Identification in Sequential First-Price Auctions: First-Round Bids Suffice *

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Abstract

We extend previous results on the identification of the distribution of valuations using bidding information to the case of sequential first-price auctions. We present the conditions that the distribution of bids must verify. It is concluded that in our model, the first round bids contain all the information needed to ensure identification.

KEYWORDS: Sequential First-Price Auctions, Non-parametric Identification.

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1 Introduction

Guerre, Perrigne and Vuong (2000) state the conditions under which the distribution of valuations (say F_v) of a group of I bidders in a single-unit first-price auction can be identified from bidding information. Remarkably, their result is non-parametric in the sense that identification does not rely on a parametric assumption on F_v.

This paper extends such a result to the case of sequential first-price auctions of a number of "replicas" of a good. We find that the first-round bids contain all the information needed for inference on F_v. In particular, the bids collected in subsequent auctions do not provide additional information about F_v beyond what is already implied in the first-round bids.

2 The Sequential First-Price Auction

Consider the sequential auction of two[1] "replicas" of an object to I agents ($I \geq 2$). Since items are identical, each agent $i = 1, \ldots, I$ has a unique valuation v_i for the first and second object.

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[1] The number of objects is assumed to be two without loss of generality. Our results can be extended to the context of an arbitrary but finite number of objects

These valuations are private knowledge and are assumed to be independently distributed across bidders. Let F_v denote[2] the distribution function of valuations with support $[\underline{v}, \bar{v}] \in \mathbb{R}_+$. It is assumed that F_v lies in $\mathcal{P}$, the set of absolutely continuous distribution functions on an interval in $\mathbb{R}_+$. Let $v_{(I)} \geq v_{(I-1)} \geq \ldots \geq v_{(1)}$ denote the ordered sample of valuations. Furthermore, assume that agents demand just one unit of the good. Thus, the first-round winner quits for the second round. Finally, the winning bid for the first auction is publicly announced before the start of the second one.

Weber (1983) proved the existence of a unique symmetric Bayesian-Nash equilibrium in this model. The equilibrium is characterized by a pair $(b_1^*(\cdot), b_2^*(\cdot))$ of continuous and monotonically increasing bidding functions in the $[\underline{v}, \bar{v}]$ domain. This equilibrium can be characterized using backward-induction reasoning. Fix a bidder i and consider the second auction. Since the first-round winning bid (b_1, say) is publicly announced, i can know the maximum valuation $v_{(I)}$ by simply computing $b_1^{*-1}(b_1)$. Conditional on this information, her best reply in the second round is given by the solution of

$$\underset{b_{2i}}{\text{Max:}}\ \Pi\left(v_i, b_{2i}\right) \equiv \begin{cases} \left(v_i - b_{2i}\right) \left\{ \frac{F_v\left[b_2^{*-1}(b_{2i})\right]}{F_v\left[v_{(I)}\right]} \right\}^{I-2} & \forall b_{2i} \leq b_1^*(v_{(I)}) \\ \left(v_i - b_{2i}\right) & \forall b_{2i} > b_1^*(v_{(I)}). \end{cases} \tag{1}$$

Thus, in the first round, the optimal bid for a bidder i must be the solution of

$$\underset{b_{1i} \in [\underline{v}, v_i]}{\text{Max:}}\ \Pi\left(v_i, b_{1i}\right) \equiv \left(v_i - b_{1i}\right) F_v \left[b_1^{*-1}\left(b_{1i}\right)\right]^{I-1} + \int\limits_{b_1^{*-1}(b_{1i})}^{\bar{v}} \left(v_i - b_2^*(s)\right) \mathrm{d}F_v\left(s\right)^{I-1}. \tag{2}$$

After some algebra (see Laffont, Loisel and Robert, 2001 for details), it can be verified that the first-order conditions of (1) and (2) are given by

$$\frac{\left[v_i - b_2^*\left(v_i\right)\right]\left(I-2\right) f_v\left(v_i\right)}{F_v\left(v_i\right) b_2^{\prime *}\left(v_i\right)} = 1, \text{ and} \tag{3}$$

$$\frac{\left[b_2^*\left(v_i\right) - b_1^*\left(v_i\right)\right]\left(I-1\right) f_v\left(v_i\right)}{F_v\left(v_i\right) b_1^{\prime *}\left(v_i\right)} = 1, \tag{4}$$

respectively. The boundary conditions $b_1^*(\underline{v}) = b_2^*(\underline{v}) = \underline{v}$ define the solutions $b_j^*(v) = E\left(v_{(I-2)} \mid v_{(I-j+1)} = v\right)$ for all $v \in [\underline{v}, \bar{v}]$ and $j = 1, 2$.

3 Identification of F_v through bidding data

At most times, when working with field-data, only bids are observable. A fundamental issue then, is to know whether certain structural elements, like F_v, can be recovered from the available information. In this paper, the available information is assumed to be the vectors $\mathbf{b_1} \in \mathbb{R}_+^I$ and $\mathbf{b_2} \in \mathbb{R}_+^{I-1}$, namely the bids collected in a sequential first-price auction for the

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[2] Throughout this paper we follow the convention that lower-case letters denote the density of the corresponding upper-case distribution function.

first and second round, respectively. Let $\mathcal{I} \equiv \{1, \ldots, I\}$ denote the set of subindexes for the I bidders and let $w \in \mathcal{I}$ be the subindex for the first-round winner. Furthermore, let $\tilde{\mathcal{I}}$ be the set of subindexes of the remaining $I-1$ players in the second round, i.e., $\tilde{\mathcal{I}} = \mathcal{I} \setminus w$.

Let $\mathbf{G}(\cdot)$ denote the joint distribution of $(\mathbf{b}_1, \mathbf{b}_2)$ and let $\mathbf{G}_1(\cdot)$ be the distribution of $\mathbf{b}_1$, $\mathbf{G}_2(\cdot)$ the distribution of $\mathbf{b}_2$, $G_1(\cdot)$ the marginal distribution of b_{1i} for each $i \in \mathcal{I}$, and $G_2(\cdot)$ the marginal distribution of b_{2i} for each $i \in \tilde{\mathcal{I}}$. Hence, if the bids are sampled in equilibrium, we have $G_1(b_{1i}) = \text{Prb}\left[b_1^{*-1}(v_i) \leq b_{1i}\right] = F_v\left[b_1^*(b_{1i})\right] = F_v(v_i)$ for any $b_{1i} \in [\underline{b}, \bar{b}] \equiv [\underline{v}, b_1^*(\bar{v})]$. Therefore, $g_1(b_{1i})/G_1(b_{1i}) = \left[1/b_1^{*'}(v_i)\right] f_v(v_i)/F_v(v_i)$. By substituting in (4),

$$b_{2i} = \varsigma\left(b_{1i}, G_1, I\right) \equiv b_{1i} + \frac{1}{I-1} \frac{G_1\left(b_{1i}\right)}{g_1\left(b_{1i}\right)}. \tag{5}$$

Moreover, $g_2(b_{1i})/G_2(b_{1i}) = \left[1/b_2^{*'}(v_i)\right] f_v(v_i)/F_v(v_i)$. By substituting this expression and equation (5) in (3) and arranging terms, we get

$$v_i = \xi\left(b_{1i}, \mathbf{G}, I\right) \equiv \varsigma\left(b_{1i}, G_1, I\right) + \frac{1}{I-2} \frac{G_2\left[\varsigma\left(b_{1i}, G_1, I\right)\right]}{g_2\left[\varsigma\left(b_{1i}, G_1, I\right)\right]}. \tag{6}$$

Equation (6) is the equivalent of equation (3) in Guerre, Perrigne and Vuong (2000) in our sequential auction context. It implies that if $(\mathbf{b}_1, \mathbf{b_2})$ were observed in equilibrium, then the valuation v_i of a bidder i who has bid b_{1i} must satisfy (6) for any $i \in \mathcal{I}$. Put simply, once $\mathbf{G}$ and I are known, (6) allows us to recover the unobservable valuations using the observed bids of the first round only. This idea is at the heart of our result, which reads as follows:

Theorem 1. *Let $\mathbf{G}(\cdot) \in \mathcal{P}^{2I-1}$ with support $[\underline{b}, \bar{b}]^{2I-1}$. There exists a unique distribution of valuations $F_v(\cdot) \in \mathcal{P}$ such that $\mathbf{G}(\cdot)$ is the joint distribution of first and second-round bids arising in equilibrium if and only if:*

C1) $\mathbf{G}(\mathbf{b}_1, \mathbf{b}_2) = \prod\limits_{i \in \mathcal{I}} G_1\left(b_{1i}\right)$ *for all* $b_{1i} \in [\underline{b}, \bar{b}]^I$, $b_{2j} \geq \varsigma\left(b_{1j}, G_1, I\right)$, $j \in \tilde{\mathcal{I}}$ *and* 0 *otherwise.*

C2) *The function $\xi\left(., \mathbf{G}, I\right)$ is strictly increasing in $[\underline{b}, \bar{b}]$ and its inverse is differentiable in $[\underline{v}, \bar{v}] = \left[\xi\left(\underline{b}, \mathbf{G}, I\right), \xi\left(\bar{b}, \mathbf{G}, I\right)\right]$.*

Proof. First, we prove necessity. Note that $b_{1i} \equiv b_1^*(v_i)$ are independent since the v_i's are independently sampled. Therefore, $\mathbf{G}(\mathbf{b}_1, \mathbf{b}_2) = \prod_{i \in \mathcal{I}} G_1\left(b_{1i}\right) \Pr\left[b_2^*\left(v_j \leq\right) b_{2j} \forall j \in \tilde{\mathcal{I}} \middle| b_1^*\left(v_i \leq\right) b_{1i} \forall i \in \mathcal{I}\right]$. Since $b_{2j} = \varsigma\left(b_{1j}, G_1, I\right)$, then the former conditional probability equals one when $b_{2j} \geq \varsigma\left(b_{1j}, G_1, I\right)$ and zero otherwise: in other words, conditional on $\mathbf{b}_1$, $\mathbf{b}_2$ is a degenerated random vector. This proves the necessity of C1. With respect to C2, (6) shows that $b_1^{*-1}(b) = \xi(b, \mathbf{G}, I)$ for all $b \in [\underline{b}, \bar{b}]$. Since $b_1^{*-1}(\cdot)$ is strictly increasing and $b_1^*(\cdot)$ is differentiable, C2 must hold.

Now we turn to sufficiency. First, C1 implies that the b_{1i} are independent with distribution G_1. Note that $\lim_{b \downarrow \underline{b}} \xi\left(b, \mathbf{G}, I\right) = \underline{b}$ because $\lim_{b \downarrow \underline{b}} G_2(b)/g_2(b) = \lim_{b \downarrow \underline{b}} G_1(b)/g_1(b) = 0$. Now, define $F(\cdot) = G_1\left[\xi^{-1}\left(\cdot, \mathbf{G}, I\right)\right]$ in $[\underline{v}, \bar{v}] \equiv \left[\underline{b}, \xi\left(\bar{b}, \mathbf{G}, I\right)\right]$. Since $\xi(\cdot, \mathbf{G}, I)$ and $G_1(\cdot)$ are both strictly increasing then F_v is a valid distribution and is strictly increasing in $[\underline{v}, \bar{v}]$. Hence, $F_v \in \mathcal{P}$. It remains to be proven that this distribution can generate $\mathbf{G}(\cdot)$ when valuations from it are sampled. This amounts to proving that $G_1(\cdot) = F_v\left[b_1^{*-1}(\cdot)\right]$ where $b_1^*(\cdot)$ solves (4). By construction, $G_1(\cdot)$ equals $F_v(\xi(\cdot, G, I)$. Thus, we only need to prove that $\xi^{-1}(\cdot, G, I)$ solves

$$\left\{\varsigma\left[\xi^{-1}\left(v,\mathbf{G},I\right)\right]-\xi^{-1}\left(v,\mathbf{G},I\right)\right\}\frac{(I-1)g_1\left[\xi^{-1}\left(v,\mathbf{G},I\right)\right]}{G_1\left[\xi^{-1}\left(v,\mathbf{G},I\right)\right]} \tag{7}$$

for all $v \in [\underline{v},\bar{v}]$ which in fact is so by definition of $\xi(\cdot,\mathbf{G},I)$ in (6). Finally, uniqueness is based on the fact that, given any $F_v(\cdot) \in \mathcal{P}$ it can be constructed as $F_v(\cdot) = G_1\left[\xi^{-1}\left(\cdot,\mathbf{G},I\right)\right]$. Since $\xi^{-1}(\cdot,\mathbf{G},I)$ is uniquely determined by G_1, then F_v is uniquely determined by G_1, which by virtue of C1, is equivalent to saying that it is uniquely determined by $\mathbf{G}(\cdot)$. $\square$

4 Conclusions

We derive the necessary and sufficient conditions for the identification of the distribution of valuations of a group of bidders who participate in a sequential first-price auction when all of the bids have been collected. Our result can be regarded as an extension of the conditions in Guerre, Perrigne and Vuong (2000) for the single-unit case. It is immediate to notice that our result collapses to their theorem 1 when the number of items sold is reduced to one.

Because bids in equilibrium are a function of individual valuations and items are considered identical, first-round bids contain all the relevant information about F_v and, in particular, second-round bids should be a degenerate random vector in equilibrium, conditional on the realization of the first-round bids. The conditions in theorem 1 are not very restrictive: in fact, there is a wide range of distribution families that satisfy C1 and C2, in particular, any log-concave distribution.

Theorem 1 can be used as a starting point for several exercises in the analysis of sequential descending-price auctions that use field-data. First, a useful preliminary test should verify that the distribution of bids verifies the conditions in theorem 1. Secondly, equation (6) may be used for the non-parametric estimation of F_v through kernel techniques.

References

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