

# Auctioning Horizontally Differentiated Items

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## Abstract

This paper characterizes all equilibrium reserve prices set by 2 sellers auctioning horizontally differentiated items. When sellers cooperate, the weakest bidder(s) gets no rents. Moreover, bidders are efficiently allocated: they attend the seller for which they represent the most. When sellers compete and products are sufficiently differentiated multiple equilibria arise. For each, the weakest bidder extracts no rents. However, the allocation of bidders is not always efficient. For closer substitutes there is a unique equilibrium. The weakest type gathers participation rents. Efficiency is restored with the indifferent type generating the same total surplus no matter which seller he attends.

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# 1 Introduction

While recent events, such as the FCC auction, drove the attention of economists towards the allocation of complements, much less is known about auctioning substitutes. Yet, in many instances, items perceived as substitutes are auctioned. In Dublin many second hand houses are sold by means of an English auction. Several procurement and construction projects are often allocated by sealed-bid auctions. Car rental companies and governmental agencies auction their used cars in the United States.

In each of these cases, the auctioneers face the same pool of bidders. Thus, the number of bidders a particular auction attracts and the revenue it will raise critically depends on what other items are available and on how these are auctioned. This interdependence of the auctions' performances complicates substantially the auction design problem a seller faces. Burguet (2003) and Parlane (2001), analyze the optimal allocation of substitutes by a monopolistic seller. This paper contributes to the existing literature by considering the case of competing auctioneers.

There are only few papers examining markets with competing sellers who use auctions instead of prices. McAfee (1993), Peters and Serinov (1997) and Peters (1997) address competing auctioneers holding homogeneous items. These papers examine markets with large numbers of sellers and buyers. The advantage of this assumption is that it gently restricts the strategic interactions between the market participants which allows the authors to solve for the equilibrium. The drawback of this approach is that it is not clear whether the results are applicable to markets with a limited number of sellers.

In this paper we consider a market with 2 sellers. Thus, strategic interactions between sellers will play a more prominent role. The cost of this is that to make the problem tractable we have to restrict attention to competition in reserve prices in English auctions. In this sense, the paper is closest to Burguet and Sákovics (1999) who analyze similar issues in a market for homogeneous items by restricting seller's strategies to reserve prices in second price auctions. It departs from it as I introduce horizontal product differentiation.

The model considered is based on the Hotelling (1929) linear model. There are  $n$  buyers, located in between the 2 sellers. Each buyer privately observes his location which gives a measure for his willingness to pay for each

item. Sellers announce simultaneously their reserve prices which constitutes the lowest acceptable bid for their English auction. Given the reserve prices and their valuation, the buyers decide on which auction to attend. We consider that auctions are simultaneous so that one buyer can attend at most one seller. We search for sub-game perfect equilibria in which sellers perfectly anticipate the buyers' behavior which is described in section 3.

In section 4, we assume sellers set their reserve prices cooperatively and use these results as a benchmark. Under cooperation, the reserve prices achieve an efficient allocation of bidders and fully extracts the weakest bidder's rents. More precisely, reserve prices allocate bidders with priority based on the expected marginal revenue (or surplus) each type is associated with. In section 5 we consider competing sellers. The cost of competition depends on the degree of product differentiation. When products are sufficiently differentiated, there exists a multiplicity of equilibria. For each of these, the weakest, indifferent bidder gets no rents. Under some distribution of types (for instance when no seller benefits from a denser market) the cooperative reserve prices still form an equilibrium. However it is no longer unique, and in any other equilibria bidders are not efficiently distributed. When products become closer substitutes there is a unique equilibrium. Competition induces each seller to leave some rents to the weakest bidder in order to prevent him from attending his opponent's auction. These rents increase as products become more homogeneous. The allocation of bidders is efficient in that the indifferent bidder is the one generating the same expected *total* surplus no matter which seller he attends. This surplus is now the sum of marginal revenue and participation rents. From the sellers' point of view the equilibrium reserve prices are sub-optimal as they could both benefit from increasing their reserve prices to the valuations of the weakest type.

Section 6 answers two subsidiary, yet interesting questions. First, I explain why the analogy between auction theory and monopoly pricing established in Bulow and Roberts (1989) cannot be extended to this duopoly setting. The argument consists in showing that a marginal change in the reserve price affects the revenue even conditional on having attracted two (and exactly) bidders, despite the fact that neither pays the reservation prices upon winning. Second, I compare the equilibrium reserve prices calculated in section 5 with those holding under sequential (monopolistic) auctioning. We show that reserve prices under duopoly can be higher than the monopolistic ones either to reduce or save on the participation rents left to the indifferent bid-

der or simply because the presence of a substitute acts as a screening device which eliminates the weakest bidders.

Section 7 presents a conclusion.

## 2 The model

Consider 2 risk neutral sellers (seller 1 and seller 2) each possessing a single item. Assume that the item has no value for the seller. The sellers face a market composed of  $n$  risk neutral consumers. These consumers consider the items to be horizontally differentiated. Each consumer is characterized by his taste ( $\theta \in [0, 1]$ ) which gives a measure for his willingness to pay for each item. A consumer with taste  $\theta$  is willing to pay  $v_i(\theta)$  for seller  $i$ 's item ( $i = 1, 2$ ). Let

$$\begin{aligned} v_1(\theta) &= 1 - t\theta, \\ v_2(\theta) &= 1 - t(1 - \theta), \end{aligned}$$

where  $t \in (0, 1]$  measures product differentiation<sup>1</sup>. Graphically, this situation can be represented as a Hotelling (1929) model of horizontal product differentiation with sellers located at the extremities of a line of length 1.

Each buyer privately observes his own location. However, it is common knowledge that tastes are identically and independently distributed according to a distribution function  $F(\theta)$  defined over  $[0, 1]$ . Let  $F(\cdot)$  be continuously differentiable and let  $f(\theta)$  denote the density function. Assume that  $f(\theta) > 0$  almost everywhere.

We assume that the sellers auction their items simultaneously by means of an English auction. More precisely, each seller asks initially for the reservation price. If there is excess demand, a seller raises the price until he faces a single buyer willing to pay the offer price. Each seller's strategic variable is his reservation price. Finally, assume that buyers who do not attend any of the 2 auctions receive a reservation utility ( $\underline{U}$ ) such that  $\underline{U} = 0$ .

Throughout the paper we will use the following notation:

Notation: Let  $\gamma_1$  and  $\gamma_2$  denote the reservation prices set by seller 1 and seller 2 respectively. Let  $(r_1, r_2) \in ]0, +\infty) \times (-\infty, 1[$  be defined such that  $v_i(r_i) \equiv \gamma_i$  ( $i = 1, 2$ ). Finally let  $R = (r_1, r_2)$ .

Notice that for any  $(\gamma_1, \gamma_2)$  there exists a unique  $(r_1, r_2)$  such that  $v_i(r_i) \equiv \gamma_i$  ( $i = 1, 2$ ). Thus, considering  $\gamma_i$  ( $i = 1, 2$ ) as seller  $i$ 's strategic variable is equivalent to considering  $r_i$  ( $i = 1, 2$ ) as seller  $i$ 's strategic variable.

The timing of the game is the following. First, Nature draws each buyer's taste (refer to as a type). Then, both sellers simultaneously announce their reservation prices. And finally, the buyers decide on which auction to attend. I will assume that auctions are simultaneous so that a buyer can only deal with one of the two sellers. Trade (if any) takes place once the auction is concluded.

### 3 The buyers' game

Consider a buyer who attends one of the 2 auctions. If he is the only one attending the auction, he wins and pays the reservation price. If there is excess demand at the reservation price, a dominant strategy for a participating bidder consists in dropping out whenever the price reaches his true valuation. This decision forms a dominant strategy equilibrium.

The following proposition characterizes the auctions' attendance.

*Proposition 1: The buyers' decision regarding which seller to attend are such that:*

*-If reserve prices are such that  $r_1 \leq r_2$  then all bidders with type  $\theta \leq r_1$  attend seller 1 and all bidders such that  $\theta \geq r_2$  attend seller 2. Buyers for which  $\theta \in ]r_1, r_2[$  do not participate.*

*-If reserve prices are such that  $r_1 > r_2$  there always exists a threshold value  $\theta_R \in [0, 1]$  such that the following strategy forms a symmetric Nash equilibrium: all buyers with a valuation  $\theta \leq \theta_R$  attend seller 1, while all buyers with a valuation  $\theta \geq \theta_R$  attend seller 2.*

*The variable  $\theta_R$  characterizes the indifferent type. It is uniquely defined by:*

$$(1 - F(\theta_R))^{n-1} (v_1(\theta_R) - \gamma_1) = F^{n-1}(\theta_R) (v_2(\theta_R) - \gamma_2). \quad (1)$$

*We have  $\theta_R \in [0, 1]$  for any  $(r_1, r_2) \in ]0, +\infty) \times (-\infty, 1[$  such that  $r_1 > r_2$ .*

*Proof: see Appendix 1.*

The variable  $\theta_R$  determines each seller's market share. Equation (1) states that the indifferent buyer is the one for whom the expected surplus upon getting item 1 equals the expected surplus upon getting item 2.

## 4 The sellers' game

Bulow and Roberts (1989) show that optimal auction theory is equivalent to standard monopoly third degree price discrimination theory. It shows that the optimal reserve price aims at preventing the entry of any bidder whose type is associated with a negative marginal revenue (under zero marginal cost). Using their approach and results from Myerson (1981), I first characterize the profit maximizing monopolistic reserve prices. Since bidders are symmetric, we can assume that each seller faces a unique representative buyer.

Assume seller 1 is the only seller. At any given price  $p_1 = v_1(\theta)$  the (expected) quantity sold is  $q_1 = F(\theta)$ . Thus, we can express seller 1's total revenue as a function of quantity as:

$$TR_1(q_1) = q_1 [1 - tF^{-1}(q_1)].$$

Differentiating the above expression leads to

$$MR_1(\theta, t) = 1 - t \left( \theta + \frac{F(\theta)}{f(\theta)} \right), \quad (2)$$

which is seller 1's marginal revenue from type  $\theta$ .

Let us use a similar procedure for seller 2. At any given price  $p_2 = v_2(\theta)$ , the (expected) quantity sold is  $q_2 = 1 - F(\theta)$ . Seller 2's total revenue can be expressed as:

$$TR_2(q_2) = q_2 [1 - t(1 - F^{-1}(1 - q_2))],$$

and, seller 2's marginal revenue as:

$$MR_2(\theta, t) = 1 - t \left[ 1 - \left( \theta - \frac{1 - F(\theta)}{f(\theta)} \right) \right]. \quad (3)$$

Thus, a type  $\theta$  bidder is accounted for as type  $\left( \theta + \frac{F(\theta)}{f(\theta)} \right)$  for seller 1 and as type  $\left( \theta - \frac{1 - F(\theta)}{f(\theta)} \right)$  for seller 2. (This is well known and points at the fact that informational rents are needed to guarantee incentive compatibility.) We make the following assumption before pursuing the analysis:

*Regularity assumption: The functions  $\left(x + \frac{F(x)}{f(x)}\right)$  and  $\left(x - \frac{1 - F(x)}{f(x)}\right)$  are increasing<sup>2</sup> and continuous.*

Under the above regularity condition, the optimal monopolistic reserve prices are set as follows. If a seller's marginal revenue is positive over the entire interval of types, he does not restrict entry and sets  $\gamma_i = (1 - t)$  ( $i = 1, 2$ ) so as to extract the weakest bidder's rents. If, however, the marginal revenue is negative for some (weak) types, the optimal reserve price should guarantee that they do not participate. If  $MR_1(\theta, t)|_{\theta=1} < 0$  then the regularity condition ensures that there exists  $\theta_1$  such that  $MR_1(\theta_1, t) = 0$  and the optimal monopolistic reserve price is  $\gamma_1^m = v_1(\theta_1)$ . Similarly, if  $MR_2(\theta, t)|_{\theta=0} < 0$  then the regularity condition ensures that there exists  $\theta_2$  such that  $MR_2(\theta_2, t) = 0$  and seller 2's optimal monopolistic reserve price is given by  $\gamma_2^m = v_2(\theta_2)$ .

We now move to the case of simultaneous auctioning of the 2 items. Let  $\hat{\theta}$  be characterized as follows:

$$MR_1(\hat{\theta}, t) = MR_2(\hat{\theta}, t).$$

Under the regularity assumption,  $\hat{\theta}$  exists and is uniquely defined. Moreover, note that  $\hat{\theta}$  is independent of  $t$  as it solves:

$$1 - 2\hat{\theta} + \frac{1 - 2F(\hat{\theta})}{f(\hat{\theta})} \equiv 0.$$

Whether  $MR_i(\hat{\theta}, t) \geq 0$  ( $i = 1, 2$ ), depends on  $t$ . Let  $\bar{t} > 0$  be such that  $MR_i(\hat{\theta}, \bar{t}) = 0$  ( $i = 1, 2$ ). Whether  $\bar{t} < 1$  depends on the distribution function. Under the uniform distribution  $\bar{t} = 1$ . For any distribution function such that the median (denoted  $\theta_M$ ) is  $1/2$  we have  $\bar{t} < 1$  only when the population is polarized. Figure 1 represents the marginal revenues in both cases:  $t \leq \bar{t}$  and  $t > \bar{t}$ .

Figure 1.

As it appears in this figure, and from now on,  $\theta_1$  and  $\theta_2$  will refer to the unique solution to  $MR_i(\theta_i, t) = 0$  for  $i = 1, 2$ .

*Proposition 2: For any  $t > \bar{t}$ , the optimal reserve prices are  $\gamma_1 = v_1(\theta_1)$  and  $\gamma_2 = v_2(\theta_2)$  whether sellers compete or cooperate. The market shares do not overlap. (If  $\bar{t} \geq 1$ , such equilibria fail to exist.)*

*Proof: see Appendix 2.*

If the distribution function is such that  $\bar{t} < 1$  then there exists a range of product differentiation for which seller 1 would have no interest in capturing any of seller 2's market and vice-versa (under the regularity condition). In other words, for sufficiently differentiated items sellers act as monopolies over their non-overlapping market shares.

For any  $t < \bar{t}$  the optimal monopolistic reserve prices no longer form a solution as the monopolistic market shares would overlap. The next section characterizes the reserve prices set cooperating sellers. It provides a benchmark to which we can compare the competitive outcome.

## 4.1 Cooperating sellers

Consider that the 2 sellers decide on their reserve prices cooperatively, maximizing joint profit. Let  $\gamma_i^c$  refer to seller  $i$ 's cooperative reserve price.

*Proposition 3: For any  $t \leq \bar{t}$  the optimal reserve prices for each object are such that the market is entirely covered. The indifferent type extracts no rents and bidders are efficiently allocated with priority based on the expected surplus each generates.*

*More precisely, we have  $\gamma_1^c = v_1(r^c)$  and  $\gamma_2^* = v_2(r^c)$  where  $r^c \in ]0, 1[$  is unique and characterized by*

$$(1 - F(r^c))^{n-1} MR_1(r^c, t) = (F(r^c))^{n-1} MR_2(r^c, t). \quad (4)$$

*Proof: See appendix 3.*

Notice first that for any  $t \leq \bar{t}$ , the sellers want to cover the entire market. Indeed, when setting reserve prices such that  $r_1 < r_2$ , we either have  $MR_1(r_1, t) > 0$  or  $MR_2(r_2, t) > 0$  or both<sup>3</sup>. It follows that any such  $R$  cannot be maximizing the sellers joint profits as they would benefit from decreasing the reserve price for which the marginal revenue is positive. It is obvious that any reservation prices such that  $r_1 > r_2$  cannot be optimal either. The sellers would be better-off raising both reserve prices to  $v_i(\theta_R)$  ( $i = 1, 2$ ) since they would not affect each item's market share and yet extract more surplus through higher reserve prices. Thus, we must have  $r_1 = r_2 = r$  to maximize joint profits.

A "natural" candidate for seller  $i$ 's optimal reserve price is  $\gamma_i = v_i(\widehat{\theta})$

( $i = 1, 2$ ). If set, each bidder would attend the seller for whom they represent the most. The following lemma states when this will form an equilibrium

*Lemma 1: Bidders are allocated with priority based on their marginal revenue only (i.e.  $r^* = \hat{\theta}$ ) if and only if  $\theta_M = 1/2$ , where  $\theta_M$  refers to the median type ( $F(\theta_M) = 1/2$ ).*

(Proof: Notice that  $\hat{\theta} = r^c = \theta_M$  if and only if  $\theta_M = 1/2$ .)

When no seller benefits from a denser market, the optimal reserve prices under cooperation are equal ( $\gamma_1^c = \gamma_2^c = 1 - \frac{1}{2}t$ ) and bidders are allocated on the basis of the marginal revenue, or surplus they generate. In any other cases, the probability with which the marginal bidder gets the item comes into play. Assume, for instance, that  $\theta_M < \frac{1}{2}$  meaning that item 1 benefits from a denser market. In that case, cooperating sellers set the marginal bidder, type  $r^c$ , such that  $\theta_M < r^c < \hat{\theta}$ . We have  $MR_1(r^c, t) > MR_2(r^c, t)$ . Although type  $r^c$  is worth more to seller 1, he is less likely to win item 1 than he is to win item 2.

The efficient allocation of bidders relies on the regularity assumption. Indeed, under this assumption the function

$$h(x) = (1 - F(x))^{n-1} MR_1(x, t) - (F(x))^{n-1} MR_2(x, t),$$

is decreasing in  $x$ . From seller  $i$ 's point of view, type  $\theta$  generates a surplus equal to  $MR_i(\theta, t)$  if he gets item  $i$  and 0 otherwise.

Thus,  $(1 - F(x))^{n-1} MR_1(x, t)$  is the expected surplus from type  $\theta$  when all bidders attend item 1's auction. Similarly,  $(F(x))^{n-1} MR_2(x, t)$  is the expected surplus from type  $\theta$  when all bidders attend the auction for item 2. In equilibrium, the marginal bidder is the one generating the same expected surplus.

## 5 Competing sellers.

In that section we search for sub-game perfect Nash equilibrium where sellers perfectly anticipate the behavior of bidders and take as given the reserve of their opponent. For any  $t \leq \bar{t}$ , the only possible equilibria are such that

the entire market is covered. To characterize such equilibria, we need to introduce additional functions. Let  $g_1(\theta, t)$  and  $g_2(\theta, t)$  be defined as:

$$g_1(\theta, t) = MR_1(\theta, t) [1 - F(\theta)]^{n-1} - t \frac{[F(\theta)]^n}{f(\theta)}, \quad (5)$$

and

$$g_2(\theta, t) = MR_2(\theta, t) [F(\theta)]^{n-1} - t \frac{[1 - F(\theta)]^n}{f(\theta)}. \quad (6)$$

The first order conditions from the expected profits maximization problems lead to:

$$n(1 - F(\theta_R))^{n-1} [v_1(\theta_R) - \gamma_1] = g_1(\theta_R, t),$$

and

$$n(F(\theta_R))^{n-1} [v_2(\theta_R) - \gamma_2] = g_2(\theta_R, t).$$

Since (1) must hold in equilibrium, the optimal reserve prices must be such that

$$g_1(\theta_R, t) = g_2(\theta_R, t). \quad (7)$$

Moreover, in any equilibrium where reserve prices are set such that  $r_1 > r_2$ , the indifferent type must get a non-negative expected utility to participate. Thus, in equilibrium we must have:

$$g_i(\theta_R, t) \geq 0 \text{ for } i = 1, 2. \quad (8)$$

Depending on whether (7) and (8) can both hold, two different types of equilibria arise depending on how close substitutes the items are. For both the market is entirely covered. Let  $\gamma_i^*$  denote seller  $i$ 's reserve price in equilibrium ( $i = 1, 2$ ).

*Proposition 4:* *There exists  $\underline{t} \in ]0, \bar{t}[$  such that for any given degree of product differentiation  $t < \underline{t}$  there exists a unique equilibrium. It is such that the market is entirely covered (i.e.  $r_1^* > r_2^*$ ) and each seller leaves some participation rents to the indifferent (weakest) type ( $\gamma_i^* < v_i(\theta_{R^*})$  for  $i = 1, 2$ ). However, bidders are efficiently allocated with priority based on the expected total surplus each type is associated with.*

*More precisely, the optimal reserve prices are such that*

$$\begin{cases} \gamma_1^* = v_1(\theta_{R^*}) - \frac{g_1(\theta_{R^*}, t)}{nt [1 - F(\theta_{R^*})]^{n-1}} \\ \gamma_2^* = v_2(\theta_{R^*}) - \frac{g_2(\theta_{R^*}, t)}{nt [F(\theta_{R^*})]^{n-1}}. \end{cases}$$

The indifferent consumer  $\theta_{R^*}$  is defined such that:

$$\begin{aligned} & [1 - F(\theta_{R^*})]^{n-1} \left\{ MR_1(\theta_{R^*}, t) + \left[ v_1\left(\theta_{R^*} - \frac{1 - F(\theta_{R^*})}{f(\theta_{R^*})}\right) - \gamma_1^* \right] \right\} \\ & = [F(\theta_{R^*})]^{n-1} \left\{ MR_2(\theta_{R^*}, t) + \left[ v_2\left(\theta_{R^*} + \frac{F(\theta_{R^*})}{f(\theta_{R^*})}\right) - \gamma_2^* \right] \right\} \quad (9) \end{aligned}$$

*Proof:* See Appendix 4.

The cost of competition for close substitutes consists in the necessity to leave rents to the indifferent, weakest, type. These rents aim at dissuading the marginal bidder from attending the other seller's auction. They are given by

$$v_1\left(\theta_{R^*} - \frac{1 - F(\theta_{R^*})}{f(\theta_{R^*})}\right) - \gamma_1^*$$

if he attends seller 1's auction, and

$$v_2\left(\theta_{R^*} + \frac{F(\theta_{R^*})}{f(\theta_{R^*})}\right) - \gamma_2^*$$

if he attends the auction for item 2. It is trivial to show that these rents increase as products becomes closer substitutes.

Interestingly, note that a seller takes into account what the marginal bidder is worth to his opponent to evaluate the participation rents he must leave to his weakest bidder. More precisely, to dissuade the marginal bidder from attending his opponent, seller 1 takes into account the fact that type  $\theta_{R^*}$  is worth  $\left(\theta_{R^*} - \frac{1 - F(\theta_{R^*})}{f(\theta_{R^*})}\right)$  to seller 2. Similarly, seller 2's treats type  $\theta_{R^*}$  as  $\left(\theta_{R^*} + \frac{F(\theta_{R^*})}{f(\theta_{R^*})}\right)$  which represents what that type is worth to seller 1.

Equation (9) shows that an efficient allocation of bidders occurs in equilibrium since the indifferent type is such that the expected total surplus he generates upon getting item 1 is equal to the expected total surplus he generates upon getting item 2. These surpluses now include the rents that the weakest, indifferent bidder can extract.

From the sellers' point of view however, competition pushes them towards

some “sub-optimal” equilibrium. Clearly both sellers would be better-off setting their reserve prices equal to the valuations of the indifferent type.

Finally, for  $t \in [\underline{t}, \bar{t}]$  a different type of equilibria arise.

*Proposition 5: For any  $t \in ]\underline{t}, \bar{t}[$  there exists multiple equilibria. In any of these equilibria the indifferent bidder gets no rents, but bidders may no longer be efficiently allocated.*

*For any given  $t \in ]\underline{t}, \bar{t}[$  there exists a non-empty range of types  $\Theta_t$  defined as follows:*

$$\Theta_t = \{\theta \in [0, 1] : MR_i(\theta, t) \geq 0 \text{ and } g_i(\theta, t) \leq 0 \text{ for } i = 1, 2\}.$$

*Any  $\gamma_1^* = v_1(r^*)$  and  $\gamma_2^* = v_2(r^*)$  with  $r^* \in \Theta_t$  forms a Nash equilibrium.*

*Proof: See Appendix 5.*

At the solution, a seller’s revenue is not differentiable. It forms a peak. Figure 2 describes the equilibrium:

Figure 2.

For any  $t \in ]\underline{t}, \bar{t}[$ , we have  $MR_i(r^*, t) \geq 0$  ( $i = 1, 2$ ). Thus, if alone, each seller would set his reserve price so as to extend his market share. However, products are sufficiently differentiated for seller  $i$  ( $i = 1, 2$ ) to desire a high enough reserve price in a duopoly situation to save on participation rents that must be awarded to the indifferent bidder when  $r_i > r_j$ .

Thus, the only cost of competition is to reach a potentially inefficient allocation of bidders. As lemma 4 states below, the cooperative reserve prices may still form an equilibrium.

*Lemma 3: For some distribution functions, the equilibrium reserve prices reached under cooperation (with  $r_1 = r_2 = r^c$ ) belong to the set of non-cooperative equilibria. This is true under any distribution function such that  $\theta_M = \frac{1}{2}$ .*

*Proof:* When  $\theta_M = \frac{1}{2}$ , we have  $r^c = \hat{\theta} = \frac{1}{2}$  and  $g_1(\frac{1}{2}, t) = g_2(\frac{1}{2}, t)$  for any  $t$ . Finally, figure 4 shows that for any  $t \in [\underline{t}, \bar{t}]$ , we have  $\frac{1}{2} \in \Theta_t$  as  $g_i(\frac{1}{2}, t) < 0$  for  $i = 1, 2$ .

Finally lemma 5 completes the characterization of the optimal reserve prices by looking specifically at the cases where  $t = \bar{t}$  and  $t = \underline{t}$ .

*Lemma 4:* At  $t = \bar{t}$  the unique solution is such that  $\gamma_1^* = v_1(r^*)$  and  $\gamma_2^* = v_2(r^*)$  with  $r^* = \hat{\theta}$ . At  $t = \underline{t}$ , the unique solution is such that  $\gamma_1^* = v_1(\theta_{\underline{t}})$  and  $\gamma_2^* = v_2(\theta_{\underline{t}})$  with  $\theta_{\underline{t}}$  defined as:

$$g_1(\theta_{\underline{t}}, \underline{t}) = g_2(\theta_{\underline{t}}, \underline{t}).$$

*(The proof is trivial.)*

Solutions are somehow continuous. For  $t = \bar{t}$ , the entire market is just covered, and the reserve prices are the optimal monopolistic reserve prices. As  $t = \underline{t}$  the market is also entirely covered. The set  $\Theta_{\underline{t}} = \theta_{\underline{t}}$  and the equilibrium reserve prices are as described in proposition 5.

## 6 Subsidiary analysis

In the previous section the optimal reserve prices were characterized for all possible degrees of product differentiation. In what follows, I answer two questions. The first is whether the analogy between monopoly pricing and auction theory established in Bulow and Roberts (1989) can be extended to this duopoly setting. The second aims at comparing the monopolistic reserve prices (set when products are sold sequentially) with the ones that will hold under simultaneous auctioning.

The answer to the first question is no, the analogy established in Bulow and Roberts (1989) cannot be extended to this duopoly situation. Applying the Bulow and Roberts (1989) approach consists in constructing each seller's residual demand considering that a bidder's opportunity cost of not attending one seller's auction is not only his valuation but this minus what he expects at the other auction. The solution to the above problem does not coincide with the equilibrium reserve prices found in the previous section. The reason is that the analogy established in Bulow and Roberts (1989) is present only when a marginal change in the reserve price affects the revenue only when a single bidder attended the auction. Consider the case of a monopolistic seller. If, prior to the change, the seller attracted more than a single bidder, then it is quite obvious that a marginal change in his reserve price will not affect his revenue. When competing sellers are considered, a marginal change in the reserve price may affect a seller's revenue even conditional on him attracting more than a single seller initially. In Burguet and Sákovicš (1999) this is the

case as reserve prices not only determine the price paid by a single bidder but also the composition of demand. In the case analyzed here, a marginal change in the reserve price affects a seller's revenue conditional on gathering one but also two, and exactly two, bidders.

The first order condition taken at all  $r_1 > r_2$  from maximizing seller 1's revenue can be written as:

$$\begin{aligned} \frac{\partial \pi_1}{\partial r_1} = & n(1 - F(\theta_R))^{n-1} F(\theta_R) \left[ v_1(r_1) \frac{f(\theta_R)}{F(\theta_R)} \frac{\partial \theta_R}{\partial r_1} - t \right] \\ & + \frac{n(n-1)}{2} (1 - F(\theta_R))^{n-2} (F(\theta_R))^2 \left[ 2(v_1(\theta_R) - \gamma_1) \frac{f(\theta_R)}{F(\theta_R)} \frac{\partial \theta_R}{\partial r_1} \right]. \end{aligned}$$

The first term expresses the marginal change in revenue when a single bidder was present at the auction. The probability of attracting one and only one bidder is given by  $n(1 - F(\theta_R))^{n-1} F(\theta_R)$ .

The term  $\frac{n(n-1)}{2} (1 - F(\theta_R))^{n-2} (F(\theta_R))^2$  is the probability that exactly 2 bidders attended the auction. In that case, the marginal gains are twice  $(v_1(\theta_R) - \gamma_1)$ . In any equilibrium where  $r_1 > r_2$  seller 1's market share is given by  $[0, \theta_R]$  where  $\theta_R < r_1$ . All of the bidders he gathers have valuations *strictly* greater than the reserve price. If seller 1 attracts a single bidder, the latter pays the reserve price  $\gamma_1$ . If he attracts 2 bidders, the minimum bid is at least equal to  $v_1(\theta_R) > \gamma_1$ . Thus, seller 1's revenue is discontinuous as we move from attracting 1 to 2 bidders. Attracting 2 bidders allows him to save  $(v_1(\theta_R) - \gamma_1)$ , where  $v_1(\theta_R)$  is potentially the losing bid and therefore the price. Conditional on attracting 3 bidders, marginal changes in the reserve price no longer affects the seller's revenue. The lowest possible bid (potentially equal to  $v_1(\theta_R)$ ) is never the price.

The comparison between the sequential monopolistic reserve prices with the ones holding under simultaneous auctioning is difficult to perform in general. In what follows I consider the case where types are uniformly distributed over  $[0, 1]$ . Under the uniform distribution, seller 1 and 2's reserve prices are always equal. Let  $\gamma^m$  denote to monopolistic reserve price and  $\gamma^d$  the one arising in a duopoly.

Under the uniform distribution we have

$$MR_1(\theta, t) = 1 - 2t\theta,$$

and

$$MR_2(\theta, t) = 1 - 2t(1 - \theta).$$

The monopolistic reserve price is given by

$$\gamma^m = \begin{cases} \frac{1}{2} & \text{for } t \geq \frac{1}{2} \\ (1-t) & \text{for } t < \frac{1}{2}. \end{cases}$$

Indeed, for any  $t < \frac{1}{2}$  the marginal revenue of a seller is positive for all types and a seller would not desire to restrict entry. Yet, to maximize profits, he charges the weakest bidder his valuation so as to extract all his rents. For any  $t \geq \frac{1}{2}$ , a seller sets his reserve price so as to prevent entry of any type associated with a negative marginal revenue. Seller 1 sets  $\gamma_1 = \frac{1}{2}$  to exclude any  $\theta > \frac{1}{2t}$ , while seller 2 sets  $\gamma_2 = \frac{1}{2}$  to get rid off any  $\theta$  such that  $(1-\theta) > \frac{1}{2t}$ .

We now consider duopoly. Under the uniform distribution, we have  $\bar{t} = 1$  while  $\underline{t} = 2/3$ . For any  $t < 2/3$ , we have  $\theta_{R^*} = \frac{1}{2}$  and

$$\gamma^d = \frac{n-1}{n} - t \frac{n-3}{2n},$$

For any  $t > 2/3$ , there exists multiple equilibria but for each of these we know (cf. proposition 5) that the equilibrium reserve price will be higher than the monopolistic reserve price. Thus, for any  $t \geq \frac{1}{2}$  we have  $\gamma^d > \gamma^m$ . For any  $t < \frac{1}{2}$  and any  $n$ , there exists  $t_n$ , with  $0 < t_n < \frac{1}{2}$  such that  $\gamma^d > \gamma^m$  for all  $t > t_n$  and  $\gamma^d < \gamma^m$  for all  $t < t_n$ . Provided product differentiation is large enough, reserve prices are greater under duopoly. The presence of a substitute drags away from an auction weak bidders who prefer the other item. This allows the seller to set higher reserve prices. For close substitutes, the reserve prices are used to provide participation rents to the marginal bidder (type  $\theta_R$ ). As a seller increases his reserve price he loses some market share but gains on the participation rents left to the indifferent bidder. The more identical the items the more each seller cares about his market share and sets his reserve price below the monopolistic one. For sufficiently differentiated items reducing the marginal bidder's rents becomes a priority and the duopolistic reserve prices are greater than the monopolistic ones. Note finally that the more bidders, the narrower the range of product differentiation for which market share is a priority.

## 7 Conclusion

In this paper I have characterized all equilibrium reserve prices resulting from auctioning simultaneously two horizontally differentiated items. Product differentiation adds an interesting dimension. For sufficiently differentiated items and under some distribution functions, competition between sellers may not have any impact on the choice of the sellers' optimal reserve prices. They may be able to set their monopolistic reserve prices and have non-overlapping market shares. These equilibria do not always exist. In general, sellers will cover the entire market.

When sellers cooperate, the reserve prices are set equal to the weakest bidder's valuations so that he extracts no rents. Moreover, bidders are efficiently allocated: they attend the seller for they represent the most. When sellers compete and products are sufficiently differentiated multiple equilibria arise. For each, the weakest bidder extracts no rents. However, the allocation of bidders is not necessarily efficient. For close substitutes there is a unique equilibrium. The weakest type gathers participation rents. Efficiency is restored with the indifferent type generating the same total surplus no matter which seller he attends.

I also show that the analogy established in Bulow and Roberts (1989) does not hold in this setting. Finally, reserve prices under competition are greater than the ones set when a single item is auctioned for items sufficiently differentiated. This is so because the presence of a substitute reduces attendance at an auction by eliminating the weakest bidders or because sellers have an incentive to save on participation rents. For very close substitutes sellers care about market shares and set reserve prices below the monopolistic ones.

Several interesting extensions could be considered. First, a natural question is what mechanism would sellers use in equilibrium if this was part of their strategic choices. Note that such an issue is complex. The consideration of horizontally differentiated substitutes potentially triggers countervailing incentives. Indeed, as a buyer overstates (or understates) his type he sends a positive signal to one seller and a negative signal to the other seller. If sellers cooperate, it can be in their interest to resort to inefficient allocations (see, for instance, Parlane (2001)) to make use of these countervailing incentives so as to save on informational rents. Thus it is not clear whether the best reply to an efficient mechanism, such the English auction, is also an efficient mechanism. As we add competition, the sub-game between buyers is to be

analyzed with caution. Indeed, even if a seller takes into account whatever the other does in a Nash approach, a buyer's expected payoff from attending the opponent's auction depends critically on what both sellers propose. One possibility to analyze equilibrium auctions would be to restrict the class of mechanisms they can implement.

A second interesting extension would be to analyze sequential versus simultaneous auctioning. If attending an auction is free for the bidders, they would attend both auctions as long as their valuation is not below any of the reserve prices and provided they did not win the initial auction. Whether sequential auctioning is always better is not clear. When the number of buyers is large the greater competition may work in favour of the sequential auctioning. When there are less buyers the simultaneous auctioning presents an advantage. In the case of cooperating sellers, the optimal reserve prices in sequential auctioning are lower than the ones they would set for each object if sold simultaneously. As the number of buyers decreases, the probability of having a bidder paying the actual reserve price increases. Thus, there may be some incentive for a cooperating sellers to sell the items simultaneously.

## 8 Appendix

- Appendix 1: Proof of proposition 1.

If reserve prices are such that  $r_1 \leq r_2$ , the proof is trivial since the reservation utility is zero.

Let the reserve prices be such that  $r_1 > r_2$ .

Claim 1: For any  $R \in ]0, +\infty) \times (-\infty, 1[$  with  $r_1 > r_2$ ,  $\theta_R$ , defined by (1), is unique and always within the interval  $[0, 1]$ .

Proof: Consider the function

$$H(\theta) = (1 - F(\theta))^{n-1} (r_1 - \theta) - F^{n-1}(\theta) (\theta - r_2),$$

defined over  $[0, 1]$ . Given (1), we have:

$$H(\theta_R) \equiv 0.$$

Given any  $R$  such that  $r_1 > r_2$ , we have<sup>4</sup>  $H(0) > 0$  and  $H(1) < 0$ . Since  $H(\cdot)$  is continuous, there exists at least one value  $\theta_R \in [0, 1]$  for which  $H(\theta_R) = 0$ . For any  $\theta \geq \min\{r_1, 1\}$ ,  $H(\theta) < 0$ . For any  $\theta \leq \max\{0, r_2\}$ ,  $H(\theta) > 0$ . Thus, all solutions to  $H(\theta) = 0$  lie within the range  $[r_2, r_1]$ . Given this, we have

$$\left. \frac{dH}{d\theta} \right|_{\theta=\theta_R} < 0.$$

Thus, there exists at most one  $\theta_R \in [0, 1]$  such that  $H(\theta_R) \equiv 0$ .

Claim 2: The buyers' strategy depicted in proposition 1 forms a Nash equilibrium.

Proof: Assume that  $(n - 1)$  buyers adopt the strategy depicted in proposition 1. Consider a buyer of type  $\theta$ . Let  $U_i(\theta)$  with  $i = 1, 2$  denote this bidder's expected payoff when attending seller  $i$  ( $i = 1, 2$ ). (We focus at the case where  $0 < r_2 < r_1 < 1$ . The extension to  $r_1 > 1$  and/or  $r_2 < 0$  is trivial.)

- If  $\theta \in [r_1, 1]$  then  $U_2(\theta) > 0 > U_1(\theta)$  : attending seller 2 is a best reply.
- If  $\theta \in [0, r_2]$  then  $U_1(\theta) > 0 > U_2(\theta)$  : attending seller 1 is a best reply.
- If  $\theta \in [r_1, r_2]$ . We have

$$U_1(\theta) = \begin{cases} (1 - F(\theta_R))^{n-1} t(r_1 - \theta) \\ \quad + \int_{\theta}^{\theta_R} t(x - \theta) (n - 1) (1 - F(x))^{n-2} f(x) dx \text{ if } \theta < \theta_R \\ (1 - F(\theta_R))^{n-1} t(r_1 - \theta) \text{ if } \theta \geq \theta_R, \end{cases}$$

and

$$U_2(\theta) = \begin{cases} (F(\theta_R))^{n-1} t(\theta - r_2) & \text{if } \theta \leq \theta_R \\ (F(\theta_R))^{n-1} t(\theta - r_2) \\ \quad + \int_{\theta}^{\theta_R} t(\theta - x) (n-1) (F(x))^{n-2} f(x) dx & \text{if } \theta > \theta_R. \end{cases}$$

For any  $\theta \in [r_1, r_2]$ , we have  $\frac{dU_1}{d\theta} < 0$  and  $\frac{dU_2}{d\theta} > 0$ . Moreover we have  $U_1(\theta_R) = U_2(\theta_R)$  by definition of  $\theta_R$ , and  $U_i(\theta_R) > 0$  for  $i = 1, 2$ . Thus for all  $\theta < \theta_R$  (respectively  $\theta > \theta_R$ ) attending seller 1 (respectively seller 2) forms a best reply. Therefore the strategy depicted in proposition 1 forms a Nash equilibrium.

- Appendix 2: Proof of proposition 2.

To clarify the presentation I will write the sellers' profits as a function of  $(r_1, r_2)$  instead of  $(\gamma_1, \gamma_2)$  and consider  $r$  as a strategic variable instead of  $\gamma$ . There is no loss in generalities in doing so.

Let  $\pi_i(r_i, r_j)$  denote seller  $i$ 's expected profit function ( $i = 1, 2$  and  $j \neq i$ ). Let  $MR_i(\cdot, t)$  ( $i = 1, 2$ ) and  $g_i(\cdot, t)$  ( $i = 1, 2$ ) be the functions defined by (2), (3), (5) and (6) in the text.

The first order condition (FOC hereafter) for each seller are given by:

Seller 1:

$$\frac{\partial \pi_1}{\partial r_1} = \begin{cases} n(1 - F(r_1))^{n-1} f(r_1) MR_1(r_1, t) & \text{for any } r_1 \leq r_2, \\ n f(\theta_R) \frac{\partial \theta_R}{\partial r_1} [g_1(\theta_R, t) - n t (1 - F(\theta_R))^{n-1} (r_1 - \theta_R)] & \text{for } r_1 > r_2. \end{cases} \quad (10)$$

Seller 2:

$$\frac{\partial \pi_2}{\partial r_2} = \begin{cases} -n (F(r_2))^{n-1} f(r_2) MR_2(r_2, t) & \text{for any } r_2 \geq r_1, \\ n f(\theta_R) \frac{\partial \theta_R}{\partial r_2} [n t (F(\theta_R))^{n-1} (\theta_R - r_2) - g_2(\theta_R, t)] & \text{for } r_2 < r_1. \end{cases} \quad (11)$$

Assume  $t > \bar{t}$ . As picture 1 (in the text) shows, there always exist  $\theta_1$  and  $\theta_2$  such that  $MR_i(\theta_i, t) = 0$  for  $i = 1, 2$  and such that  $\theta_1 < \theta_2$ .

Assume seller 2 sets  $\gamma_2 = v_2(\theta_2)$ . Over the interval  $[0, \theta_2]$   $\pi_1(r_1, \theta_2)$  reaches a maximum at  $\theta_1$  since  $\frac{\partial \pi_1}{\partial r_1} = 0$  at  $\theta_1$  and concavity is ensured under the regularity assumption. The expected profit  $\pi_1$  is continuous at  $r_1 = \theta_2$ . We

have  $\frac{\partial \pi_1}{\partial r_1} < 0$  for all  $r_1 \geq \theta_2$  since  $(r_1 - \theta_{(r_1, \theta_2)}) > 0$  and  $g_1(\theta_{(r_1, \theta_2)}, t) < MR_1(\theta_{(r_1, \theta_2)}, t) < 0$ .

Assume seller 1 sets  $\gamma_1 = v_1(\theta_1)$ . Over the interval  $[\theta_1, 1]$   $\pi_2(r_2, \theta_1)$  reaches a maximum at  $\theta_2$  since  $\frac{\partial \pi_2}{\partial r_2} = 0$  at  $\theta_2$  and concavity is ensured under the regularity condition. The expected profit  $\pi_2$  is continuous at  $r_2 = \theta_1$ . We have  $\frac{\partial \pi_2}{\partial r_2} > 0$  for all  $r_2 \leq \theta_1$  since  $(\theta_{(\theta_1, r_2)} - r_2) > 0$  and  $g_2(\theta_{(\theta_1, r_2)}, t) < MR_2(\theta_{(\theta_1, r_2)}, t) < 0$ . Thus, independently on whether they compete or maximize joints profits, seller  $i$ 's optimal reserve price when  $t > \bar{t}$  is given by  $\gamma_i = v_i(\theta_i)$  for  $i = 1, 2$ .

- Appendix 3: Proof of proposition 3.

Let  $t \leq \bar{t}$  and assume sellers maximize joint profits.

In the text we prove formally that, joint profit maximization requires  $r_1 = r_2 = r$ . Using this result, we can write the derivative of joint profits as:

$$\frac{d(\pi_1 + \pi_2)}{dr} = f(r) [(1 - F(r))^{n-1} MR_1(r, t) - (F(r))^{n-1} MR_2(r, t)].$$

Consider the function

$$h(r) = (1 - F(r))^{n-1} MR_1(r, t) - (F(r))^{n-1} MR_2(r, t).$$

Let  $\theta_i$  be such that  $MR_i(\theta_i, t) \equiv 0$  for  $i = 1, 2$ . For any  $t \leq \bar{t}$  we have  $\theta_1 \geq \theta_2$ . The function  $h(r)$  is strictly decreasing over  $[\theta_2, \theta_1]$ . We have  $h(\theta_2) > 0$  and  $h(\theta_1) < 0$ . Thus, over the range  $[\theta_2, \theta_1]$  there is one and only one value  $r^c$  such that  $h(r^c) = 0$ . For any  $r < \theta_2$ , we have  $h(r) > 0$ , and for any  $r > \theta_1$  we have  $h(r) < 0$ . Thus,  $r^c$  is a unique solution to  $\frac{d(\pi_1 + \pi_2)}{dr} = 0$  such that

$$\frac{d\pi}{dr} \begin{cases} > 0 \\ < 0 \end{cases} \text{ as } r \begin{cases} < \\ > \end{cases} r^c.$$

Thus the sellers' joint profits reach a maximum at  $r = r^c$ .

- Appendix 4: Proof of proposition 4.

Let again  $\theta_i$  be defined such that  $MR_i(\theta_i, t) = 0$  and let  $t < \bar{t}$ . For any such  $t$ , we have  $\theta_1 > \theta_2$ . The first step consists in proving that there exists a range of degree of product differentiation below which both, (7) and (8) can hold.

*Lemma 2: For any  $t \in ]0, \bar{t}[$  there exists a unique  $\theta_t \in [\theta_2, \theta_1]$  for which  $g_1(\theta_t, t) = g_2(\theta_t, t)$ . Moreover, there exists a unique  $\underline{t} \in ]0, \bar{t}[$  such that for all  $t \leq \underline{t}$ ,  $g_i(\theta_t, t) > 0$  for  $i = 1, 2$ . The variable  $\underline{t}$  solves*

$$g_1(\theta_{\underline{t}}, \underline{t}) = g_2(\theta_{\underline{t}}, \underline{t}) = 0.$$

**Proof**

(1) Existence and uniqueness of  $\theta_t$ . The functions  $g_1(\theta, t)$  and  $g_2(\theta, t)$  are continuous in  $\theta$ . We have  $g_1(0, t) = g_2(1, t) = 1$  while  $g_1(\theta_1, t) < 0$  and  $g_2(\theta_2, t) < 0$ . Thus, for any  $t \in ]0, \bar{t}[$  there exists at least one  $\theta_t \in [\theta_2, \theta_1]$  such that:

$$g_1(\theta_t, t) - g_2(\theta_t, t) = 0.$$

Over the range  $[\theta_2, \theta_1]$ , we have  $MR_i(\theta, t) \geq 0$  for  $i = 1, 2$ . This implies that, over that range,

$$\frac{\partial}{\partial \theta} [g_1(\theta, t) - g_2(\theta, t)] < 0.$$

Thus,  $\theta_t$  is unique.

(1) Existence and uniqueness of  $\underline{t}$ . Let  $\underline{t}$  denote the degree of product differentiation such that:  $g_1(\theta_{\underline{t}}, \underline{t}) = g_2(\theta_{\underline{t}}, \underline{t}) = 0$ . Let  $G(t) \equiv g_i(\theta_t, t)$  where  $i = 1$  or  $2$ . The function  $G(t)$  is continuous. As  $t \rightarrow 0$ ,  $G(t) > 0$  since  $g_i(\theta, 0) > 0$  for any  $\theta \in ]0, 1[$ . Furthermore we necessarily have  $G(\bar{t}) < 0$  since we have  $g_i(\theta, t) \leq MR_i(\theta, t)$ . Thus there exists at least one  $\underline{t} \in [0, \bar{t}]$  such that  $G(\underline{t}) = 0$ .

To prove uniqueness, we shall prove that  $G(t)$  is decreasing in  $t$ . We have

$$\frac{dG}{dt} = \frac{\partial g_i}{\partial \theta} \frac{d\theta_t}{dt} + \frac{\partial g_i}{\partial t} \Big|_{\theta=\theta_t}.$$

Using the implicit function theorem we have

$$\frac{d\theta_t}{dt} = - \frac{\frac{\partial g_1}{\partial t} - \frac{\partial g_2}{\partial t}}{\frac{\partial g_1}{\partial \theta} - \frac{\partial g_2}{\partial \theta}}.$$

Since  $\left(\frac{\partial g_1}{\partial \theta} - \frac{\partial g_2}{\partial \theta}\right) < 0$  it is true that

$$\text{sign of } \frac{dG}{dt} = \text{sign of } \frac{\partial g_i}{\partial \theta} \left( \frac{\partial g_1}{\partial t} - \frac{\partial g_2}{\partial t} \right) + \frac{\partial g_i}{\partial t} \left( \frac{\partial g_2}{\partial \theta} - \frac{\partial g_1}{\partial \theta} \right).$$

Over the range  $[\theta_2, \theta_1]$  we have  $\frac{\partial g_1}{\partial \theta} < 0$  and  $\frac{\partial g_2}{\partial \theta} > 0$ . We always have  $\frac{\partial g_i}{\partial t} < 0$  for  $i = 1, 2$ . Thus, whether one sets  $i = 1$  or  $i = 2$  in the expression above, we always have that the sign of  $\frac{dG}{dt}$  is negative.

Finally since  $g_i(\theta_{\bar{t}}, \bar{t}) < 0$ , it must be that  $\underline{t} < \bar{t}$ .

We may now prove that the reservation prices described in proposition 4 form an equilibrium. Let  $t \leq \underline{t}$ . Consider  $(\gamma_1^*, \gamma_2^*)$  presented in proposition 4. They are such that the *FOC* (10) and (11) are satisfied. Moreover, since these reserve prices are set such that  $\theta_{R^*} = \theta_t$  for any  $t < \underline{t}$ , the additional restrictions (7) and 8) hold. Thus, all we must prove is that they also satisfy second order conditions. In order to do so, we need more precise information on the functions  $g_i(\theta, t)$  ( $i = 1, 2$ ).

(1) There is a unique  $\tilde{\theta}_1$  such that  $g_1(\tilde{\theta}_1, t) = 0$  for any  $t$ .

Proof: We have  $g_1(0, t) = 1$ , and  $g_1(1, t) = -\frac{t}{f(1)} < 0$ . Since  $g_1$  is a continuous function, there exists at least one  $\tilde{\theta}_1$  such that  $g_1(\tilde{\theta}_1, t) = 0$ . Moreover, because  $g_1(\theta, t)$  is decreasing in  $\theta$  for all  $\theta$  such that  $MR_1(\theta, t) \geq 0$  and since  $MR_1(\tilde{\theta}_1, t) > 0$ , we always have  $\frac{d}{d\theta}g_1(\theta, t) < 0$  at  $\tilde{\theta}_1$ . Thus,  $\tilde{\theta}_1$  is unique.

(2) There is a unique  $\tilde{\theta}_2$  such that  $g_2(\tilde{\theta}_2, t) = 0$  for any  $t$ .

Proof: We have  $g_2(1, t) = 1$ , and  $g_2(0, t) = -\frac{t}{f(0)} < 0$ . Since  $g_2$  is a continuous function, there exists at least one  $\tilde{\theta}_2$  such that  $g_2(\tilde{\theta}_2, t) = 0$ . Moreover, because the function  $g_2(\theta, t)$  is increasing in  $\theta$  for all  $\theta$  such that

$MR_2(\theta, t) \geq 0$  and since  $MR_2(\tilde{\theta}_2, t) > 0$ , we always have  $\frac{d}{d\theta}g_2(\theta, t) > 0$  at  $\tilde{\theta}_2$ . Thus,  $\tilde{\theta}_2$  is unique.

Given the previous points we know that the curves of marginal revenues and  $g_i(\cdot)$  interact as follows for  $t < \underline{t}$ :

Figure 3.

We can now prove that the proposed solution maximizes each seller's expected profit. For any of these reserve prices we have  $r_2^* < \theta_t < r_1^*$ .

Consider seller 1. He takes as given  $r_2^*$ , such that  $r_2^* < \theta_t$ . Clearly, we have  $\frac{\partial \pi_1}{\partial r_1} > 0$  for all  $r_1 < r_2^*$  since  $MR_1(r, t) > 0$  for all  $r \leq \theta_t$ . For any  $r_1 \in [r_2^*, +\infty)$ , we have (given (10)):

$$\frac{\partial \pi_1}{\partial r_1} = nf(\theta_R) \frac{\partial \theta_R}{\partial r_1} (1 - F(\theta_R))^{n-1} H_1(\theta_R, r_1, t) = 0 \text{ at } r_1 = r_1^* \text{ and } R = (r_1, r_2^*),$$

with

$$H_1(\theta_R, r_1, t) = \left[ MR_1(\theta_R, t) - t \frac{F(\theta_R)}{f(\theta_R)} \left[ \frac{F(\theta_R)}{1 - F(\theta_R)} \right]^{n-1} - nt(r_1^* - \theta_R) \right].$$

Since  $0 < \frac{\partial \theta_R}{\partial r_i} < 1$  (trivial), and under the regularity assumption we have

$\frac{\partial H_1}{\partial r_1} < 0$ . Thus:

$$\frac{\partial \pi_1}{\partial r_1} > 0 \text{ as } r_1 < r_1^* \\ \frac{\partial \pi_1}{\partial r_1} < 0 \text{ as } r_1 > r_1^*$$

which proves that  $r_1^*$  is best reply to  $r_2^*$ .

Consider now seller 2. He takes as given  $r_1^*$ , such that  $r_1^* > \theta_t$ . Clearly, we have  $\frac{\partial \pi_2}{\partial r_2} < 0$  for all  $r_2 > r_1^*$  since  $MR_2(r, t) > 0$  for all  $r \geq \theta_t$ . For any  $r_2 \in (-\infty, r_1^*]$ , we have (given(11)) :

$$\frac{\partial \pi_2}{\partial r_2} = nf(\theta_R) \frac{\partial \theta_R}{\partial r_1} (F(\theta_R))^{n-1} H_2(\theta_R, r_2, t) = 0 \text{ at } r_2 = r_2^* \text{ and } R = (r_1^*, r_2),$$

with

$$H_2(\theta_R, r_2, t) = \left[ nt(\theta_R - r_2) - MR_2(\theta_R, t) + t \frac{1 - F(\theta_R)}{f(\theta_R)} \left[ \frac{1 - F(\theta_R)}{F(\theta_R)} \right]^{n-1} \right].$$

Since  $0 < \frac{\partial \theta_R}{\partial r_i} < 1$ , and under the regularity assumption we have  $\frac{\partial H}{\partial r_2} < 0$ .

Thus:

$$\frac{\partial \pi_2}{\partial r_2} > 0 \text{ as } r_2 < r_2^* \\ \frac{\partial \pi_2}{\partial r_2} < 0 \text{ as } r_2 > r_2^*$$

which proves that  $r_2^*$  is best reply to  $r_1^*$ .

(Expression (9) can be found easily using (7).)

- Appendix 6: Proof of proposition 5.

Let  $t \in ]\underline{t}, \bar{t}[$ . For such  $t$ , the marginal revenue and  $g_i(., .)$  curves interact as follows:

Figure 4.

- (1) The set  $\Theta_t$  is always non-empty. Let

$$\Omega_i = \{\theta : MR_i(\theta, t) \geq 0 \text{ and } g_i(\theta, t) \leq 0\}$$

for  $i = 1, 2$ . Since  $g_1(\theta, t)$  is decreasing over  $[0, \theta_1]$  with  $g_1(\theta_1, t) < 0$ ,  $\Omega_1$  is non empty and  $\Omega_1 = [\tilde{\theta}_1, \theta_1]$ , with  $g_1(\tilde{\theta}_1, t) = 0$ . Since  $g_2(\theta, t)$  is increasing over  $[\theta_2, 1]$  with  $g_2(\theta_2, t) < 0$ ,  $\Omega_2$  is also non-empty and  $\Omega_2 = [\theta_2, \tilde{\theta}_2]$  with  $g_2(\tilde{\theta}_2, t) = 0$ .

By definition we have  $\Theta_t = \Omega_1 \cap \Omega_2$ . As  $t < \bar{t}$  we have  $\theta_2 < \theta_1$ . Thus,  $\Theta_t$  would be empty if and only if  $\tilde{\theta}_2 < \tilde{\theta}_1$ . However, for  $t > \underline{t}$ , we have  $g_i(\theta_t, t) < 0$  for  $i = 1, 2$  and therefore  $\tilde{\theta}_2 > \tilde{\theta}_1$ .

Consider now any  $r_2^* \in \Theta_t$ . For any  $r_1 < r_2^*$  we have  $\frac{\partial \pi_1}{\partial r_1} > 0$  since  $MR_1(r, t) > 0$  for all  $r < r_2^*$ . For all  $r_1 > r_2^*$  we have  $(\theta_{(r_1, r_2^*)} - r_1) < 0$  and  $g_1(\theta_{(r_1, r_2^*)}, t) < 0$  which implies that  $\frac{\partial \pi_1}{\partial r_1} < 0$  for all  $r_1 \geq r_2^*$ . At  $r_2^*$ ,  $\pi_1$  is not differentiable and forms a peak since  $\left. \frac{\partial \pi_1}{\partial r_1} \right|_{r_2^* - \varepsilon} > \left. \frac{\partial \pi_1}{\partial r_1} \right|_{r_2^* + \varepsilon}$  as  $\varepsilon \rightarrow 0$ . Yet, it also forms a maximum (as it appears on picture 2 in the text)..

Consider now any  $r_1^* \in \Theta_t$ . For any  $r_2 > r_1^*$  we have  $\frac{\partial \pi_2}{\partial r_2} < 0$  since  $MR_2(r, t) > 0$  for all  $r > r_1^*$ . For all  $r_2 < r_1^*$  we have  $(\theta_{(r_1^*, r_2)} - r_2) > 0$  and  $g_2(\theta_{(r_1^*, r_2)}, t) < 0$  which implies that  $\frac{\partial \pi_2}{\partial r_2} > 0$  for all  $r_2 < r_1^*$ . Once again at  $r_2 = r_1^*$  the profit function forms a peak.

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# Figures

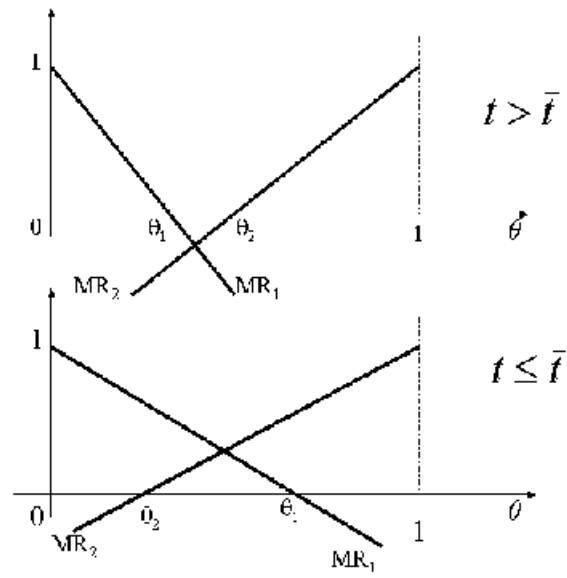


Figure 1:

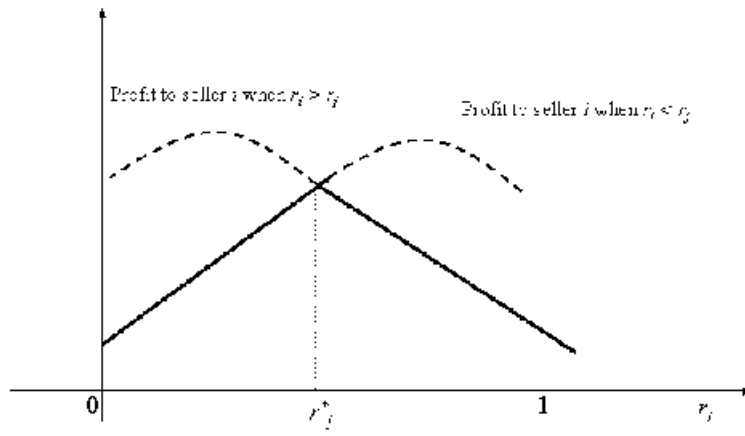


Figure 2: Illustration of proposition 5.

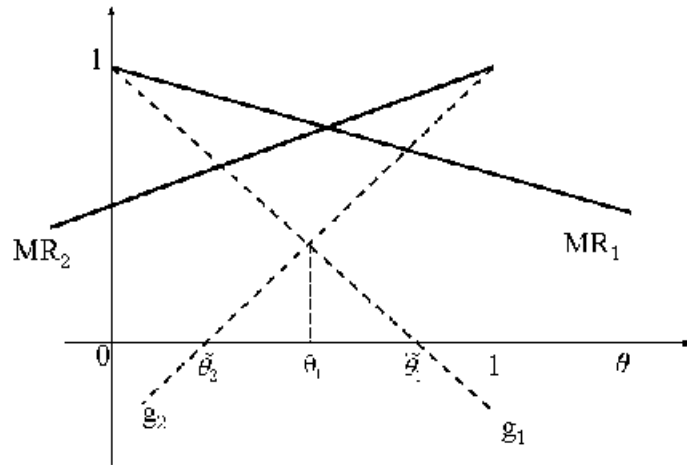


Figure 3:

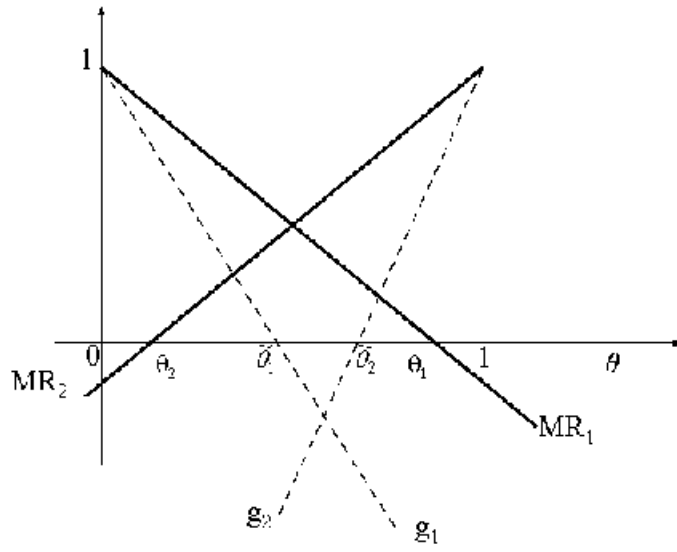


Figure 4:

## Footnotes.

1- Results would hold if one extended this model to a situation where  $v_1(\theta) = v - t\theta$ , and  $v_2(\theta) = v - t(1 - \theta)$  as long as  $v \geq t$ . Similarly, they would hold if one considered quadratic transport costs.

2-Allowing for nondecreasing functions instead may lead to multiplicity of equilibria.

3-The remaining possibility:  $MR_i(r_i, t) < 0$  for  $i = 1, 2$  requires  $r_1 > r_2$ .

4-Note that  $r_1 > 0$  and  $r_2 < 1$ .