f(v)dv
\end{aligned}$$

Proof. For (5) matters are relatively simple. Expected revenue in the first round is simply given as

$$ER_1(\hat{v}) = B(\hat{v})(1 - F_{1,n}(\hat{v})) + \int_0^{\hat{v}} b^1(v)n(F(\hat{v}) - F(v))f_{1,n-1}(v)dv$$

since revenue is $B(\hat{v})$ if the highest-valuation bidder has a valuation that exceeds \hat{v} , and $b^1(v)$ if the second-highest type is x and the highest is somewhere between v and \hat{v} . Substituting for $B(\hat{v})(1 - F_{1,n}(\hat{v}))$ from (4), for $b^1(v)$ from (1), and noting that $f_{1,n-1}(v) = (n-1)F_{1,n-2}(v)f(v)$, expected revenue in the first round can be written as in (5).

For (6) things are somewhat more complicated, and we relegate most of the details to the Appendix. To give a gist of the argument, we note that the price in the second round, p_2 , can be either kv_1 , v_2 , kv_2 or v_3 . Now, arrange bidders in descending order, $v_1 > v_2 > \dots > v_n$. Then, the four possibilities contribute the following to expected revenue in stage two.

1. *The price in stage two is kv_1 .* This requires that bidder 1 wins stage one (which may involve accepting the buy-out price) and is the runner-up in stage two, that is, $v_2 > kv_1 > v_3$. Hence, we get a revenue contribution corresponding to E_1 , where v may be thought of as v_1 .

If $v_1 = v$ is less than \hat{v} , bidder 1 will win stage one, and the price in stage two is kv if $v_2 > kv_1 > v_3$, which explains the first term. If $v_1 > \hat{v}$ but $kv_1 < \hat{v}$, the price in stage two is kv_1 only if there is at most one rival with valuation between \hat{v} and v_1 (and the rest have valuations below v_1), and that rival lost the lottery for the buy-out price. This explains the second term. The third comes from the fact that if $kv_1 > \hat{v}$, it is possible that the price in stage two is kv_1 even if several rivals have valuations above \hat{v} . However, if $i+1$ rivals have valuations above \hat{v} , bidder 1 only wins stage one with probability $1/(i+2)$. This explains the last term.

2. *The price in stage two is v_2 .* This requires that bidder 2 does not win stage one, and that she is the runner up in stage 2. This occurs if either (i) bidder 1 wins stage one and $kv_1 > v_2$, or if (ii) someone other than bidder 1 or bidder 2 wins stage 1, which in turn means that $v_2 > v_3 > \hat{v}$. Thus, we get the contribution E_2 , where v may be thought of as v_2 .

The first two terms capture the possibility that bidder 1 wins stage one and stage 2, which requires that his type exceeds $m(v_2)$. Notice that if there are $i+2$ bidders in total with valuations above \hat{v} , the

probability that bidder 1 wins stage one is $1/(i+2)$. The third term is for cases in which neither bidder 1 nor bidder 2 win stage one, which clearly requires not only that v_1 and v_2 are above \hat{v} , but also that at least one other bidder has a valuation in excess of \hat{v} . If there are i such rivals, $i > 0$, the probability that neither bidder 1 nor bidder 2 win stage one is $i/(i+2)$.

3. *The price in stage two is kv_2 .* This requires that bidder 2 wins stage one (implying that $v_2 > \hat{v}$) and is the runner-up in stage two ($kv_2 > v_3$). Consequently, we get the contribution E_3 , where v may be thought of as v_2 .

If $v_2 > \hat{v}$, but $kv_2 < \hat{v}$, the price in stage two can be kv_2 only if bidder 1 is the only rival with valuation above \hat{v} , in which case stage one is won by bidder 2 with probability $1/2$ (the first term). On the other hand, if $kv_2 > \hat{v}$, the price in stage two may be kv_2 even if several rivals have valuations above \hat{v} .

4. *The price in stage two is v_3 .* For the price to be v_3 , bidder 3 must lose stage one and be the runner-up in stage two. Furthermore, if bidder 3 is the runner up in stage two, either bidder 1 or bidder 2 must have won stage one. However, notice that if $v_1, v_2 > m(v_3)$, both would be willing to pay more for the second unit than v_3 . Hence, to get a price of v_3 , it must be the case that either (i) both bidder 1 and bidder 2 have valuations between v_3 and $m(v_3)$, in which case whoever wins stage one has a willingness to pay below v_3 , or (ii) bidder 1's valuation exceeds $m(v_3)$, but bidder 2 wins stage one and has a valuation between v_3 and $m(v_3)$, implying that his bid in stage two is below v_3 . We get the contribution E_4 , where v may be thought of as v_3 .

The first two terms in E_4 are related to the first possibility described,²¹ while the last two relate to the second possibility. In particular, if $v_3 = v$, and both v_1 and v_2 are between v and $m(v)$, the price in stage two must be v_3 if either bidder 1 or bidder 2 won stage 1 (the first two terms). A price of $v_3 = v$ is also possible if v_1 is above $m(v)$, but then it requires that v_2 is between v and $m(v)$, and that bidder 2 wins stage one (the last two terms). ■

3.2 Optimal first-round buy-out price and the path of revenues

We first show that it is optimal for the first seller to introduce a buy-out price which is accepted with positive probability, i.e., that he is better off targeting some $\hat{v} < \bar{v}$.²²

Proposition 4 *The optimal buy-out price is accepted with positive probability. That is, $\hat{v} < \bar{v}$.*

Proof. See Appendix. ■

Thus, the ability to offer a buy-out price strictly improves expected revenue to the first seller. Recalling that prices are increasing in the bench-mark model, this leads to the question of whether the buy-out price may even allow the first seller to overtake the second seller in terms of expected revenue. The following proposition answers this question in the negative, and we conclude that it is impossible for the first seller to “level the playing field” by introducing a buy-out price.

Proposition 5 (Increasing Prices) *Regardless of the buy-out price, expected revenue is strictly increasing over the sequence of auctions, that is, $ER_2(\hat{v}) > ER_1(\hat{v}), \forall \hat{v} \in [0, \bar{v}]$.*

²¹In the second term in E_4 we could have written the permutations and combinations as

$$P_{n,3} \sum_{i=0}^{n-3} K_{n-3,i} \cdot \frac{2}{i+3} \cdot \frac{1}{2}$$

This captures more explicitly that either bidder 1 or bidder 2 wins, while $i+3$ bidders accept the buy-out price. The probability that either bidder 1 or bidder 2 wins is then $\frac{2}{i+3}$. See the Appendix for further details.

²²In the proof of this result, the reader should note that the assumptions needed for showing the usefulness of a buy-out price are more demanding for $n > 2$ than for $n = 2$. In particular, $f'(\cdot)$ is assumed to exist for the case with more than two bidders.

Proof. We can write expected revenue in the first auction as

$$ER_1(\hat{v}) = \int_0^{\hat{v}} b^1(x)n(n-1)(F(\hat{v}) - F(x))F^{n-2}(x)f(x)dx + B(\hat{v})(1 - F^n(\hat{v}))$$

or, after inserting $B(\hat{v})$, as

$$\begin{aligned} ER_1(\hat{v}) &= \int_0^{\hat{v}} b^1(x)n(n-1)(F(\hat{v}) - F(x))F^{n-2}(x)f(x)dx \\ &+ \int_0^{\hat{v}} b^1(x)n(n-1)(1 - F(\hat{v}))F^{n-2}(x)f(x)dx \\ &+ \int_{\hat{v}}^{m(\hat{v})} \left(kxF_{1,n-2}(kx) + \int_{kx}^{\hat{v}} yf_{1,n-2}(y)dy \right) \frac{1}{2}n(n-1)(1 - F(\hat{v}))f(x)dx \\ &+ \hat{v} \left[1 - \left(F^n(\hat{v}) + n(1 - F(\hat{v}))F^{n-1}(\hat{v}) + \frac{1}{2}n(n-1)(1 - F(\hat{v}))F^{n-2}(\hat{v})(F(m(\hat{v})) - F(\hat{v})) \right) \right] \end{aligned}$$

Now, in order to compare $ER_1(\hat{v})$ with expected revenue in the second auction, we identify four possible events:

(i) All n buyers have type below \hat{v} . The probability of this event is $F^n(\hat{v})$, or

$$\int_0^{\hat{v}} n(n-1)(F(\hat{v}) - F(x))F^{n-2}(x)f(x)dx.$$

Notice the relationship to the first term in $ER_1(\hat{v})$

(ii) Precisely one buyer has type above \hat{v} . The probability of this event is clearly $n(1 - F(\hat{v}))F^{n-1}(\hat{v})$. Again, notice the relationship to the second term in $ER_1(\hat{v})$.

(iii) Precisely 2 buyers have type above \hat{v} , but whichever of the two who wins stage one has a type not exceed $m(\hat{v})$, implying that his bid, and the price, in stage two will be below \hat{v} . If $m(\hat{v}) = \bar{v}$, the probability of this event is

$$n(n-1)\frac{1}{2}F^{n-2}(\hat{v})(1 - F(\hat{v}))^2,$$

which is the probability mass in the third term in $ER_1(\hat{v})$, when $m(\hat{v}) = \bar{v}$. When $m(\hat{v}) = \hat{v}/k$, the probability is

$$n(n-1)F^{n-2}(\hat{v})(1 - F(m(\hat{v})))\left(F(m(\hat{v})) - F(\hat{v})\right)\frac{1}{2} - n(n-1)\frac{1}{2}F^{n-2}(\hat{v})(F(m(\hat{v})) - F(\hat{v}))^2,$$

since it is the probability that the buyer with the highest type has type above $m(\hat{v})$, yet he loses to a buyer with type in the range \hat{v} to $m(\hat{v})$ (the probability of which is .5), less the probability that both of the buyers who are willing to accept $B(\hat{v})$ has type in the range \hat{v} to $m(\hat{v})$. Notice that this probability can be rewritten as

$$\frac{1}{2}n(n-1)F^{n-2}(\hat{v})(1 - F(\hat{v}))(F(m(\hat{v})) - F(\hat{v})),$$

which is easily seen to be the probability mass in the third term of $ER_1(\hat{v})$.

(iv) All other possibilities not included in (i), (ii), (iii). The probability of this is clearly the term in brackets in the fourth term in $ER_1(\hat{v})$.

Only in case (i) will the buy-out price not be accepted in stage one. In this case, if x denotes the second-highest type, the price in stage one is $b^1(x)$, and we know that *the price in stage two is at least $b^1(x)$* , since it would have been precisely $b^1(x)$ if the highest type had been precisely x rather than higher than x . Hence, in case (i), *expected revenue in stage two exceeds the first term in $ER_1(\hat{v})$* . We will proceed in this manner to show that, case for case, expected revenue is higher in stage two than in stage one.

Regarding the second case, consider the following. If precisely one buyer has type above \hat{v} and the runner-up has type x , $x \in [0, \hat{v})$, *the price in stage two must be at least $b^1(x)$* , again because it would have been precisely $b^1(x)$ if the highest type had been x rather than above \hat{v} , where $\hat{v} > x$.

In the third case, let x be the type of the winner of stage one, who, by assumption, has type between \hat{v} and $m(\hat{v})$. Then, it is easily seen that *the term in parenthesis is the price in stage two*.

Finally, notice that in cases (i), (ii) and (iii), the price in stage two is at most \hat{v} , but in all other cases (i.e. case (iv)) *the price in stage two must be at least \hat{v}* .

We have now shown that, case for case, expected revenue in stage two is higher than revenue in stage one, which concludes the proof. ■

3.3 Market efficiency

The use of a buy-out price by the first seller has implications for the overall market. In this section, we study how the allocation of objects is affected, and how overall revenue and revenue to the second seller change.

First, notice that when $n > 2$, any $\hat{v} < \bar{v}$ has the potential to cause the final allocation to be inefficient. The reason is that the bidder with the third-highest valuation may win the first auction by accepting the buy-out price, which is clearly inefficient. Similarly, it is possible that the bidder with the second-highest valuation wins the first auction by accepting the buy-out price in cases where it is efficient for the bidder with the highest type to win both objects.

Of course, when $n = 2$ only the second possibility is relevant. In this case, observe that $\hat{v} \geq k\bar{v}$ does not lead to inefficiency. If bidder 2, rather than bidder 1, wins stage one, we can conclude that $v_2 \geq \hat{v} \geq k\bar{v} \geq kv_1$, so it is efficient for each bidder to win one unit. Hence, the buy-out price may lead to a change in the order in which units are won, but not to a change in the final allocation. However, if $\hat{v} < k\bar{v}$ it may be efficient for the bidder with the highest type to win both objects, but the competing bidder may end up purchasing the first object by accepting the buy-out price.

For $n = 2$, Kirkegaard and Overgaard (2003) have shown that the *optimal* \hat{v} is strictly below $k\bar{v}$.²³ Therefore, the use of a buy-out price by the first seller leads to inefficiency.

Corollary 1 *The sequence of auctions is inefficient when the first seller sets the buy-out price optimally.*

Next, we argue that the introduction of a buy-out price by the first seller is detrimental to overall revenue. Subject to a standard regularity condition, Kirkegaard (2004) establishes that any efficient mechanism is revenue superior to any mechanism, in which it is never the case that a bidder wins two units more often than is efficient.²⁴ This result applies to the sequence of auctions with a buy-out price in stage one, since it may award a unit to a bidder who would not have received one in an efficient mechanism, and it never awards two units to a bidder who should have received none or one. In particular, it is evident from the preceding discussion that if the sequence is inefficient, it is because a bidder wins one unit, when he would have won none in an efficient mechanism.

²³The example in Section 5 illustrates that this is not always the case when $n > 2$.

²⁴The regularity condition is the well known condition that $v - (1 - F(v))/f(v)$ is strictly increasing. An additional assumption on marginal utility is needed, but it is satisfied by the assumption used here, namely that the second unit consumed is worth k times as much as the first unit consumed. The result also holds when $k \in \{0, 1\}$.

Proposition 6 (Kirkegaard (2004)) *If demand is regular, any $\hat{v} < \bar{v}$ leads to a strict loss in (expected) overall revenue, when $n > 2$. When $n = 2$, any $\hat{v} < k\bar{v}$ (including the optimal \hat{v}) leads to a strict loss in (expected) overall revenue, but overall revenue is unaffected if $\hat{v} \geq k\bar{v}$.*

For $n = 2$, Kirkegaard and Overgaard (2003) show that even if the regularity condition is violated, the sequence of auctions with a buy-out price yields lower revenue than without a buy-out price.

Proposition 6 implies that the use of a buy-out price in the first auction can only be explained as long as the two units are owned by competing sellers. If the units are owned by the same seller, his objective is clearly to maximize overall revenue. In this case, however, Kirkegaard and Overgaard (2003) show that the seller can benefit from offering a buy-out price in the second auction, rather than in the first. The reason is that such a buy-out price would favour the bidder who won the first auction, and may lead him to win two units in cases where he should have won only one. A similar outcome occurs in a monopoly with quantity discounts, in the sense that efficiency can be improved by taking a unit away from someone who bought two, and give it to someone who bought none. Since quantity discounts generally improve monopoly profits, this explains why a buy-out price in the second auction can be beneficial to a single seller with two units.²⁵

For competing sellers, however, the *optimal* buy-out price was shown to strictly improve expected revenue to the first seller. Nevertheless, it is not hard to see that a *non-optimal* buy-out price may hurt the first seller (consider, for example, $\hat{v} = 0$). To examine the effects of a buy-out price in the first stage on the second seller, notice that the second seller is influenced only if the first stage was won by someone other than the bidder with the highest type. There are two possible cases.

Arranging bidders in descending order, $v_1 > v_2 > \dots > v_n$, the first possibility to be studied is that the first stage is won by bidder 2 (he accepts the buy-out price). In this case, bidder 1 wins stage two, and the price must be $\max\{kv_2, v_3\}$. Then, the second seller is adversely affected. If $kv_1 > v_2$, bidder 1 would have won stage two (and one) in the absence of a buy-out price in stage one, but at a price of $v_2 > \max\{kv_2, v_3\}$. If $v_2 > kv_1$, the second stage would have been won by bidder 2, at a price of $\max\{kv_1, v_3\} \geq \max\{kv_2, v_3\}$. Heuristically, observe that as k declines, the second event, $v_2 > kv_1$, is more likely to occur for given v_1, v_2 , and that $\max\{kv_1, v_3\}$ as well as $\max\{kv_2, v_3\}$ both equals v_3 for sufficiently small k . Hence, when k is small, the second seller is not affected very much when the first stage was won by bidder 2.

The second possibility is that the first stage was won by a bidder other than bidder 1 or 2. Then, bidder 1 will win stage two, at a price of v_2 . Without a buy-out price in stage one, the price in stage two would also have been v_2 if $kv_1 > v_2$, but it would only have been $\max\{kv_1, v_3\} < v_2$ if $v_2 > kv_1$. Hence, when some bidder other than bidder 1 or 2 wins stage one, the second seller is better off. Again, notice that as k declines, the second event, $v_2 > kv_1$, is more likely for given v_1, v_2 , and the gain in this case, $v_2 - \max\{kv_1, v_3\}$, increases as k decreases. Thus, when stage one is won by someone other than bidder 1 or 2, the second seller is better off the smaller k is.

Thus, the effect of a buy-out price in stage one on the second seller is ambiguous. However, when the *optimal* buy-out price is offered in stage one, the second seller must be worse off. The reason is that the sum of revenues declines, while the first seller, by definition, is better off. Also, observe that for $n = 2$, the second seller is unambiguously worse off for *any* buy-out price.

Corollary 2 *For $n = 2$, the second seller is worse off regardless of the buy-out price employed by the first seller. When $n > 2$, the second seller is worse off when the first seller uses the optimal buy-out price.*

Nevertheless, there are cases, for $n > 2$, when the second seller is better off when the first seller uses a

²⁵Contrast this to the case of quantity premia, where efficiency can be improved by taking one unit from someone who bought one unit, and give it to someone else who also bought only one unit, but who would value a second unit highly. The sequence of auctions with a buy-out price in the first auction gives rise to the same kind of inefficiency as quantity premia, and since these are seldomly profitable, the negative conclusion in Proposition 6 is explained.

non-optimal buy-out price.²⁶ It is clear from the arguments above that as k decreases, it becomes more likely that any change to the second seller's revenue will be a positive change. Likewise, the lower \hat{v} is, the more likely it is that the first stage is won by a bidder other than bidder 1.²⁷ So, to illustrate that the second seller can be made better off, consider the case where k is negligible and $\hat{v} = 0$ (implying the first seller earns nothing). Then, the first stage is won by bidder 1 or bidder 2 with probability $2/n$, in which case revenue in stage two is about v_3 (since k is small), and won by another bidder with probability $(n-2)/n$, in which case revenue is v_2 . Hence, expected revenue in stage two, for given v_1, v_2, \dots, v_n , is approximately

$$\frac{2}{n}v_3 + \frac{n-2}{n}v_2,$$

which is obviously larger than v_3 , the approximate price without a buy-out price in stage one.

It is now easy to see why *multi-unit demand is necessary* in order to explain the use of a buy-out price in the first auction. When $n > 2$ and bidders have unit demands, $k = 0$, the above arguments show that the second seller is unambiguously better off when a buy-out price is used in stage one. Combining this with the fact that overall revenue declines leads to the conclusion that the first seller is worse off for any $\hat{v} < \bar{v}$, and therefore will elect not to use a buy-out price. If $n = 2$ there is no excess demand when $k = 0$, and both sellers will earn zero revenue, regardless of the buy-out price.

Likewise, it is *necessary that marginal utility is strictly decreasing*, or $k < 1$. If demand is flat, $k = 1$, revenue in the second auctions is v_2 independently of the outcome in the first auction, and by Proposition 6 it follows that the first seller is worse off, should he decide to offer a buy-out price. Alternatively, this can be interpreted as saying that for the first seller to offer a buy-out price, *it is necessary that there are future auctions*. The reason is that when $k = 1$, bidder behavior in the second auction is independent of the outcome of the first auction, so when deciding on a strategy in the first auction, bidders need not take into consideration that there are future auctions. Hence, the first auction might as well be a one-shot auction.²⁸

4 Discussion

In this section, we first provide an alternative, and very brief, proof that a buy-out price is profitable when $n = 2$. Thereafter, we turn to a discussion of the modelling assumptions and the robustness of our results in the general case.

4.1 Buy-out prices with two bidders

Assuming, without loss of generality, that $v_1 > v_2$, the first auction is won by bidder 1 when there is no buy-out price, and with only two bidders it follows that the price in the second auction is $\min\{kv_1, v_2\}$. With a buy-out price, however, the first stage may be won by bidder 2, in which case revenue in the second stage is $kv_2 < \min\{kv_1, v_2\}$. Hence, as mentioned earlier, the second seller is worse off when the first seller offers a buy-out price.

Next, notice that if $\hat{v} \geq k\bar{v}$, the final allocation is the same as without the buy-out price, although the order in which bidders win units may be reversed. Now, since the allocation is the same in the two

²⁶Since overall revenue declines, and the second seller is better off, the first seller must have offered a buy-out price that leads him to earn lower revenue.

²⁷For $n = 2$, this leads Kirkegaard and Overgaard (2003) to conclude that the second seller's revenue is increasing in \hat{v} , because if there is a change to the second seller's revenue, it is a unambiguously a change for the worse when $n = 2$. This result does not hold for $n > 2$.

²⁸In one-shot auctions with risk neutral and patient bidders, it is well known that the second price sealed bid auction maximizes expected revenue absent reserve prices. Hence, there is no incentive to modify it with a buy-out price. As mentioned earlier, a buy-out price can also be explained by relaxing the assumptions of risk neutrality or patience.

different mechanisms, overall revenue must also be the same, by the *Revenue Equivalence Theorem*. As overall revenue is unchanged, and the second seller is worse off, the first must be better off. Consequently, it pays for the first seller to offer a buy-out price with $\hat{v} \in [k\bar{v}, \bar{v})$. In fact, it can be shown that the optimal \hat{v} is below $k\bar{v}$ (see Kirkegaard and Overgaard (2003) and the example in the next section).

4.2 English auctions

Formally, we have modelled the two auctions as second-price, *sealed-bid* auctions. This allows us to focus on the role of buy-out prices and economize somewhat on the discussion of the auctions, since a strategy in the first auction is simply a bid.

In contrast, a strategy in the first of two *English* auctions is more complex, as it specifies when to stop bidding, contingent on the information at the time, which may include whether and when other bidders have dropped out. Février, Roos and Visser (2003) derive the equilibrium strategies in this case.

Without a buy-out price, the sequence of auctions are revenue equivalent stage by stage, regardless of whether they are modelled as second-price, sealed-bid auctions or English auctions.

Similarly, with a buy-out price, it can be shown that the payoff to a type \hat{v} bidder from accepting the buy-out price is the same for either auction format, and so is the payoff from rejecting. Hence, $B(\hat{v})$ does not depend on the auction format. Likewise, for given \hat{v} , the expected revenue in the two auctions are independent of the auction format.

4.3 Permanent buy-out prices

In this paper we have chosen to focus on *eBay*'s version of the buy-out price. The buy-out price is temporary on *eBay*, whereas *Yahoo!* offers a permanent buy-out price (termed the *Buy Price*). Reynolds and Wooders (2003) compare the two types of buy-out prices²⁹ in one-shot auctions.

With a permanent buy-out price the auction can be thought of as an English auction, where, at any time, a bidder can choose to accept the buy-out price, rather than continuing the bidding process. Then, bidders with very high valuations may accept the buy-out price immediately. Bidders with lower valuations initially ignore the buy-out price, but as bidding in the English auction progresses, they become more and more pessimistic about the severity of the competition and eventually accept the buy-out price (given that it is lower than the valuation). The higher the valuation, the sooner the buy-out price is accepted.

If the buy-out price is very high, not even a bidder with valuation \bar{v} will accept it immediately. As the price in the English auction increases, however, the buy-out price may be accepted. If so, it is accepted by the bidder with the highest valuation, and the auction is efficient. On the other hand, if the buy-out price is such that it would be accepted by a bidder with valuation in the interval $[\hat{v}, \bar{v}]$ in an *eBay* auction, the same bidder accepts it immediately in the *Yahoo!* auction. If it is not accepted immediately, it may be accepted later on, by the high-valuation bidder. Hence, $B(\hat{v})$ causes the same type of inefficiency whether it is a permanent or temporary buy-out price.

By the *Revenue Equivalence Theorem*, the *eBay* and *Yahoo!* auction formats yield the same revenue, for a given $B(\hat{v})$. Thus, our results are also valid when the buy-out price is permanent. In fact, we conjecture that the results thus far are robust to small changes in the extensive form of the game. The reason is straightforward, and relies only on the possibility that the buy-out price may affect who wins the first auction.

In particular, consider the case $n = 2$ and assume that in the equilibrium (on which bidders coordinate) of the particular game, there is a \hat{v} such that stage one is won by the bidder with the highest valuation, if at most one bidder has a valuation that exceeds \hat{v} , but that there is a strictly positive probability that it is won by the other bidder otherwise. In this event, the second seller is worse off (because competition

²⁹Reynolds and Wooders (2003) assume there are two bidders. Hidvégi, Wang and Whinston (2003) analyze the permanent buy-out price with an arbitrary number of bidders.

is diminished in stage two). At the same time, however, we know that overall revenue is unchanged if $\hat{v} \geq k\bar{v}$ (because the sequence of auctions is efficient in this case). Consequently, when \hat{v} is sufficiently high, the first seller is better off with a buy-out price as long as the identity of the winner in stage one changes with positive probability.

4.4 Left-overs from previous paper (to be pruned)

4.4.1 Buy-outs, inefficiency and revenue non-equivalence

As mentioned, the *Revenue Equivalence Theorem* reveals that if two mechanisms yield the same allocation, expected revenue in the two mechanisms must also be the same. Since the outcome of the bench-mark model is efficient, it follows that introducing a buy-out price changes total revenue if and only if³⁰ the resulting allocation is inefficient.

Introducing a buy-out price in the *first* auction results in the following kind of inefficiency: an agent may win one item when he would have won none without the buy-out price. A buy-out price in the *second* auction may cause another type of inefficiency: an agent may win both items, when he would have won exactly one without a buy-out price. In the latter case, an agent who would have won one unit in an efficient mechanism risks not winning one at all.

To understand the consequences of these different kinds of inefficiencies, it is useful to exploit the similarities between monopoly pricing and auctions³¹. When a monopolist faces agents with multi-unit demands, it is well known that the optimal pricing schedule may involve quantity discounts. These discounts enable the monopolist to sell several units to agents with high *marginal revenue* on all units, without at the same time selling to agents with low marginal revenue on some units. Whether agents have unit or multi-unit demands, it is well understood that the key ingredient in the monopolist's optimization problem is marginal revenue.

Now, the expression for what amounts to marginal revenue of a bidder with valuation v in an auction is

$$J(v) = v - \frac{1 - F(v)}{f(v)}$$

for the first unit, and it can easily be shown that marginal revenue is $kJ(v)$ for the second unit.³² The expected revenue to the seller is then

$$E \left[\sum_{i=1}^2 (q_i^1(v_1, v_2)J(v_i) + q_i^2(v_1, v_2)kJ(v_i)) \right] - 2EU(0, 0) \quad (7)$$

where $q_i^j(v_1, v_2)$ is the probability that agent i wins at least j units, given that the two agents are of type v_1 and v_2 , respectively. The last term is the expected rent obtained by an agent of type $\underline{v} = 0$ in the mechanism. (7) is the counterpart of the revenue for a monopolist, who earns the area under the marginal revenue curve.

Clearly, if $EU(0, 0)$ is the same across different mechanisms, and if these mechanisms implement the same allocation, (i.e., the same $q_i^j(v_1, v_2)$), expected revenue must be the same. This is the *Revenue Equivalence Theorem*.

³⁰This assumes that an agent of type \underline{v} is indifferent between the two mechanisms. We will return to this point momentarily.

³¹These similarities were first pointed out by Bulow and Roberts (1989) for auctions with unit demand, see also Bulow and Klemperer (1996) and Klemperer (1999). Maskin and Riley (1989) draw parallels between auctions with multi-unit demand and non-linear pricing. For more on the latter, see also Kirkegaard (2004).

³²For a derivation of $J(v)$, see Myerson (1981) or Bulow and Roberts (1989). Since willingness-to-pay for a second unit is k times that for the first unit, it is unsurprising that marginal revenue of the second unit is k times marginal revenue of the first unit, see Kirkegaard (2004).

We are now equipped to provide an alternative proof of why overall revenue declines when an optimally chosen buy-out price is introduced by the first seller. Given that $0 < \hat{v} < k\bar{v}$, the allocation changes as a consequence of the buy-out price, if the winner of stage 1 would not have won a unit at all in the efficient allocation. If the winner of stage 1 has valuation v , this happens when $v < k\bar{v}$, and if the rival bidder has valuation $x \in (\frac{v}{k}, \bar{v})$. In this case the revenue *gain* from the winner of the first auction (who should have won no item) is simply $J(v)$. The revenue *loss* from the loser of the first auction (who will only win the second item, when he should have won both) is $E(kJ(x) \mid x > \frac{v}{k})$, which we can write as

$$\int_{\frac{v}{k}}^{\bar{v}} kJ(x) \frac{f(x)}{1 - F(\frac{v}{k})} dx = v$$

Now, $J(v) < v$, and we conclude that, given the event that the allocation has changed, the marginal revenue gained falls short of the marginal revenue lost. Thus, overall revenue decreases since the first term in (7) declines, while second term is unchanged. Hence, it is not profitable to allow an agent to win *one* unit too often, compared to the efficient allocation.

4.4.2 One-shot auctions

We have already argued that when $k = 1$ (horizontal demands), stage 1 is equivalent to a one-shot auction. In one-shot auctions, revenue is clearly maximized by allocating the object to the agent with the highest marginal revenue. When the agent with the highest valuation is also the agent with the highest $J(v)$, that is, when $J(v)$ is increasing in v , this is accomplished with an efficient mechanism. However, when $J(v)$ is non-monotonic, it is impossible to always give the object to the agent with the highest marginal revenue. The reason is that the auctioneer must respect the incentive compatibility constraints when designing his mechanism. To satisfy these, it is necessary that the probability of winning the object is non-decreasing in the valuation.

In the cases where $J(v)$ is non-monotonic, the rules of the optimal mechanism³³ ensure that the probability of winning is constant over a subset of valuations. That is, agents with different valuations have the same probability of winning, and therefore contributes marginally the same to revenue. Hence, the optimal mechanism is said to “iron” the marginal revenue curve. Now, we observe that the buy-out price is a crude way of ironing the marginal revenue curve, since all agents with valuation above \hat{v} have the same probability of winning in a one-shot auction. It is crude because the interval on which marginal revenue is ironed in an optimal mechanism is always interior, whereas the buy-out price also bundles valuations close to and including \bar{v} with lower valuations.

Since buy-out prices offer some (excessive) ironing, it is perhaps not obvious whether or not buy-out prices can increase revenue when $J(v)$ is non-monotonic and $k = 1$. However, our model is sufficiently general to encompass these situations, and we can therefore conclude that buy-out prices are counterproductive even when some ironing is called for, precisely because the ironing is too crude. We stress this, since we are not aware of any papers on auctions (or monopoly) showing that “ironing” may be counterproductive, if it is too crude in the sense of this paper. Among the related papers the model of Budish and Takeyama (2001) is discrete, while Reynolds and Wooders (2003) assume uniformly distributed valuations. Ironing is not an issue in either of these specifications. Mathews (2002) also assumes uniform distributions, but he remarks that his results hold for any distribution, though without referring to ironing.

Thus, in one-shot auctions, buy-out prices are unprofitable when utilized in the way assumed so far, even if the optimal auction involves ironing of the marginal revenue curve. However, a more sophisticated design can combine buy-out prices and reserve prices to maximize revenue in these cases.

³³See Myerson (1981) or Bulow and Roberts (1989).

As mentioned, the optimal interval on which ironing should be performed is interior. Let this interval³⁴ be given by $[\hat{v}, v^r]$, $0 < \hat{v} < v^r < \bar{v}$. Consider the following auction for one object, which takes place in two stages. First, an auction with a reserve price is staged. If the object is not sold, another auction is staged, in which a buy-out price is available. The reserve price and the buy-out price should jointly be set in such a way that a bidder bids in the first stage if and only if his valuation is larger than v^r , and such that the buy-out price in stage 2 is accepted if and only if the bidder has a valuation that exceeds \hat{v} .^{35,36}

Now, the auction is efficient if at most one bidder has a valuation in the interval $[\hat{v}, v^r]$. Otherwise, however, both bidders will not participate in the first stage, but will instead accept the buy-out price in stage 2. Hence, the bidders have an even chance of winning, and this chance is, importantly, independent of the exact valuation. In other words, the marginal revenue curve has been optimally ironed, and it follows that the proposed two stage auction maximizes revenue.³⁷

5 Example (incomplete)

To add some further insights into the results above, in this section we consider the *uniform* case with $v \in [0, 1]$, that is, $\bar{v} = 1$, $f(v) = 1$ and $F(v) = v$.

5.1 No buy-out

Assume first that there is no buy-out price. Bidding in the first round as captured by (1) reduces to

$$b^1(v) = \frac{n-2+k^{n-1}}{n-1}v = v - \frac{1-k^{n-1}}{n-1}v$$

where we note that $b^1(v) \rightarrow \frac{n-2}{n-1}v$ for $k \rightarrow 0$ (*single-unit* demand),³⁸ $b^1(v) \rightarrow v$ for $k \rightarrow 1$ (*horizontal* demand) and $b^1(v) \rightarrow v$ for $n \rightarrow \infty$ (*unlimited* competition). The expected revenues in the two auctions reduce to

$$ER_1^{SSP} = \frac{n-2+k^{n-1}}{n+1}$$

and

$$ER_2^{SSP} = \frac{n-2+k^{n-1}}{n+1} + \frac{n-1}{n+1}k^{n-1}(1-k) = ER_1^{SSP} + \frac{n-1}{n+1}k^{n-1}(1-k)$$

while total revenue is

$$\sum_{t=1}^2 ER_t^{ssp} = \frac{2(n-1) + (n+1 - (n-1)k)k^{n-1}}{n+1}$$

We plot ER_1^{SSP} and ER_2^{SSP} against k for different n in Fig. 1 - 4, where ER_2^{SSP} is the *fat* line, while ER_1^{SSP} is *thin*.

³⁴We assume there is only one such interval. However, it should be obvious how to extend the following mechanism if there are more.

³⁵To achieve this, it is necessary that the buy-out price in stage 2 is known before stage 1 commences.

³⁶It is straightforward to show that such a combination of a reserve price and a buy-out price exists.

³⁷To be precise, the auction is optimal among all auctions that sell the object with probability one. Notice that the reserve price in this auction does not serve to ration output. Obviously, such a reserve could be added to the second stage of the auction.

³⁸This is a special case of an example developed by Krishna (2002, Example 15.2, p. 219).

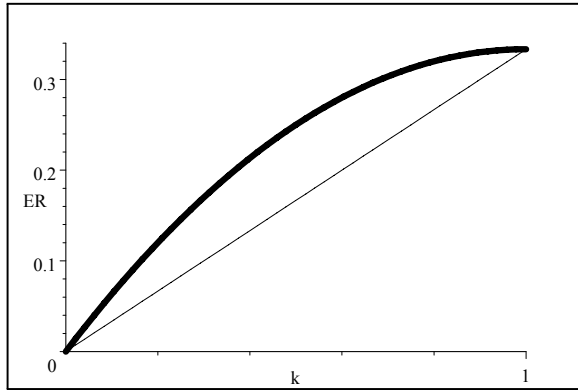


Fig. 1: SSP auctions with two bidders

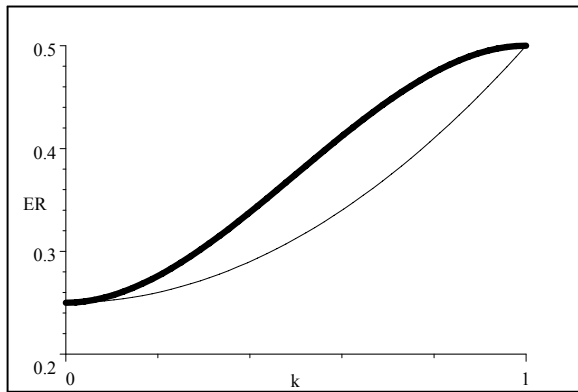


Fig. 2: SSP auctions with three bidders

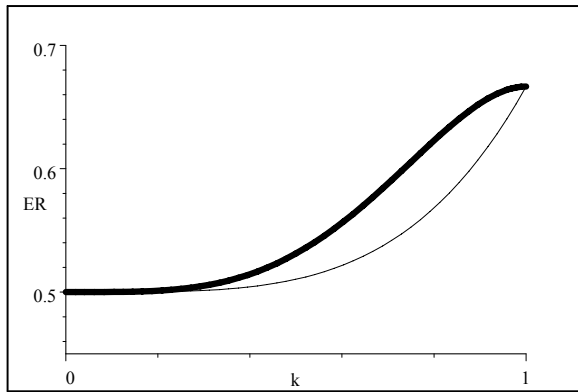


Fig. 3: SSP auctions with five bidders

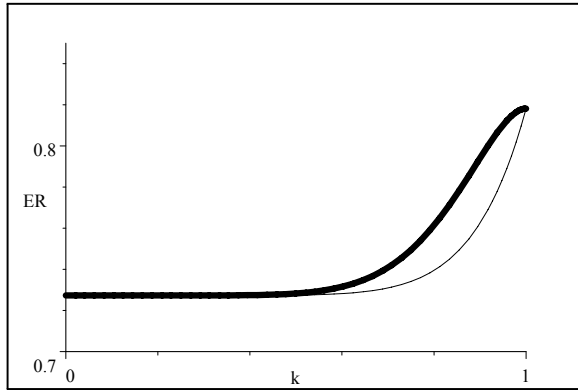


Fig. 4: SSP auctions with ten bidders

The *ratio* between expected revenues in the first and second auction

$$RR = \frac{ER_1^{SSP}}{ER_2^{SSP}} = \frac{n - 2 + (k)^{n-1}}{n - 2 + (k)^{n-1} + (n - 1)(k)^{n-1}(1 - k)}$$

is illustrated in Fig. 5.

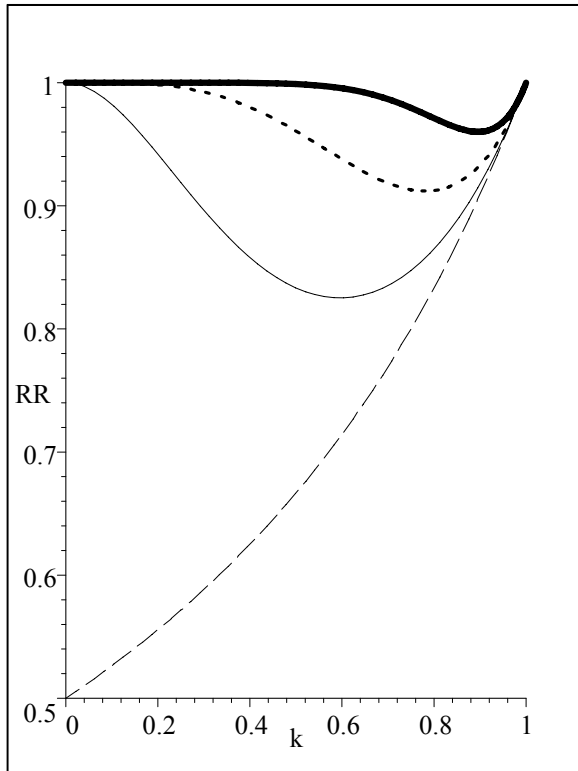


Fig. 5: The revenue-ratio in two SSP auctions
($n = 2$ - dashed, $n = 3$ - thin, $n = 5$ - dotted, $n = 10$ - fat)

For $n = 2$, note the discontinuity at $k = 0$. When $k = 0$, both sellers earn nothing, that is, the same. However, when k is small, but strictly positive, we observe that the winner of the first auction is very unlikely also to be the winner of the second auction. Hence, the expected revenue in the first auction is k times (the expected value of) the second highest valuation, while the expected revenue in the second auction is *approximately* k times (the expected value of) the highest valuation. For the uniform case with $n = 2$, the ratio between the expected value of the highest ($2/3$) and the expected value of the second highest valuation ($1/3$) is exactly $1/2$. For $n > 2$, there are no such discontinuities.

From this example it is immediate that the difference in expected revenues can be significant unless k is close to one (demands are near-horizontal) or, for $n > 2$, if k is close to zero (near unit-demands). Any difference, of course, disappears as the number of bidders increases without bound.

5.2 Buy-out

Now, consider introducing a buy-out price in the example.

First round bidding is captured by **(to be added)**

Given \hat{v} , with uniform distributions on the unit interval, the revenues in (5) and (6) reduce to

$$ER_1(\hat{v}) = \left\{ \begin{array}{l} \hat{v} - \frac{n}{2}\hat{v}^{n-1} + (n-2)\hat{v}^n - \frac{(n-1)(n-2)}{2(n+1)}\hat{v}^{n+1} \\ + \frac{k^{n-1}}{2}(1-\hat{v} + \hat{v}^n - \frac{n-1}{n+1}\hat{v}^{n+1}) \\ \hat{v} - \frac{n}{2}\hat{v}^{n-1} + (n-2)\hat{v}^n - \frac{(n-1)(n-2)}{2(n+1)}\hat{v}^{n+1} \\ + \frac{k^{n-1}}{2}(1-\hat{v} + \hat{v}^n - \frac{n-1}{n+1}\hat{v}^{n+1}) \\ - \frac{1}{2k}(1-\hat{v})(k^n - \hat{v}^n - n\hat{v}^{n-1}(k-\hat{v})) \end{array} \right\} \begin{array}{l} \text{if } \hat{v} \geq k \\ \\ \\ \text{if } \hat{v} \leq k \end{array}$$

and **(to be added)**

$$ER_2(\hat{v}) = \left\{ \begin{array}{l} \hat{v} \geq k \\ \hat{v} \leq k \end{array} \right.$$

5.2.1 Two bidders

With two bidders, the revenues are

$$ER_1(\hat{v}) = \left\{ \begin{array}{l} \frac{k}{3} + \frac{k}{6}(1-\hat{v})^3 \\ \frac{k}{3} + \frac{k}{6}(1-\hat{v})^3 - \frac{1}{2k}(k-\hat{v})^2(1-\hat{v}) \end{array} \right. \begin{array}{l} \hat{v} \geq k \\ \hat{v} \leq k \end{array}$$

and

$$ER_2(\hat{v}) = \left\{ \begin{array}{l} \frac{k}{3} + \frac{k}{6}(1-\hat{v})^3 + \frac{k}{3}((1-k) - (1-\hat{v})^3) \\ \frac{k}{3} + \frac{k}{6}(1-\hat{v})^3 - \frac{1}{2k}(k-\hat{v})^2(1-\hat{v}) \\ + \frac{k}{3}((1-k) - (1-\hat{v})^3) + \frac{1}{6k}(k-\hat{v})^2(3(1-\hat{v}) + (k-\hat{v})) \end{array} \right\} \begin{array}{l} \hat{v} \geq k \\ \hat{v} \leq k \end{array}$$

The latter can be written as

$$ER_2(\hat{v}) = \left\{ \begin{array}{l} ER_1(\hat{v}) + \frac{k}{3}((1-k) - (1-\hat{v})^3) \\ ER_1(\hat{v}) + \frac{k}{3}((1-k) - (1-\hat{v})^3) + \frac{1}{6k}(k-\hat{v})^2(3(1-\hat{v}) + (k-\hat{v})) \end{array} \right. \begin{array}{l} \hat{v} \geq k \\ \hat{v} \leq k \end{array}$$

The sum of (expected) revenues is

$$\sum_{t=1}^2 ER_t(\hat{v}) = \left\{ \begin{array}{l} \frac{k(3-k)}{3} = \sum_{t=1}^2 ER_t^{ssp} \\ \frac{k(3-k)}{3} - \frac{(k-\hat{v})^2(2(1-\hat{v})+1-k)}{6k} < \sum_{t=1}^2 ER_t^{ssp} \end{array} \right. \begin{array}{l} \hat{v} \geq k \\ \hat{v} < k \end{array}$$

reflecting that efficiency and “revenue equivalence” is lost, once the buy-out price is set such that \hat{v} drops below k ($= k\bar{v}$).

To get a feel for the dependence of revenues on the buy-out price, consider the special case where $k = \frac{1}{2}$. Figure 6 plots $ER_1(\hat{v})$ (*thin*), $ER_2(\hat{v})$ (*dots*) and their sum (*fat*) against \hat{v} .

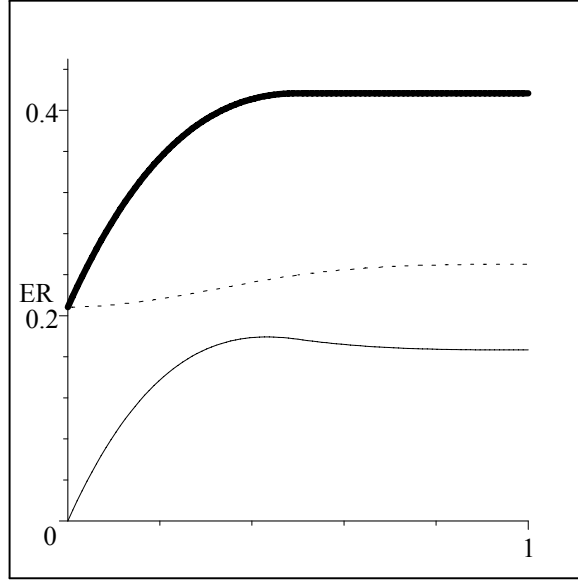


Fig. 6: Revenues against \hat{v}

Next, turn to the optimal value of the cut-off, hence, the first-round buy-out price. From Corollary 1 and Proposition 6 we know that $\hat{v} < k\bar{v} = k$, for any $k \in (0, 1)$. Maximizing

$$ER_1(\hat{v}) = \frac{k}{3} + \frac{k}{6} (1 - \hat{v})^3 - \frac{1}{2k} (k - \hat{v})^2 (1 - \hat{v})$$

with respect to \hat{v} gives the optimal cut-off valuation from the perspective of the first seller

$$v^* = \frac{1 + 2k - k^2}{3 - k^2} - \frac{((1 + 2k - k^2)^2 - 2k(3 - k^2))^{1/2}}{3 - k^2} < k = k\bar{v}$$

and the associated, optimal buy-out price, $B(v^*)$ is given by

$$B(v^*) = \frac{k}{2(1 + v^*)} \left((1 + (v^*)^2) - \left(1 - \frac{v^*}{k}\right)^2 \right)$$

We can substitute v^* into the revenue expressions, and Fig. 7 illustrates how $ER_1(v^*)$ (*thin*) and $ER_2(v^*)$ (*fat*) vary with k .

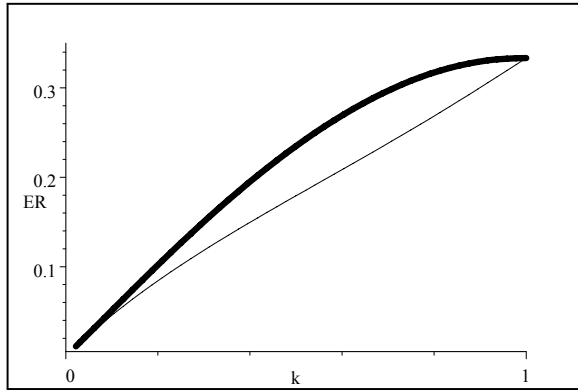


Fig. 7: Revenues with optimal buy-out

The ratio between the expected revenues given an optimally chosen buy-out price, $RR(BO) = \frac{ER_1(v^*)}{ER_2(v^*)}$, is illustrated in the following figure

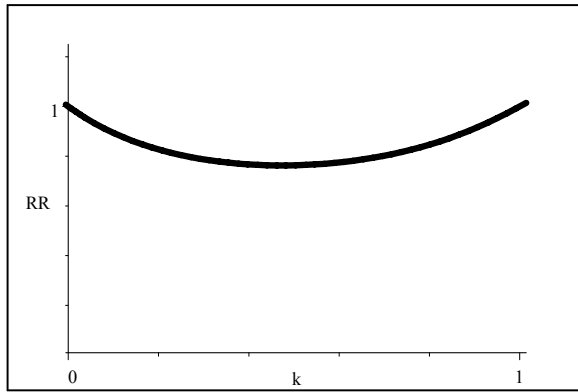


Fig. 8: Revenue ratio with optimal buy-out

We can compare with the case of two straight second-price auctions illustrated in Fig. 1 and Fig. 5. In Fig. 9 we merge the information in Fig. 1 and Fig. 7. The dashed lines are for two straight second-price auctions, while the solid lines are for the case where the first seller chooses the buy-out price to implement v^* .

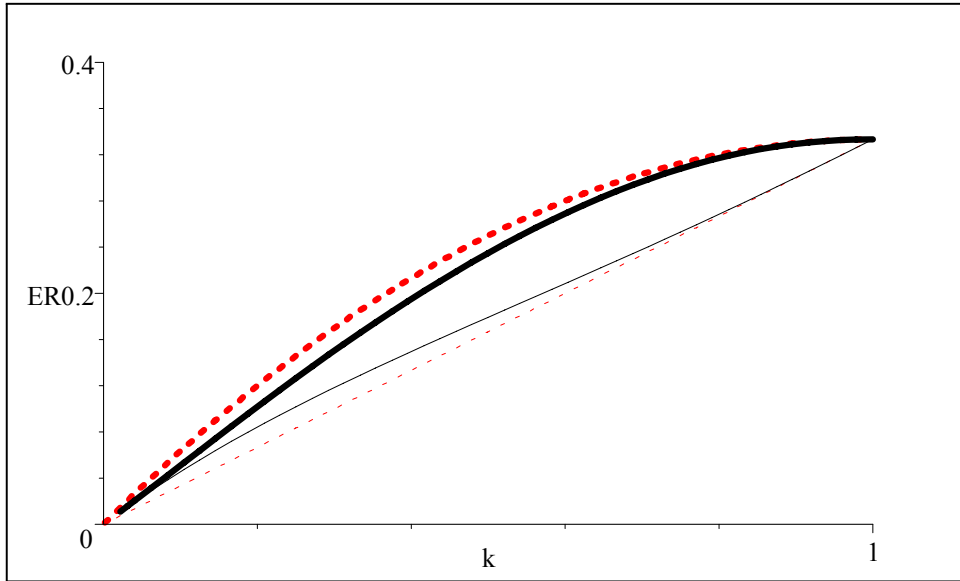


Fig. 9: Comparison of auction revenues

Fig. 10 merges the information from Fig. 5 and Fig. 7, and the *thin* line is for two straight second-price auctions, while the *heavy* line is associated with an optimal buy-out price.

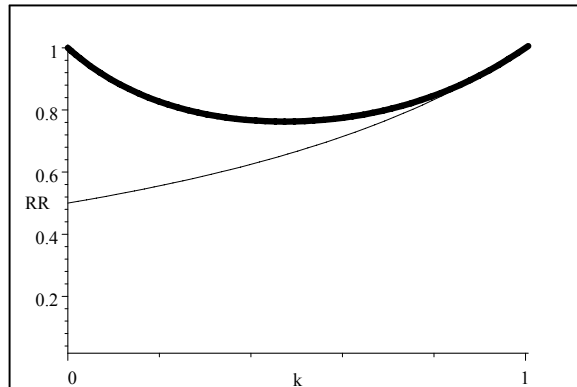


Fig. 10: Revenue Ratios

Finally, in Fig. 11 we plot the percentage gain to the first seller from an optimally chosen buy-out compared to the straight second-price auction, $G = 100 \times \frac{ER_1(v^*) - ER_1^{SSP}}{ER_1^{SSP}}$.

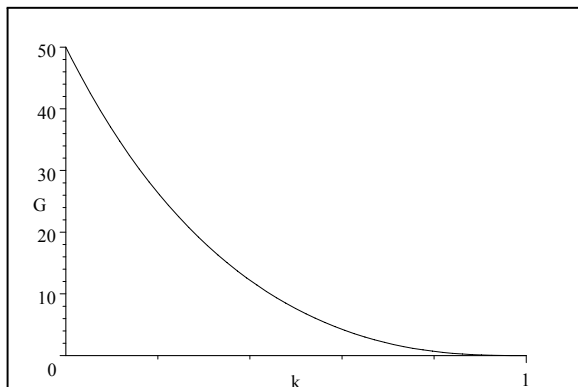


Fig. 11: Percentage gain from buy-out price

The last three figures essentially illustrate that the value from the perspective of the first seller of introducing a buy-out price is substantial when the individual demand functions are relatively steep (k small). When demands are steep, and there are only two bidders, the competition for the first object will be weak. It follows that the first seller has a strong incentive to try to improve his position in this case by introducing a suitably chosen buy-out price. The following table captures central features of the example in an alternative way.

| k | ER_1^{SSP} | v^* | $B(v^*)$ | $ER_1(v^*)$ | G |
|------|--------------|---------|----------|-------------|-------|
| 0.01 | 0.00333 | 0.00995 | 0.00495 | 0.00495 | 48.65 |
| 0.10 | 0.03333 | 0.09549 | 0.04597 | 0.04558 | 36.75 |
| 0.25 | 0.08333 | 0.22618 | 0.10623 | 0.10176 | 22.12 |
| 0.50 | 0.16667 | 0.43308 | 0.20404 | 0.17931 | 7.58 |
| 0.75 | 0.25000 | 0.66667 | 0.32222 | 0.25309 | 1.24 |

Recall that in this example revenue equivalence and efficiency is lost when \hat{v} is set below $k = k\bar{v}$. Hence, a comparison of the first and third column is indicative of the inefficiency when \hat{v} is set optimally. For example, when $k = k\bar{v} = \frac{1}{2}$ the optimal \hat{v} is approximately 0.43, which implies that there is a small, but “non-trivial”, probability that the final allocation is inefficient. Note that $k = \frac{1}{2}$ implies that $ER_2^{SSP} - ER_1^{SSP} = \frac{1}{3}k(1 - k)$ is maximized. When the first seller sets the optimal buy-out price $B(v^*) \approx 0.2$, he manages to increase his expected revenue by 7.58%, while total revenue falls by only 0.58%.

5.2.2 Three bidders

(to be added)

5.2.3 Discussion

(to be added)

6 Concluding Remarks

In this paper we sought to explain the use of buy-out prices by observing that online auction markets are dynamic, with players knowing that goods not presently on the market are likely to be offered in the

future and with buyers displaying multi-unit demands. It was shown that there is an incentive for an early seller to offer a buy-out price that is accepted with positive probability. Suitably chosen, such a buy-out price will increase revenue in the early auction. However, revenue in subsequent auctions will decrease, as will the sum of revenues. When the buy-out price in the early auction is chosen optimally, the resulting string of auctions is inefficient, in the sense that the item on offer may go to a buyer who would have received none in any efficient selling mechanism.

References

- Bagnoli, M., and T. Bergstrom, 1989, Log-Concave Probability and Its Applications, draft, University of Michigan.
- Black, J., and D. de Meza, 1992, Systematic Price Differences Between Successive Auctions Are No Anomaly, *Journal of Economics and Management Strategy* 1: 607-628.
- Budish, E., and L. Takeyama, 2001, Buy Prices in Online Auctions: Irrationality on the Internet?, *Economics Letters* 73: 325-333.
- Bulow, J., and P. Klemperer, 1996, Auctions vs. Negotiations, *American Economic Review* 86: 180-194.
- Bulow, J., and J. Roberts, 1989, The Simple Economics of Optimal Auctions, *Journal of Political Economy* 97: 1060-1090.
- Février, P., W. Roos and M. Visser (2003), The Revenue-Effects of the Buyer's Option in Multi-Unit Ascending Auctions: The Case of Wine Auctions at Drouot, draft, CREST-LEI, Paris.
- Hidvégi, Z., W. Wang and A. Whinston, 2003, Buy-Price English Auction, draft, University of Texas, Austin.
- Katzman, B., 1999, A Two Stage Sequential Auction with Multi-Unit Demands, *Journal of Economic Theory* 86: 77-99.
- Kirkegaard, R., 2004, Inefficiency and Non-Linear Pricing in the Optimal Multi-Unit Auction, draft, University of Aarhus.
- Kirkegaard, R., and P.B. Overgaard, 2003, Buy-Out prices in Online Auction: Multi-Unit Demand, draft, University of Aarhus.
- Klemperer, P., 1999, Auction Theory: A Guide to the Literature, *Journal of Economic Surveys* 13: 227-286.
- Krishna, V., 2002, *Auction Theory*, Academic Press, San Diego: CA.
- Lucking-Reiley, D., 2000, Auctions on the Internet: What's Being Auctioned and How?, *Journal of Industrial Economics* 48: 227-252.
- Maskin, E., and J. Riley, 1989, Optimal Multi-Unit Auctions, in F. Hahn (ed.), *The Economics of Missing Markets, Information and Games*, Oxford University Press, Oxford: UK.
- Mathews, T., 2002, Buyout Options in Internet Auction Markets, unpublished Ph.D. thesis, SUNY, Stony Brook.
- Myerson, R., 1981, Optimal Auction Design, *Mathematics of Operations Research* 6: 58-73.
- Reynolds, S., and J. Wooders, 2003, Ascending Bid Auctions with a Buy-Now Price, WP #2003-01, University of Arizona, Tucson.
- Weber, R., 1983, Multiple-Object Auctions, in R. Engelbrecht-Wiggans, M. Shubik and R. Stark (eds.), *Auctions, Bidding and Contracting: Uses and Theory*, New York University Press, New York: NY.

Appendix

Proof of Lemma 2. Since we have already derived the bidding strategies in the two auctions, arriving at the expected revenues is straightforward (though, somewhat tedious).

ER_1^{ssp}

In the first auction, bidders are bidding according to $b^1(v)$, and it follows that expected revenue is

$$ER_1^{ssp} = \int_0^{\bar{v}} b^1(v) f_{2,n}(v) dv = \int_0^{\bar{v}} \left(\frac{kvF_{1,n-2}(kv) + \int_{kv}^v y f_{1,n-2}(y) dy}{F_{1,n-2}(v)} \right) f_{2,n}(v) dv$$

or

$$ER_1^{ssp} = \int_0^{\bar{v}} kv \frac{F_{1,n-2}(kv)}{F_{1,n-2}(v)} f_{2,n}(v) dv + \int_0^{\bar{v}} \left(\int_{kv}^v y f_{1,n-2}(y) dy \right) \frac{f_{2,n}(v)}{F_{1,n-2}(v)} dv \quad (8)$$

Now, the first term, t_1 , in (8) can be rewritten as

$$t_1 = \int_0^{\bar{v}} kv \frac{F^{n-2}(kv)}{F^{n-2}(v)} f_{2,n}(v) dv \quad (9)$$

Noting that

$$\frac{f_{2,n}(v)}{F_{1,n-2}(v)} = \frac{n(n-1)(1-F(v))F^{n-2}(v)f(v)}{F^{n-2}(v)} = n(n-1)(1-F(v))f(v)$$

the second term, t_2 , in (8) can be rewritten as

$$\begin{aligned} t_2 &= n(n-1) \int_0^{\bar{v}} \left(\int_{kv}^v y f_{1,n-2}(y) dy \right) (1-F(v)) f(v) dv \\ &= n(n-1) \int_0^{\bar{v}} \left(\int_{kv}^v y f_{1,n-2}(y) dy \right) f(v) dv \\ &\quad - \frac{n(n-1)}{2} \int_0^{\bar{v}} \left(\int_{kv}^v y f_{1,n-2}(y) dy \right) 2F(v) f(v) dv \end{aligned}$$

Changing the order of integration, we can write the last expression as

$$\begin{aligned} &n(n-1) \int_0^{\bar{v}} y f_{1,n-2}(y) \left(\int_y^{m(y)} f(v) dv \right) dy - \frac{n(n-1)}{2} \int_0^{\bar{v}} y f_{1,n-2}(y) \left(\int_y^{m(y)} 2F(v) f(v) dv \right) dy \\ &= n(n-1) \int_0^{\bar{v}} y f_{1,n-2}(y) [F(v)]_y^{m(y)} dy - \frac{n(n-1)}{2} \int_0^{\bar{v}} y f_{1,n-2}(y) [F^2(v)]_y^{m(y)} dy \\ &= n(n-1) \int_0^{k\bar{v}} y f_{1,n-2}(y) \left[F\left(\frac{y}{k}\right) - F(y) \right] dy + n(n-1) \int_{k\bar{v}}^{\bar{v}} y f_{1,n-2}(y) [1 - F(y)] dy \\ &\quad - \frac{n(n-1)}{2} \int_0^{k\bar{v}} y f_{1,n-2}(y) \left[F^2\left(\frac{y}{k}\right) - F^2(y) \right] dy - \frac{n(n-1)}{2} \int_{k\bar{v}}^{\bar{v}} y f_{1,n-2}(y) [1 - F^2(y)] dy \end{aligned}$$

$$\begin{aligned}
&= n(n-1) \int_0^{k\bar{v}} y f_{1,n-2}(y) \left[F\left(\frac{y}{k}\right) - F(y) - \frac{1}{2}\left(F\left(\frac{y}{k}\right) + F(y)\right)\left(F\left(\frac{y}{k}\right) - F(y)\right) \right] dy \\
&\quad + n(n-1) \int_{k\bar{v}}^{\bar{v}} y f_{1,n-2}(y) \left[1 - F(y) - \frac{1}{2}(1 + F(y))(1 - F(y)) \right] dy \\
&= n(n-1) \int_0^{k\bar{v}} y(n-2)F^{n-3}(y)f(y)\frac{1}{2}\left(F\left(\frac{y}{k}\right) - F(y)\right)\left(2 - F\left(\frac{y}{k}\right) - F(y)\right)dy \\
&\quad + n(n-1) \int_{k\bar{v}}^{\bar{v}} y(n-2)F^{n-3}(y)f(y)\frac{1}{2}(1 - F(y))^2 dy \\
&= \int_0^{k\bar{v}} y \frac{\left(F\left(\frac{y}{k}\right) - F(y)\right)\left(2 - F\left(\frac{y}{k}\right) - F(y)\right)}{(1 - F(y))^2} \frac{n(n-1)(n-2)}{2} (1 - F(y))^2 F^{n-3}(y) f(y) dy \\
&\quad + \int_{k\bar{v}}^{\bar{v}} y \frac{n(n-1)(n-2)}{2} (1 - F(y))^2 F^{n-3}(y) f(y) dy \\
&= \int_0^{\bar{v}} y \frac{\left(F(m(y)) - F(y)\right)\left(2 - F(m(y)) - F(y)\right)}{(1 - F(y))^2} \frac{n(n-1)(n-2)}{2} (1 - F(y))^2 F^{n-3}(y) f(y) dy
\end{aligned}$$

where $m(y) = \min\{\frac{y}{k}, \bar{v}\}$, which implies that we can write

$$t_2 = \int_0^{\bar{v}} y \cdot \frac{(1 - F(y))^2 - (1 - F(m(y)))^2}{(1 - F(y))^2} \cdot f_{3,n}(y) dy \quad (10)$$

>From (9) and (10) we get

$$\begin{aligned}
ER_1^{ssp} &= t_1 + t_2 \\
&= \int_0^{\bar{v}} kv \frac{F^{n-2}(kv)}{F^{n-2}(v)} f_{2,n}(v) dv \\
&\quad + \int_0^{\bar{v}} v \cdot \frac{(1 - F(v))^2 - (1 - F(m(v)))^2}{(1 - F(v))^2} \cdot f_{3,n}(v) dv
\end{aligned}$$

as stated in the lemma.

ER_2^{ssp}

The derivation of the expected revenue in the second auction is a little more complicated. However, bidding strategies are simple, in the sense that all bidders will simply bid their *relevant* valuations. Thus, bidder i will bid $b^2(v_i) = kv_i$, if he won the first auction, and $b^2(v_i) = v_i$ otherwise. This will allow us to write ER_2^{ssp} as the sum of three terms following from three types of events: (1) the winner of the first auction wins again, (2) the winner of the first auction does not win again but is the runner-up and (3) the winner of the first auction does not win again nor is the runner up.

In the following x denotes the valuation of the winner of the first auction, while y_1 and y_2 denote the valuations of his strongest and second-strongest rivals, respectively. Thus, $x \geq y_1 \geq y_2$.

First term. The winner of the first auction wins again.

This requires $kx \geq y_1 \geq y_2$, and we can write the associated expected revenue as

$$\begin{aligned}
T_1 &= n(n-1) \int_0^{\bar{v}} \int_0^{kx} y_1 F^{n-2}(y_1) f(y_1) f(x) dy_1 dx \\
&= n(n-1) \int_0^{k\bar{v}} y_1 F^{n-2}(y_1) f(y_1) \left(\int_{\frac{y_1}{k}}^{\bar{v}} f(x) dx \right) dy_1
\end{aligned}$$

$$T_1 = n(n-1) \int_0^{k\bar{v}} y_1 (1 - F(\frac{y_1}{k})) F^{n-2}(y_1) f(y_1) dy_1 \quad (11)$$

Second term. The winner of the first auction is the runner-up in the second auction.

This requires $y_1 \geq kx \geq y_2$, and we can write the associated expected revenue as

$$\begin{aligned} T_2 &= n(n-1)(n-2) \int_0^{\bar{v}} \int_{kx}^x \int_0^{kx} kx F^{n-3}(y_2) f(y_2) f(y_1) f(x) dy_2 dy_1 dx \\ &= n(n-1) \int_0^{\bar{v}} \int_{kx}^x kx \left(\int_0^{kx} (n-2) F^{n-3}(y_2) f(y_2) dy_2 \right) f(y_1) f(x) dy_1 dx \\ &= n(n-1) \int_0^{\bar{v}} \int_{kx}^x kx \left(\int_0^{kx} \frac{dF^{n-2}(y_2)}{dy_2} dy_2 \right) f(y_1) f(x) dy_1 dx \\ &= n(n-1) \int_0^{\bar{v}} \int_{kx}^x kx F^{n-2}(kx) f(y_1) f(x) dy_1 dx \\ &= n(n-1) \int_0^{\bar{v}} kx F^{n-2}(kx) \left(\int_{kx}^x f(y_1) dy_1 \right) f(x) dx \end{aligned}$$

$$T_2 = n(n-1) \int_0^{\bar{v}} kx (F(x) - F(kx)) F^{n-2}(kx) f(x) dx \quad (12)$$

Third term. The winner of the first auction neither wins nor is the runner-up in the second auction.

This requires $y_1 \geq y_2 \geq kx$, and we can write the associated expected revenue as

$$\begin{aligned} T_3 &= n(n-1)(n-2) \int_0^{\bar{v}} \int_{kx}^x \int_{kx}^{y_1} y_2 F^{n-3}(y_2) f(y_2) f(y_1) f(x) dy_2 dy_1 dx \\ &= n(n-1)(n-2) \int_0^{\bar{v}} \int_{kx}^x y_2 F^{n-3}(y_2) f(y_2) \left(\int_{y_2}^x f(y_1) dy_1 \right) f(x) dy_2 dx \\ &= n(n-1)(n-2) \int_0^{\bar{v}} \int_{kx}^x y_2 F^{n-3}(y_2) f(y_2) (F(x) - F(y_2)) f(x) dy_2 dx \\ &= n(n-1)(n-2) \int_0^{\bar{v}} \int_{kx}^x y_2 F^{n-3}(y_2) f(y_2) F(x) f(x) dy_2 dx \\ &\quad - n(n-1)(n-2) \int_0^{\bar{v}} \int_{kx}^x y_2 F^{n-2}(y_2) f(y_2) f(x) dy_2 dx \end{aligned}$$

Changing the order of integration, we can rewrite this as

$$\begin{aligned} T_3 &= \frac{n(n-1)(n-2)}{2} \int_0^{k\bar{v}} y_2 F^{n-3}(y_2) f(y_2) \left(\int_{y_2}^{\frac{y_2}{k}} 2F(x) f(x) dx \right) dy_2 \\ &\quad + \frac{n(n-1)(n-2)}{2} \int_{k\bar{v}}^{\bar{v}} y_2 F^{n-3}(y_2) f(y_2) \left(\int_{y_2}^{\bar{v}} 2F(x) f(x) dx \right) dy_2 \\ &\quad - n(n-1)(n-2) \int_0^{k\bar{v}} y_2 F^{n-3}(y_2) f(y_2) \left(\int_{y_2}^{\frac{y_2}{k}} f(x) dx \right) dy_2 \\ &\quad - n(n-1)(n-2) \int_{k\bar{v}}^{\bar{v}} y_2 F^{n-3}(y_2) f(y_2) \left(\int_{y_2}^{\bar{v}} f(x) dx \right) dy_2 \end{aligned}$$

$$\begin{aligned}
T_3 &= \frac{n(n-1)(n-2)}{2} \int_0^{k\bar{v}} y_2 F^{n-3}(y_2) f(y_2) (F^2(\frac{y_2}{k}) - F^2(y_2)) dy_2 \\
&+ \frac{n(n-1)(n-2)}{2} \int_{k\bar{v}}^{\bar{v}} y_2 F^{n-3}(y_2) f(y_2) (1 - F^2(y_2)) dy_2 \\
&- n(n-1)(n-2) \int_0^{k\bar{v}} y_2 F^{n-3}(y_2) f(y_2) F(y_2) (F(\frac{y_2}{k}) - F(y_2)) dy_2 \\
&- n(n-1)(n-2) \int_{k\bar{v}}^{\bar{v}} y_2 F^{n-3}(y_2) f(y_2) F(y_2) (1 - F(y_2)) dy_2
\end{aligned}$$

$$\begin{aligned}
T_3 &= \frac{n(n-1)(n-2)}{2} \int_0^{k\bar{v}} y_2 (F^2(\frac{y_2}{k}) - F^2(y_2)) F^{n-3}(y_2) f(y_2) dy_2 \\
&- \frac{n(n-1)(n-2)}{2} \int_0^{k\bar{v}} y_2 2F(y_2) (F(\frac{y_2}{k}) - F(y_2)) F^{n-3}(y_2) f(y_2) dy_2 \\
&+ \frac{n(n-1)(n-2)}{2} \int_{k\bar{v}}^{\bar{v}} y_2 (1 - F^2(y_2)) F^{n-3}(y_2) f(y_2) dy_2 \\
&- \frac{n(n-1)(n-2)}{2} \int_{k\bar{v}}^{\bar{v}} y_2 2F(y_2) (1 - F(y_2)) F^{n-3}(y_2) f(y_2) dy_2
\end{aligned}$$

$$\begin{aligned}
T_3 &= \frac{n(n-1)(n-2)}{2} \int_0^{k\bar{v}} y_2 (F^2(\frac{y_2}{k}) - F^2(y_2) - 2F(y_2)F(\frac{y_2}{k}) + 2F^2(y_2)) F^{n-3}(y_2) f(y_2) dy_2 \\
&+ \frac{n(n-1)(n-2)}{2} \int_{k\bar{v}}^{\bar{v}} y_2 (1 - F^2(y_2) - 2F(y_2) + 2F^2(y_2)) F^{n-3}(y_2) f(y_2) dy_2
\end{aligned}$$

$$\begin{aligned}
T_3 &= \frac{n(n-1)(n-2)}{2} \int_0^{k\bar{v}} y_2 (F(\frac{y_2}{k}) - F(y_2))^2 F^{n-3}(y_2) f(y_2) dy_2 \\
&+ \frac{n(n-1)(n-2)}{2} \int_{k\bar{v}}^{\bar{v}} y_2 (1 - F(y_2))^2 F^{n-3}(y_2) f(y_2) dy_2
\end{aligned}$$

or

$$T_3 = \frac{n(n-1)(n-2)}{2} \int_0^{\bar{v}} y_2 (F(m(y_2)) - F(y_2))^2 F^{n-3}(y_2) f(y_2) dy_2 \quad (13)$$

where $m(y_2) = \min\{\frac{y_2}{k}, \bar{v}\}$. Collecting the terms in (11), (12) and (13), we get

$$\begin{aligned}
ER_2^{ssp} &= T_1 + T_2 + T_3 \\
&= n(n-1) \int_0^{k\bar{v}} y_1 (1 - F(\frac{y_1}{k})) F^{n-2}(y_1) f(y_1) dy_1 \\
&+ n(n-1) \int_0^{\bar{v}} kx (F(x) - F(kx)) F^{n-2}(kx) f(x) dx \\
&+ \frac{n(n-1)(n-2)}{2} \int_0^{\bar{v}} y_2 (F(m(y_2)) - F(y_2))^2 F^{n-3}(y_2) f(y_2) dy_2
\end{aligned}$$

We can further rewrite this as

$$\begin{aligned}
ER_2^{ssp} &= \int_0^{k\bar{v}} y_1 \cdot \frac{1 - F(\frac{y_1}{k})}{1 - F(y_1)} \cdot n(n-1)(1 - F(y_1))F^{n-2}(y_1)f(y_1)dy_1 \\
&+ \int_0^{\bar{v}} kx \cdot \frac{F(x) - F(kx)}{1 - F(x)} \cdot \frac{F^{n-2}(kx)}{F^{n-2}(x)} n(n-1)(1 - F(x))F^{n-2}(x)f(x)dx \\
&+ \int_0^{\bar{v}} y_2 \frac{(F(m(y_2)) - F(y_2))^2}{(1 - F(y_2))^2} \frac{n(n-1)(n-2)}{2} (1 - F(y_2))^2 F^{n-3}(y_2)f(y_2)dy_2
\end{aligned}$$

or, in more compact form, as

$$\begin{aligned}
ER_2^{ssp} &= \int_0^{k\bar{v}} v \cdot \frac{1 - F(\frac{v}{k})}{1 - F(v)} \cdot f_{2,n}(v)dv \\
&+ \int_0^{\bar{v}} kv \cdot \frac{F(v) - F(kv)}{1 - F(v)} \cdot \frac{F^{n-2}(kv)}{F^{n-2}(v)} f_{2,n}(v)dv \\
&+ \int_0^{\bar{v}} v \frac{(F(m(v)) - F(v))^2}{(1 - F(v))^2} f_{3,n}(v)dv
\end{aligned}$$

or

$$\begin{aligned}
ER_2^{ssp} &= \int_0^{\bar{v}} v \cdot \frac{1 - F(m(v))}{1 - F(v)} \cdot f_{2,n}(v)dv \\
&+ \int_0^{\bar{v}} kv \cdot \frac{F(v) - F(kv)}{1 - F(v)} \cdot \frac{F^{n-2}(kv)}{F^{n-2}(v)} f_{2,n}(v)dv \\
&+ \int_0^{\bar{v}} v \frac{(F(m(v)) - F(v))^2}{(1 - F(v))^2} f_{3,n}(v)dv
\end{aligned}$$

as stated in the lemma. ■

Proof of Lemma 3. In equilibrium, the bidder bids $B(\hat{v})$, as does every rival with valuation above \hat{v} . If more than one bidder bid $B(\hat{v})$, the winner is determined by a fair lottery. Hence, the probability of winning when submitting a bid of $B(\hat{v})$ is

$$\begin{aligned}
\Pr(W|\hat{v}) &= \sum_{i=0}^{n-1} \binom{n-1}{i} (1 - F(\hat{v}))^i F(\hat{v})^{n-1-i} \frac{1}{i+1} \\
&= \frac{1}{n(1 - F(\hat{v}))} \sum_{i=0}^{n-1} \frac{n \times (n-1)!}{(i+1)!(n-(i+1))!} (1 - F(\hat{v}))^{i+1} F(\hat{v})^{n-(i+1)} \\
&= \frac{1}{n(1 - F(\hat{v}))} \sum_{j=1}^n \binom{n}{j} (1 - F(\hat{v}))^j F(\hat{v})^{n-j}
\end{aligned}$$

Then, the lemma follows by applying the Binomial Theorem. ■

Proof of Proposition 2. In relation to Proposition 2, it remains to prove that $v \neq \hat{v}$ has no incentive to deviate. We sketch the argument.

First, consider $v > \hat{v}$. Expected payoff by accepting B is

$$\begin{aligned}
EU(B, v) &\geq \Pr(W|\hat{v})(v - B) + \int_0^{\min\{kv, \hat{v}\}} (kv - x)f_{1,n-1}(x)dx \\
&+ \sum_{i=1}^{n-1} \binom{n-1}{i} (1 - F(\hat{v}))^i F(\hat{v})^{n-1-i} \left(1 - \frac{1}{i+1}\right) EU_2(v|L, \hat{v}, i)
\end{aligned} \tag{14}$$

where the first term comes from the fact that the buyer may win stage one at the buy-out price, and the second term from the fact that he wins both stages with probability one if the highest rival type is lower than $\min\{kv, \hat{v}\}$. In the third term, $EU_2(v|L, \hat{v}, i)$ denotes the expected payoff in stage 2, given the buyer *lost* stage 1 and there are i rivals with type above \hat{v} . The third term is then explained by noticing that

$$1 - \frac{1}{i+1}$$

is the probability that the buyer loses stage one when faced with i rivals who are also willing to accept B . Finally, the inequality in (14) derives from the possibility that the buyer may win both stages even if his highest rival has type above \hat{v} . Naturally, this requires that $kv > \hat{v}$, so this possibility does not exist for type \hat{v} buyers, for whom $ER(B, v)$ therefore equals the right hand side of (14).

If the buyer rejects B , on the other hand, we have already established that he will choose to outbid anybody with valuation below \hat{v} in stage one, if the buy-out price is not accepted by a rival. This means that expected payoff is

$$\begin{aligned} EU(NB, v) &\geq \int_0^{\hat{v}} (v - b(x))f_{1,n-1}(x)dx + \int_0^{\min\{kv, \hat{v}\}} (kv - x)f_{1,n-1}(x)dx \\ &\quad + \sum_{i=1}^{n-1} \binom{n-1}{i} (1 - F(\hat{v}))^i F(\hat{v})^{n-1-i} EU_2(v|L, \hat{v}, i) \end{aligned}$$

The first two terms are self explanatory, and the third captures the fact that the buyer is bound to lose stage one if there is one or more rivals with type above \hat{v} . By construction, $EU(B, \hat{v}) - EU(NB, \hat{v}) = 0$, so we need only show that

$$\begin{aligned} EU(B, v) - EU(NB, v) &\geq \Pr(W|\hat{v})(v - B) - \int_0^{\hat{v}} (v - b^1(x))f_{1,n-1}(x)dx \\ &\quad - \sum_{i=1}^{n-1} \binom{n-1}{i} (1 - F(\hat{v}))^i F(\hat{v})^{n-1-i} \frac{1}{i+1} EU_2(v|L, \hat{v}, i) \end{aligned}$$

is increasing in v . The derivative equals

$$\Pr(W|\hat{v}) - F^{n-1}(\hat{v}) - \sum_{i=1}^{n-1} \binom{n-1}{i} (1 - F(\hat{v}))^i F(\hat{v})^{n-1-i} \frac{1}{i+1} EU_2'(v|L, \hat{v}, i)$$

Assuming for the moment that $EU_2'(v|L, \hat{v}, i) \leq 1$, this is at least

$$\Pr(W|\hat{v}) - F^{n-1}(\hat{v}) - \sum_{i=1}^{n-1} \binom{n-1}{i} (1 - F(\hat{v}))^i F(\hat{v})^{n-1-i} \frac{1}{i+1} = 0$$

implying that $EU(B, v) - EU(NB, v) \geq 0$ for all $v > \hat{v}$. To see that $EU_2'(v|L, \hat{v}, i) \leq 1$, notice that $EU_2(v|L, \hat{v}, i)$ can be written in the form

$$EU_2(v|L, \hat{v}, i) = \int_{k\hat{v}}^v (v - x)h(x|L, \hat{v}, i)dx$$

where $h(\cdot|\cdot)$ is the density of the highest rival bid in stage two, given the buyer lost stage one to one of i rivals with type above \hat{v} . This density is positive on $[k\hat{v}, \max\{\hat{v}, k\bar{v}\}]$ for $i = 1$ and on $[\hat{v}, \bar{v}]$ for $i > 1$. Hence, it can then easily be verified that $EU_2'(v|L, \hat{v}, i) \leq 1$.

Next, consider $v < \hat{v}$. If the buyer chooses not to accept the buy-out price, we know that the best response in the remainder of stage 1 is to bid $b^1(v)$. Hence, we need only show that it is preferable to reject B , and subsequently bid $b^1(v)$, than to accept B . The latter strategy yields expected payoff of

$$\begin{aligned} EU(B, v) &= (v - B) \Pr(W|\hat{v}) + \int_0^{kv} (kv - x) f_{1,n-1}(x) dx \\ &\quad + \frac{1}{2}(n-1) \int_{\hat{v}}^{\max\{\hat{v}, m(v)\}} \left((v - kx) F_{1,n-2}(kx) + \int_{kx}^v (v - y) f_{1,n-2}(y) dy \right) f(x) dx \end{aligned}$$

The third term derives from the fact that if the buyer loses stage 1, there must be at least one rival with type above \hat{v} . If there is more than one, the buyer will lose stage two as well (he will be outbid). However, if there is precisely one rival with type above \hat{v} , the buyer loses stage 1 with probability .5, but in that event he has a chance of winning stage 2, provided $v > k\hat{v}$, or $\frac{v}{k} > \hat{v}$.

By rejecting B and bidding $b^1(v)$ in stage 1, expected payoff is

$$\begin{aligned} EU(NB, v) &= \int_0^v (v - b^1(x)) f_{1,n-1}(x) dx + \int_0^{kv} (kv - x) f_{1,n-1}(x) dx \\ &\quad + (n-1) \int_v^{m(v)} \left((v - kx) F_{1,n-2}(kx) + \int_{kx}^v (v - y) f_{1,n-2}(y) dy \right) f(x) dx \end{aligned}$$

Again, the third term derives from the fact that if the highest rival has type above v , the buyer loses stage 1 with probability one, but nevertheless has a chance of winning stage two, provided there is only one rival with a type exceeding v .

Subtracting the two yields

$$\begin{aligned} &EU(NB, v) - EU(B, v) \\ &= \int_0^v (v - b^1(x)) f_{1,n-1}(x) dx - (v - B) \Pr(W|\hat{v}) \\ &\quad + (n-1) \int_v^{m(v)} \left((v - kx) F_{1,n-2}(kx) + \int_{kx}^v (v - y) f_{1,n-2}(y) dy \right) f(x) dx \\ &\quad - \frac{1}{2}(n-1) \int_{\hat{v}}^{\max\{\hat{v}, m(v)\}} \left((v - kx) F_{1,n-2}(kx) + \int_{kx}^v (v - y) f_{1,n-2}(y) dy \right) f(x) dx \end{aligned}$$

which by construction is zero for $v = \hat{v}$. Differentiating with respect to v ,

$$\begin{aligned} &(v - b^1(v)) f_{1,n-1}(v) + F_{1,n-1}(v) - \Pr(W|\hat{v}) \\ &\quad - (n-1) \left((v - kv) F_{1,n-2}(kv) + \int_{kv}^v (v - y) f_{1,n-2}(y) dy \right) f(v) \\ &\quad + (n-1) \int_v^{m(v)} F_{1,n-2}(v) f(x) dx - \frac{1}{2}(n-1) \int_{\hat{v}}^{\max\{\hat{v}, m(v)\}} F_{1,n-2}(v) f(x) dx \end{aligned}$$

which, after inserting $b^1(v)$, reduces to

$$(n-1) F^{n-2}(v) \left(F(m(v)) - F(v) - \frac{1}{2} (F(\max\{\hat{v}, m(v)\}) - F(\hat{v})) \right) + F^{n-1}(v) - \Pr(W|\hat{v})$$

For $v = \hat{v}$, this is

$$(n-1) F^{n-2}(\hat{v}) \frac{1}{2} (F(m(\hat{v})) - F(\hat{v})) + F^{n-1}(\hat{v}) - \frac{1 - F^n(\hat{v})}{n(1 - F(\hat{v}))}$$

which is smaller than

$$(n-1)F^{n-2}(\hat{v})\frac{1}{2}(1-F(\hat{v})) + F^{n-1}(\hat{v}) - \frac{1-F^n(\hat{v})}{n(1-F(\hat{v}))}$$

Rewriting,

$$\frac{1}{n(1-F(\hat{v}))} \left[1 - F^n(\hat{v}) - n(1-F(\hat{v}))F^{n-1}(\hat{v}) - n(n-1)\frac{1}{2}(1-F(\hat{v}))^2F^{n-2}(\hat{v}) \right] \geq 0$$

where the inequality follows from the fact that the term in brackets equals

$$1 - \sum_{i=0}^2 \binom{n}{i} (1-F(\hat{v}))^i F^{n-i}(\hat{v}) \geq 0.$$

Finally, noticing that the second derivative of $EU(NB, v) - EU(B, v)$ is positive implies that $EU(NB, v) - EU(B, v)$ is decreasing everywhere on $v \in [0, \hat{v}]$, meaning that $EU(NB, v) - EU(B, v) \geq 0$ for all $v \leq \hat{v}$. This completes the proof of Proposition 2. ■

Proof of Proposition 3. In order to derive an expression for $ER_2(\hat{v})$, we first arrange bidders in descending order, $v_1 > v_2 > \dots > v_n$. Then, the price in the second auction, p_2 , can be either kv_1 , v_2 , kv_2 or v_3 . The four possibilities contribute to expected revenue as indicated in the following four parts of this proof. For future reference, we fix the following notation for combinations

$$K_{n,m} = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

and permutations

$$P_{n,m} = \frac{n!}{(n-m)!} = K_{n,m}m!$$

1. *The price in stage two is kv_1 .* This requires that bidder 1 (v_1) wins stage one (which may involve accepting the buy-out price), and that he is the runner-up in stage two, $v_2 > kv_1 > v_3$. In stage two, the bids are thus kv_1 , v_2 , v_3 etc. $p_2 = kv_1$ can arise if

- a. no one took B (v_1 wins stage one with probability 1),
- b. v_1 took B (v_1 wins stage one with probability 1),
- c. v_1 and v_2 took B (v_1 wins stage one with probability $\frac{1}{2}$), or
- d. at least one more than v_1 and v_2 took B (v_1 wins stage one with probability $\frac{1}{i+2}$, where $i = 1, 2, \dots, n-2$).

a. None took B . In this case we have $\hat{v} > v_1 > v_2 > \dots > v_n$ and $v_2 > kv_1 > v_3$. We get the contribution

$$E_{1a} = P_{n,2} \cdot \int_0^{\hat{v}} kv_1 F^{n-2}(kv_1)(F(v_1) - F(kv_1))f(v_1)dv_1$$

where $P_{n,2} = n(n-1)$ just captures the n ways that the highest-valuation bidder could be picked, and the remaining $n-1$ ways that the second-highest-valuation bidder could be picked.

b. Only v_1 took B . In this case we have $v_1 > \hat{v} > v_2 > \dots > v_n$ and $v_2 > kv_1 > v_3$. We get the contribution

$$E_{1b} = P_{n,2} \cdot \int_{\hat{v}}^{m(\hat{v})} kv_1 F^{n-2}(kv_1)(F(\hat{v}) - F(kv_1))f(v_1)dv_1$$

c. v_1 and v_2 took B . In this case we have $v_1 > v_2 > \hat{v} > v_3 \dots > v_n$ and $v_2 > kv_1 > v_3$. We can, thus, either have $kv_1 > \hat{v}$ or $kv_1 < \hat{v}$. We get the contribution

$$\begin{aligned} E_{1c} &= P_{n,2} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{m(\hat{v})} kv_1 F^{n-2}(kv_1)(F(v_1) - F(\hat{v}))f(v_1)dv_1 \\ &\quad + P_{n,2} \cdot \frac{1}{2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv_1 F^{n-2}(\hat{v})(F(v_1) - F(kv_1))f(v_1)dv_1 \end{aligned}$$

d. At least one more than v_1 and v_2 took B (*this disappears when $n = 2$*). In this case we have $v_1 > v_2 > v_{2+1} > \dots > v_{2+i} > \hat{v} > v_{2+i+1} > \dots > v_n$ and $v_2 > kv_1 > v_3$. We get contributions of the type (for $i = 1, 2, \dots, n-2$)

$$E_{1d}^i = P_{n,2} \cdot K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv_1 F^{n-2-i}(\hat{v})(F(kv_1) - F(\hat{v}))^i (F(v_1) - F(kv_1))f(v_1)dv_1$$

Collecting terms (for case d), we get the total contribution

$$E_{1d} = \sum_{i=1}^{n-2} T_{1d}^i = P_{n,2} \cdot \sum_{i=1}^{n-2} K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv_1 F^{n-2-i}(\hat{v})(F(kv_1) - F(\hat{v}))^i (F(v_1) - F(kv_1))f(v_1)dv_1$$

Hence, the contribution to expected revenue in stage two from all the cases where price is $p_2 = kv_1$ is given by

$$\begin{aligned} E_1 &= E_{1a} + E_{1b} + E_{1c} + E_{1d} \\ &= P_{n,2} \cdot \int_0^{\hat{v}} kv_1 F^{n-2}(kv_1)(F(v_1) - F(kv_1))f(v_1)dv_1 \\ &\quad + P_{n,2} \cdot \int_{\hat{v}}^{m(\hat{v})} kv_1 F^{n-2}(kv_1)(F(\hat{v}) - F(kv_1))f(v_1)dv_1 \\ &\quad + P_{n,2} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{m(\hat{v})} kv_1 F^{n-2}(kv_1)(F(v_1) - F(\hat{v}))f(v_1)dv_1 \\ &\quad + P_{n,2} \cdot \frac{1}{2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv_1 F^{n-2}(\hat{v})(F(v_1) - F(kv_1))f(v_1)dv_1 \\ &\quad + P_{n,2} \cdot \sum_{i=1}^{n-2} K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv_1 F^{n-2-i}(\hat{v})(F(kv_1) - F(\hat{v}))^i (F(v_1) - F(kv_1))f(v_1)dv_1 \end{aligned}$$

We can easily rewrite this as stated in the proposition

$$\begin{aligned} E_1 &= P_{n,2} \cdot \int_0^{\hat{v}} kvF(kv)^{n-2}(F(v) - F(kv))f(v)dv \\ &\quad + P_{n,2} \cdot \int_{\hat{v}}^{m(\hat{v})} kvF(kv)^{n-2} \left(F(\hat{v}) - F(kv) + \frac{1}{2}(F(v) - F(\hat{v})) \right) f(v)dv \\ &\quad + P_{n,2} \cdot \sum_{i=0}^{n-2} K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{m(\hat{v})}^{\bar{v}} kvF(\hat{v})^{n-2-i}(F(kv) - F(\hat{v}))^i (F(v) - F(kv))f(v)dv \end{aligned}$$

2. *The price in stage two is v_2 .* This requires that v_2 is the runner-up in stage two after not winning stage one. This is possible in one of two ways: A) Either v_1 won stage one and v_2 is the runner-up to kv_1

in stage two, or B) someone other than v_1 and v_2 won stage one, and v_2 is the runner-up to v_1 in stage two (note that B disappears when $n = 2$).

2A) v_1 won stage one and v_2 is runner up to kv_1 in stage two, $v_1 > kv_1 > v_2 > \dots > v_n$. Second-stage bids are kv_1, v_2, v_3 , etc. v_1 could win stage one in several ways

- a. If no one took B (v_1 wins stage one with probability 1)
- b. If v_1 took B (v_1 wins stage one with probability 1)
- c. If v_1 and v_2 took B (v_1 wins stage one with probability $\frac{1}{2}$)
- d. If at least one more than v_1 and v_2 took B (v_1 wins stage one with probability $\frac{1}{i+2}$, where $i = 1, 2, \dots, n - 2$).

Aa. None took B . In this case we have $\hat{v} > v_1 > v_2 > \dots > v_n$ and $kv_1 > v_2$. We get the contribution

$$E_{2Aa} = P_{n,2} \cdot \int_0^{k\hat{v}} v_2 F^{n-2}(v_2) (F(\hat{v}) - F(\frac{v_2}{k})) f(v_2) dv_2$$

Ab. Only v_1 took B . In this case we have $v_1 > \hat{v} > v_2 > \dots > v_n$ and $kv_1 > v_2$. Hence, we can either have $kv_1 > \hat{v}$ or $kv_1 < \hat{v}$. We get the contributions

$$\begin{aligned} E_{2Ab} &= P_{n,2} \cdot \int_0^{k\hat{v}} v_2 F^{n-2}(v_2) (1 - F(\hat{v})) f(v_2) dv_2 \\ &+ P_{n,2} \cdot \int_{k\hat{v}}^{km(\hat{v})} v_2 F^{n-2}(v_2) (1 - F(\frac{v_2}{k})) f(v_2) dv_2 \end{aligned}$$

Ac. v_1 and v_2 took B . In this case we have $v_1 > v_2 > \hat{v} > v_3 \dots > v_n$ and $kv_1 > v_2$. We get the contribution

$$E_{2Ac} = P_{n,2} \cdot \frac{1}{2} \cdot \int_{km(\hat{v})}^{k\bar{v}} v_2 F^{n-2}(\hat{v}) (1 - F(\frac{v_2}{k})) f(v_2) dv_2$$

Ad. At least one more than v_1 and v_2 took B (this disappears when $n = 2$). In this case we have $v_1 > v_2 > v_{2+1} > \dots > v_{2+i} > \hat{v} > v_{2+i+1} > \dots > v_n$ and $kv_1 > v_2$. We get contributions of the type (for $i = 1, 2, \dots, n - 2$)

$$E_{2Ad}^i = P_{n,2} \cdot K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{km(\hat{v})}^{k\bar{v}} v_2 F^{n-2-i}(\hat{v}) (F(v_2) - F(\hat{v}))^i (1 - F(\frac{v_2}{k})) f(v_2) dv_2$$

Collecting terms for this case, we get the total contribution

$$E_{2Ad} = \sum_{i=1}^{n-2} E_{2Ad}^i = P_{n,2} \cdot \sum_{i=1}^{n-2} K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{km(\hat{v})}^{k\bar{v}} v_2 F^{n-2-i}(\hat{v}) (F(v_2) - F(\hat{v}))^i (1 - F(\frac{v_2}{k})) f(v_2) dv_2$$

2B) Someone other than v_1 and v_2 won stage one, and v_2 is the runner-up to v_1 in stage two (this disappears when $n = 2$). This requires $v_1 > v_2 > v_3 > \hat{v}$, so that - at least - three bidders took B in stage one. Second-stage bids are v_1, v_2, v_3 , etc., except for some $j \neq 1, 2$ bidding kv_j . Thus, we get terms of the type (for $i = 1, 2, \dots, n - 2$)

$$E_{2B}^i = P_{n,2} \cdot K_{n-2,i} \cdot \frac{i}{i+2} \cdot \int_{\hat{v}}^{\bar{v}} v_2 F^{n-2-i}(\hat{v}) (F(v_2) - F(\hat{v}))^i (1 - F(v_2)) f(v_2) dv_2$$

where $\frac{i}{i+2}$ is the probability that someone other than v_1 and v_2 wins stage one when $v_1 > v_2 > \dots > v_i > \hat{v} > v_{i+1}$. Collecting terms from this case, we get the total contribution

$$E_{2B} = \sum_{i=1}^{n-2} E_{2B}^i = P_{n,2} \cdot \sum_{i=1}^{n-2} K_{n-2,i} \cdot \frac{i}{i+2} \cdot \int_{\hat{v}}^{\bar{v}} v_2 F^{n-2-i}(\hat{v}) (F(v_2) - F(\hat{v}))^i (1 - F(v_2)) f(v_2) dv_2$$

Hence, the contribution to expected revenue in stage two from the cases where price is $p_2 = v_2$ is given by

$$\begin{aligned}
E_2 &= E_{2Aa} + E_{2Ab} + E_{2Ac} + E_{2Ad} + E_{2B} \\
&= P_{n,2} \cdot \int_0^{k\hat{v}} v_2 F^{n-2}(v_2) (F(\hat{v}) - F(\frac{v_2}{k})) f(v_2) dv_2 \\
&\quad + P_{n,2} \cdot \int_0^{k\hat{v}} v_2 F^{n-2}(v_2) (1 - F(\hat{v})) f(v_2) dv_2 \\
&\quad + P_{n,2} \cdot \int_{k\hat{v}}^{km(\hat{v})} v_2 F^{n-2}(v_2) (1 - F(\frac{v_2}{k})) f(v_2) dv_2 \\
&\quad + P_{n,2} \cdot \frac{1}{2} \cdot \int_{km(\hat{v})}^{k\bar{v}} v_2 F^{n-2}(\hat{v}) (1 - F(\frac{v_2}{k})) f(v_2) dv_2 \\
&\quad + P_{n,2} \cdot \sum_{i=1}^{n-2} K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{km(\hat{v})}^{k\bar{v}} v_2 F^{n-2-i}(\hat{v}) (F(v_2) - F(\hat{v}))^i (1 - F(\frac{v_2}{k})) f(v_2) dv_2 \\
&\quad + P_{n,2} \cdot \sum_{i=1}^{n-2} K_{n-2,i} \cdot \frac{i}{i+2} \cdot \int_{\hat{v}}^{\bar{v}} v_2 F^{n-2-i}(\hat{v}) (F(v_2) - F(\hat{v}))^i (1 - F(v_2)) f(v_2) dv_2
\end{aligned}$$

which can be stated more succinctly as in the proposition

$$\begin{aligned}
E_2 &= P_{n,2} \cdot \int_0^{\hat{v}} v F(v)^{n-2} (1 - F(m(v))) f(v) dv \\
&\quad + P_{n,2} \cdot \sum_{i=0}^{n-2} K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{\hat{v}}^{\bar{v}} v F(\hat{v})^{n-2-i} (F(v) - F(\hat{v}))^i (1 - F(m(v))) f(v) dv \\
&\quad + P_{n,2} \cdot \sum_{i=1}^{n-2} K_{n-2,i} \cdot \frac{i}{i+2} \cdot \int_{\hat{v}}^{\bar{v}} v F(\hat{v})^{n-2-i} (F(v) - F(\hat{v}))^i (1 - F(v)) f(v) dv
\end{aligned}$$

3. *The price in stage two is kv_2 .* This requires that v_2 is the runner-up in stage two (to v_1) after winning stage one. This clearly requires (at least) that $v_1 > v_2 > \hat{v}$, and that $kv_2 > v_3$. In stage two, the bids are thus v_1, kv_2, v_3 etc. $p_2 = kv_1$ can arise if

- a. v_1 and v_2 took B (v_2 wins stage one with probability $\frac{1}{2}$), or
- b. at least one more than v_1 and v_2 took B (v_2 wins stage one with probability $\frac{1}{i+2}$, where $i = 1, 2, \dots, n-2$).

a. v_1 and v_2 took B . In this case we have $v_1 > v_2 > \hat{v} > v_3 \dots > v_n$ and $kv_2 > v_3$. We can, thus, either have $kv_2 > \hat{v}$ or $kv_2 < \hat{v}$. We get the contribution

$$\begin{aligned}
E_{3a} &= P_{n,2} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{m(\hat{v})} kv_2 F^{n-2}(kv_2) (1 - F(v_2)) f(v_2) dv_2 \\
&\quad + P_{n,2} \cdot \frac{1}{2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv_2 F^{n-2}(\hat{v}) (1 - F(v_2)) f(v_2) dv_2
\end{aligned}$$

b. At least one more than v_1 and v_2 took B (*this disappears when $n = 2$*). In this case we have $v_1 > v_2 > v_{2+1} > \dots > v_{2+i} > \hat{v} > v_{2+i+1} > \dots > v_n$ and $kv_2 > v_3$. We get contributions of the type (for $i = 1, 2, \dots, n-2$)

$$E_{3b}^i = P_{n,2} \cdot K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv_2 F^{n-2-i}(\hat{v}) (F(kv_2) - F(\hat{v}))^i (1 - F(v_2)) f(v_2) dv_2$$

Collecting terms for this case, we get the total contribution

$$E_{3b} = \sum_{i=1}^{n-2} E_{3b}^i = P_{n,2} \cdot \sum_{i=1}^{n-2} K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv_2 F^{n-2-i}(\hat{v})(F(kv_2) - F(\hat{v}))^i (1 - F(v_2)) f(v_2) dv_2$$

Hence, the contribution to expected revenue in stage two from the cases where price is $p_2 = kv_2$ is given by

$$\begin{aligned} E_3 &= E_{3a} + E_{3b} \\ &= P_{n,2} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{m(\hat{v})} kv_2 F^{n-2}(kv_2)(1 - F(v_2)) f(v_2) dv_2 \\ &\quad + P_{n,2} \cdot \frac{1}{2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv_2 F^{n-2}(\hat{v})(1 - F(v_2)) f(v_2) dv_2 \\ &\quad + P_{n,2} \cdot \sum_{i=1}^{n-2} K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv_2 F^{n-2-i}(\hat{v})(F(kv_2) - F(\hat{v}))^i (1 - F(v_2)) f(v_2) dv_2 \end{aligned}$$

or

$$\begin{aligned} E_3 &= P_{n,2} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{m(\hat{v})} kv F^{n-2}(kv)(1 - F(v)) f(v) dv \\ &\quad + P_{n,2} \cdot \sum_{i=0}^{n-2} K_{n-2,i} \cdot \frac{1}{i+2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv F^{n-2-i}(\hat{v})(F(kv) - F(\hat{v}))^i (1 - F(v)) f(v) dv \end{aligned}$$

as stated in the proposition.

4. *The price in stage two is v_3 .* Hence, v_3 must be the runner-up in stage two after losing stage one. (Note that this entire case disappears when $n = 2$) For v_3 to be the runner-up in stage two clearly requires that either v_1 or v_2 won stage one. This can happen in several ways, and we split the contributions into A) those following v_1 winning stage one and v_3 being the runner up in stage two to v_2 ($v_3 > kv_1$) and B) those following v_2 winning stage one and v_3 being the runner-up in stage two to v_1 ($v_3 > kv_2$). In A) stage-two bids are kv_1, v_2, v_3 , etc., while in B) stage-two bids are v_1, kv_2, v_3 , etc.

A) v_1 winning stage one and v_3 being the runner-up in stage two to v_2 ($v_2 > v_3 > kv_1$). This can arise if

- a. no one took B (v_1 wins stage one with probability 1),
- b. v_1 took B (v_1 wins stage one with probability 1),
- c. v_1 and v_2 took B (v_1 wins stage one with probability $\frac{1}{2}$), or
- d. at least one more than v_1 and v_2 took B (v_1 wins stage one with probability $\frac{1}{i+3}$, where $i = 0, 1, 2, \dots, n-3$).

Aa. None took B . In this case we have $\hat{v} > v_1 > v_2 > v_3 > \dots > v_n$ and $v_2 > v_3 > kv_1$. We get the contribution

$$E_{4Aa} = P_{n,3} \cdot \int_0^{\hat{v}} \int_{kv_1}^{v_1} \int_{kv_1}^{v_2} v_3 F^{n-3}(v_3) f(v_3) f(v_2) f(v_1) dv_3 dv_2 dv_1$$

where $P_{n,3} = n(n-1)(n-2)$ captures the way the highest, the second-highest and the third-highest

bidder could be picked. We can write this as

$$\begin{aligned}
E_{4Aa} &= P_{n,3} \cdot \frac{1}{2} \cdot \int_0^{\hat{v}} v_3 F^{n-3}(v_3) (F(\min\{\frac{v_3}{k}, \hat{v}\}) - F(v_3))^2 f(v_3) dv_3 \\
&= P_{n,3} \cdot \frac{1}{2} \cdot \int_0^{k\hat{v}} v_3 F^{n-3}(v_3) (F(\frac{v_3}{k}) - F(v_3))^2 f(v_3) dv_3 \\
&\quad + P_{n,3} \cdot \frac{1}{2} \cdot \int_{k\hat{v}}^{\hat{v}} v_3 F^{n-3}(v_3) (F(\hat{v}) - F(v_3))^2 f(v_3) dv_3 \\
&= P_{n,3} \cdot \frac{1}{2} \cdot \int_0^{k\hat{v}} v_3 F^{n-3}(v_3) (F(m(v_3)) - F(v_3))^2 f(v_3) dv_3 \\
&\quad + P_{n,3} \cdot \frac{1}{2} \cdot \int_{k\hat{v}}^{\hat{v}} v_3 F^{n-3}(v_3) (F(\hat{v}) - F(v_3))^2 f(v_3) dv_3
\end{aligned}$$

Ab. v_1 took B . In this case we have $v_1 > \hat{v} > v_2 > v_3 > \dots > v_n$ and $v_2 > v_3 > kv_1$. We get the contribution

$$E_{4Ab} = P_{n,3} \cdot \int_{\hat{v}}^{m(\hat{v})} \int_{kv_1}^{\hat{v}} \int_{kv_1}^{v_2} v_3 F^{n-3}(v_3) f(v_3) f(v_2) f(v_1) dv_3 dv_2 dv_1$$

which we can write as

$$E_{4Ab} = P_{n,3} \cdot \int_{k\hat{v}}^{\hat{v}} v_3 F^{n-3}(v_3) (F(\hat{v}) - F(v_3)) (F(m(v_3)) - F(\hat{v})) f(v_3) dv_3$$

Ac. v_1 and v_2 took B . In this case we have $v_1 > v_2 > \hat{v} > v_3 > \dots > v_n$ and $v_2 > v_3 > kv_1$. We get the contribution

$$E_{4Ac} = P_{n,3} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{m(\hat{v})} \int_{\hat{v}}^{v_1} \int_{kv_1}^{\hat{v}} v_3 F^{n-3}(v_3) f(v_3) f(v_2) f(v_1) dv_3 dv_2 dv_1$$

which we can write as

$$E_{4Ac} = P_{n,3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \int_{k\hat{v}}^{\hat{v}} v_3 F^{n-3}(v_3) (F(m(v_3)) - F(\hat{v}))^2 f(v_3) dv_3$$

Ad. At least one more than v_1 and v_2 took B - in particular v_3 must take B (*this disappears when $n = 2$*). In this case we have $v_1 > v_2 > v_3 > v_{3+1} > \dots > v_{3+i} > \hat{v} > v_{3+i+1} > \dots > v_n$ and $v_2 > v_3 > kv_1$. We get contributions of the type (for $i = 0, 1, 2, \dots, n-3$)

$$E_{4Ad}^i = P_{n,3} \cdot K_{n-3,i} \cdot \frac{1}{i+3} \cdot \int_{\hat{v}}^{\bar{v}} \int_{\max\{kv_1, \hat{v}\}}^{v_1} \int_{\max\{kv_1, \hat{v}\}}^{v_2} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i f(v_3) f(v_2) f(v_1) dv_3 dv_2 dv_1$$

capturing that i have valuations between \hat{v} and v_3 , while $n-3-i$ have valuations below \hat{v} . After some manipulation, we can write this as

$$E_{4Ad}^i = P_{n,3} \cdot K_{n-3,i} \cdot \frac{1}{i+3} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{\bar{v}} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i (F(m(v_3)) - F(v_3))^2 f(v_3) dv_3$$

Collecting terms (for case Ad) we get

$$E_{4Ad} = P_{n,3} \sum_{i=0}^{n-3} K_{n-3,i} \cdot \frac{1}{i+3} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{\bar{v}} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i (F(m(v_3)) - F(v_3))^2 f(v_3) dv_3$$

B) v_2 winning stage one and v_3 being the runner-up in stage two to v_1 ($v_2 > v_3 > kv_2$). This can arise if

- a. v_1 and v_2 took B (v_2 wins stage one with probability $\frac{1}{2}$), or
- b. at least one more than v_1 and v_2 took B (v_2 wins stage one with probability $\frac{1}{i+3}$, where $i = 0, 1, 2, \dots, n-3$).

Ba. v_1 and v_2 took B . In this case we have $v_1 > v_2 > \hat{v} > v_3 > \dots > v_n$ and $v_2 > v_3 > kv_2$. We get the contribution

$$E_{4Ba} = P_{n,3} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{\bar{v}} \int_{\hat{v}}^{\min\{\frac{\hat{v}}{k}, v_1\}} v_3 F^{n-3}(v_3) f(v_3) f(v_2) f(v_1) dv_3 dv_2 dv_1$$

which can be written as

$$\begin{aligned} E_{4Ba} &= P_{n,3} \cdot \frac{1}{2} \cdot \int_{k\hat{v}}^{\hat{v}} v_3 F^{n-3}(v_3) (F(m(v_3)) - F(\hat{v})) (1 - F(m(v_3))) f(v_3) dv_3 \\ &\quad + P_{n,3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \int_{k\hat{v}}^{\hat{v}} v_3 F^{n-3}(v_3) (F(m(v_3)) - F(\hat{v}))^2 f(v_3) dv_3 \end{aligned}$$

Bb. At least one more than v_1 and v_2 took B (*this disappears when $n = 2$*). In this case we have $v_1 > v_2 > v_3 > v_{3+1} > \dots > v_{3+i} > \hat{v} > v_{3+i+1} > \dots > v_n$ and $v_2 > v_3 > kv_2$. We get contributions of the type (for $i = 0, 1, 2, \dots, n-3$)

$$E_{4Bb}^i = P_{n,3} \cdot K_{n-3,i} \cdot \frac{1}{i+3} \cdot \int_{\hat{v}}^{\bar{v}} \int_{\hat{v}}^{v_1} \int_{\max\{\hat{v}, v_2\}}^{v_2} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i f(v_3) f(v_2) f(v_1) dv_3 dv_2 dv_1$$

capturing that i have valuations between \hat{v} and v_3 , while $n-3-i$ have valuations below \hat{v} . After tedious manipulation, this can be written as

$$\begin{aligned} E_{4Bb}^i &= P_{n,3} \cdot K_{n-3,i} \cdot \frac{1}{i+3} \cdot \int_{\hat{v}}^{\max\{\hat{v}, k\bar{v}\}} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i (1 - F(\frac{v_3}{k})) (F(\frac{v_3}{k}) - F(v_3)) f(v_3) dv_3 \\ &\quad + P_{n,3} \cdot K_{n-3,i} \cdot \frac{1}{i+3} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{\bar{v}} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i (F(m(v_3)) - F(v_3))^2 f(v_3) dv_3 \end{aligned}$$

or

$$\begin{aligned} E_{4Bb}^i &= P_{n,3} \cdot K_{n-3,i} \cdot \frac{1}{i+3} \cdot \int_{\hat{v}}^{\bar{v}} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i (1 - F(m(v_3))) (F(m(v_3)) - F(v_3)) f(v_3) dv_3 \\ &\quad + P_{n,3} \cdot K_{n-3,i} \cdot \frac{1}{i+3} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{\bar{v}} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i (F(m(v_3)) - F(v_3))^2 f(v_3) dv_3 \end{aligned}$$

Collecting terms (for case Bb) we get

$$\begin{aligned} E_{4Bb} &= P_{n,3} \sum_{i=0}^{n-3} K_{n-3,i} \cdot \frac{1}{i+3} \cdot \int_{\hat{v}}^{\bar{v}} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i (1 - F(m(v_3))) (F(m(v_3)) - F(v_3)) f(v_3) dv_3 \\ &\quad + P_{n,3} \sum_{i=0}^{n-3} K_{n-3,i} \cdot \frac{1}{i+3} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{\bar{v}} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i (F(m(v_3)) - F(v_3))^2 f(v_3) dv_3 \end{aligned}$$

Hence, the contribution to expected revenue in stage two from the cases where price is $p_2 = v_3$ is given

by

$$\begin{aligned}
E_4 &= E_{4Aa} + E_{4Ab} + E_{4Ac} + E_{4Ad} + E_{4Ba} + E_{4b} \\
&= P_{n,3} \cdot \frac{1}{2} \cdot \int_0^{k\hat{v}} v_3 F^{n-3}(v_3) (F(m(v_3)) - F(v_3))^2 f(v_3) dv_3 \\
&\quad + P_{n,3} \cdot \frac{1}{2} \cdot \int_{k\hat{v}}^{\hat{v}} v_3 F^{n-3}(v_3) (F(\hat{v}) - F(v_3))^2 f(v_3) dv_3 \\
&\quad + P_{n,3} \cdot \int_{k\hat{v}}^{\hat{v}} v_3 F^{n-3}(v_3) (F(\hat{v}) - F(v_3)) (F(m(v_3)) - F(\hat{v})) f(v_3) dv_3 \\
&\quad + P_{n,3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \int_{k\hat{v}}^{\hat{v}} v_3 F^{n-3}(v_3) (F(m(v_3)) - F(\hat{v}))^2 f(v_3) dv_3 \\
&\quad + P_{n,3} \sum_{i=0}^{n-3} K_{n-3,i} \cdot \frac{1}{i+3} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{\bar{v}} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i (F(m(v_3)) - F(v_3))^2 f(v_3) dv_3 \\
&\quad + P_{n,3} \cdot \frac{1}{2} \cdot \int_{k\hat{v}}^{\hat{v}} v_3 F^{n-3}(v_3) (F(m(v_3)) - F(\hat{v})) (1 - F(m(v_3))) f(v_3) dv_3 \\
&\quad + P_{n,3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \int_{k\hat{v}}^{\hat{v}} v_3 F^{n-3}(v_3) (F(m(v_3)) - F(\hat{v}))^2 f(v_3) dv_3 \\
&\quad + P_{n,3} \sum_{i=0}^{n-3} K_{n-3,i} \cdot \frac{1}{i+3} \cdot \int_{\hat{v}}^{\bar{v}} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i (1 - F(m(v_3))) (F(m(v_3)) - F(v_3)) f(v_3) dv_3 \\
&\quad + P_{n,3} \sum_{i=0}^{n-3} K_{n-3,i} \cdot \frac{1}{i+3} \cdot \frac{1}{2} \cdot \int_{\hat{v}}^{\bar{v}} v_3 F^{n-3-i}(\hat{v}) (F(v_3) - F(\hat{v}))^i (F(m(v_3)) - F(v_3))^2 f(v_3) dv_3
\end{aligned}$$

which can be rewritten as stated in the proposition

$$\begin{aligned}
E_4 &= P_{n,3} \cdot \frac{1}{2} \cdot \int_0^{\hat{v}} v F^{n-3}(v) (F(m(v)) - F(v))^2 f(v) dv \\
&\quad + P_{n,3} \sum_{i=0}^{n-3} K_{n-3,i} \cdot \frac{1}{i+3} \cdot \int_{\hat{v}}^{\bar{v}} v F^{n-3-i}(\hat{v}) (F(v) - F(\hat{v}))^i (F(m(v)) - F(v))^2 f(v) dv \\
&\quad + P_{n,3} \cdot \frac{1}{2} \cdot \int_{k\hat{v}}^{\hat{v}} v F^{n-3}(v) (F(m(v)) - F(\hat{v})) (1 - F(m(v))) f(v) dv \\
&\quad + P_{n,3} \sum_{i=0}^{n-3} K_{n-3,i} \cdot \frac{1}{i+3} \cdot \int_{\hat{v}}^{\bar{v}} v F^{n-3-i}(\hat{v}) (F(v) - F(\hat{v}))^i (1 - F(m(v))) (F(m(v)) - F(v)) f(v) dv
\end{aligned}$$

This ends the proof of Proposition 3. ■

Proof of Proposition 4. To show this result, we focus on high values of \hat{v} , by assuming that $\hat{v} > k\bar{v}$, or $m(\hat{v}) = \bar{v}$. First, for given $\hat{v} > k\bar{v}$ we can then write the first-round revenue as

$$\begin{aligned}
ER_1(\hat{v}) &= \hat{v} [1 - F^n(\hat{v}) - nF^{n-1}(\hat{v})(1 - F(\hat{v})) - \frac{1}{2}n(n-1)(1 - F(\hat{v}))^2 F^{n-2}(\hat{v})] \\
&\quad + n(n-1) \int_0^{\hat{v}} \left(kx F^{n-2}(kx) + \int_{kx}^x y(n-2) F^{n-3}(y) f(y) dy \right) (1 - F(x)) f(x) dx \\
&\quad + \frac{1}{2}n(n-1)(1 - F(\hat{v})) \int_{\hat{v}}^{\bar{v}} \left(kx F^{n-2}(kx) + \int_{kx}^{\hat{v}} y(n-2) F^{n-3}(y) f(y) dy \right) f(x) dx
\end{aligned}$$

After some manipulation, the derivative with respect to \hat{v} reduces to

$$\begin{aligned} ER'_1(\hat{v}) &= 1 - F^n(\hat{v}) - nF^{n-1}(\hat{v})(1 - F(\hat{v})) - \frac{1}{2}n(n-1)(1 - F(\hat{v}))^2F^{n-2}(\hat{v}) \\ &\quad - \frac{1}{2}n(n-1)f(\hat{v}) \int_{\hat{v}}^{\bar{v}} \left(kx F^{n-2}(kx) + \int_{kx}^{\hat{v}} y(n-2)F^{n-3}(y)f(y)dy \right) f(x)dx \\ &\quad + \frac{1}{2}n(n-1)(1 - F(\hat{v}))f(\hat{v}) \left(k\hat{v}F^{n-2}(k\hat{v}) + \int_{k\hat{v}}^{\hat{v}} y(n-2)F^{n-3}(y)f(y)dy \right) \end{aligned}$$

In the special case where $n = 2$, the first line is equal to zero, and the rest is strictly negative when $\hat{v} \in [k\bar{v}, \bar{v})$, implying that $\hat{v} = k\bar{v}$ is better than any higher \hat{v} . When $n \geq 3$, the first line is non-negative, since it equals

$$1 - \sum_{i=0}^2 \binom{n}{i} (1 - F(\hat{v}))^i F^{n-2}(\hat{v})$$

while the rest is non-positive. Hence, it is difficult to generally determine the sign of $ER'_1(\hat{v})$. However, it is easily seen that $ER'_1(\bar{v}) = 0$.

The second derivative with respect to \hat{v} equals

$$\begin{aligned} ER''_1(\hat{v}) &= -\frac{1}{2}n(n-1)(n-2)(1 - F(\hat{v}))^2F^{n-3}(\hat{v})f(\hat{v}) + \frac{1}{2}n(n-1)(1 - F(\hat{v}))f(\hat{v})kF^{n-2}(k\hat{v}) \\ &\quad - \frac{1}{2}n(n-1)f'(\hat{v}) \int_{\hat{v}}^{\bar{v}} \left(kx F^{n-2}(kx) + \int_{kx}^{\hat{v}} y(n-2)F^{n-3}(y)f(y)dy \right) f(x)dx \\ &\quad + \frac{1}{2}n(n-1)(1 - F(\hat{v}))f'(\hat{v}) \left(k\hat{v}F^{n-2}(k\hat{v}) + \int_{k\hat{v}}^{\hat{v}} y(n-2)F^{n-3}(y)f(y)dy \right) \end{aligned}$$

Again, $ER''_1(\bar{v}) = 0$. However, it is easy to evaluate the third derivative at \bar{v} ,

$$ER'''_1(\bar{v}) = -\frac{1}{2}n(n-1)(f(\bar{v}))^2kF^{n-2}(k\bar{v}) < 0.$$

This implies that immediately to the left of \bar{v} , $ER''_1(\cdot)$ is positive, which, in turn, means that $ER'_1(\cdot)$ is negative just to the left of \bar{v} . It follows that the first seller is better off with some cut-off strictly less than \bar{v} than without a cut-off. This completes the proof of Proposition 4. ■